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Gap Tauberian theorem for generalized Abel summability

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Let $\alpha > -1$. For a given series $\sum_0^\infty a_n$, write

$$A_n = \sum_{r=0}^{n} a_r \quad (n \geqslant 0),$$

$$a(t) = \sum_{n=0}^{\infty} a_n \binom{n+\alpha}{\alpha} e^{-nt} \quad (t > 0),$$

$$A(t) = \sum_{n=0}^{\infty} A_n \binom{n+\alpha}{\alpha} e^{-nt} \quad (t > 0),$$

it being tacitly assumed that the series defining $a(t)$ and $A(t)$ converge for all $t > 0$. Σa_n is said to be summable (A_α) to A if

$$(1-e^{-t})^{\alpha+1} A(t) \;\rightarrow\; A \quad \text{as} \quad t \rightarrow 0+.$$

The purpose of this note is to establish the following gap Tauberian theorem for (A_α) summability:

THEOREM. *If Σa_n satisfies a gap condition as follows:*

$$\left.\begin{array}{l} a_n = 0 \textit{ for } n \neq n_r \textit{ where } \{n_r\} \textit{ is a sequence of positive integers such that} \\ n_{r+1}/n_r \geqslant c > 1 \textit{ for all } r, \end{array}\right\} \tag{1}$$

and if Σa_n is summable (A_α) to A, then Σa_n converges to A.

Since (A_λ) summability implies (A_μ) summability for $\lambda > \mu > -1$ ((1), Theorem 2, 319), we need to (and do) consider the case $-1 < \alpha < 0$. The well known gap Tauberian theorem for (A_0) summability (viz. the special case of (2), Theorem 114, where λ_n therein are integers) is, therefore, a special case of our Theorem.

We will say that a function $s(y)$ belongs to the Tauberian class T_σ if functions $\delta = \delta(\epsilon, x)$, $\xi = \xi(\epsilon, x)$ can be defined for all x and $\epsilon > 0$ such that

$$\delta > 0, \quad |\xi - x| \leqslant \delta,$$

$$|s(y) - s(x)| \leqslant \epsilon \quad \text{for} \quad |y - \xi| < \delta.$$

This is a special case of the Tauberian class T_β considered by Pitt ((3); 7, 8; with Pitt's notation, we have taken $\mu = 0$).

LEMMA 1. *Suppose that $k(x)$, $k_1(x)$ are integrable over $(-\infty,\infty)$, that the Fourier transform $K(t)$ of $k(x)$ does not vanish for real t, that $s(x)$ is bounded and that*

$$\lim_{x\to\infty}\int_{-\infty}^{\infty} k(x-y)\,s(y)\,dy = A\int_{-\infty}^{\infty} k(y)\,dy.$$

Then
$$\lim_{x\to\infty}\int_{-\infty}^{\infty} k_1(x-y)\,s(y)\,dy = A\int_{-\infty}^{\infty} k_1(y)\,dy.$$

This is Wiener's theorem ((3); IV, Theorem 11, 51).

LEMMA 2. *Suppose $\sigma_2 > \sigma_1 \geqslant 0$,*

$$K(-i\omega) = K(t-i\sigma) = \int_{-\infty}^{\infty} e^{-\omega y}k(y)\,dy \neq 0 \quad (0 \leqslant \sigma \leqslant \sigma_1).$$

Suppose also that $s(x) = O(e^{\sigma x})$ for $x > 0$ and each fixed σ in the range $\sigma_1 < \sigma \leqslant \sigma_2$, while $s(x) = 0$ for $x \leqslant 0$,

$$g(x) = \int_{-\infty}^{\infty} k(x-y)\,s(y)\,dy \quad \textit{is bounded}$$

and $s(x)$ belongs to the Tauberian class T_σ. Then $s(x)$ is bounded.

This is a special case of a theorem of Pitt ((3), IV, Theorem 23, 72).

LEMMA 3. *If (1) is satisfied and $(1-e^{-t})^{\alpha+1}A(t) = O(1)$ $(t \to 0+)$, then $A_n = O(n^{-\alpha})$ $(n \to \infty)$.*

Proof. Since the series defining $a(t)$ and $A(t)$ are power series in e^{-t} we can write for all $t > 0$,

$$\begin{aligned} a(t) &= \sum_{n=0}^{\infty}\binom{n+\alpha}{\alpha} e^{-nt}(A_n - A_{n-1}) \quad (A_{-1} = 0),\\ &= A(t) - \sum_{n=0}^{\infty}\binom{n+\alpha}{\alpha}\left(1+\frac{\alpha}{n+1}\right) e^{-(n+1)t}A_n,\\ &= A(t) - e^{-t}A(t) - \alpha\int_t^{\infty} e^{-u}A(u)\,du. \end{aligned}$$

By hypothesis, there is a constant M such that

$$|(1-e^{-t})^{\alpha+1}A(t)| \leqslant M \quad \text{for all} \quad t > 0.$$

Hence
$$\begin{aligned} |a(t)| &\leqslant M(1-e^{-t})^{-\alpha} + |\alpha|\int_t^{\infty} e^{-u}M(1-e^{-u})^{-(\alpha+1)}\,du\\ &= M(1-e^{-t})^{-\alpha} - M\alpha\left[\frac{1-(1-e^{-t})^{-\alpha}}{-\alpha}\right] = O(1) \quad (t \to 0+). \end{aligned}$$

Applying Theorem 116 in (2) to $\Sigma b_n e^{-nt}$ where $b_n = a_n\binom{n+\alpha}{\alpha}$ it now follows that $a_n\binom{n+\alpha}{\alpha} = O(1)$ and hence that $a_n = O(n^{-\alpha})$.

Now, if $n_k \leqslant n < n_{k+1}$ we have, by (1),

$$|A_n| \leqslant \sum_{r=1}^{k}|a_{n_r}| \leqslant M_1 n_k^{-\alpha}\sum_{r=1}^{k}\left(\frac{n_k}{n_r}\right)^{\alpha} \leqslant M_1 n^{-\alpha}\sum_{r=1}^{k} c^{(k-r)\alpha} = O(n^{-\alpha}).$$

LEMMA 4. *If* $A_n = o(n)\ (n \to \infty)$, Σa_n *is summable* (A_α) *to* A *and*

$$\Psi(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\infty u^\alpha e^{-tu} \bar{A}(u)\, du \quad (t > 0), \tag{2}$$

where $\bar{A}(u) = A_n$ *for* $n \leqslant u < n+1$, *then*

(i) $\Psi(t) \to A\ (t \to 0+)$, *and*

(ii) $\Psi(t)$ *is bounded for* $0 < t < \infty$.

Proof. Part (i) is a special case of an unpublished result of C. T. Rajagopal who has kindly supplied this proof to me. Supposing that $A_0 = 0$ we can write $\Psi(t) = \sum_{n=1}^{\infty} d_n(t) A_n$ where

$$\begin{aligned} d_n(t) &= \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} \int_n^{n+1} u^\alpha e^{-tu}\, du, \\ &= \frac{t^\alpha}{\Gamma(\alpha+1)} [n+\theta_n(t)]^\alpha [e^{-nt} - e^{-(n+1)t}] \quad (0 < \theta_n(t) < 1), \\ &= \frac{t^\alpha}{\Gamma(\alpha+1)} n^\alpha \left[1 + \frac{w_n(t)}{n}\right] e^{-nt}(1-e^{-t}) \quad (|w_n(t)| \leqslant M), \\ &= t^\alpha \binom{n+\alpha}{\alpha} \left(1 + \frac{w_n'}{n}\right) \left[1 + \frac{w_n(t)}{n}\right] e^{-nt}(1-e^{-t}) \quad (|w_n'| \leqslant M'), \\ &= t^\alpha (1-e^{-t}) \left[1 + \frac{\delta_n(t)}{n}\right] \binom{n+\alpha}{\alpha} e^{-nt} \quad (|\delta_n(t)| \leqslant M''), \end{aligned}$$

M, M', M'' being independent of n and t. This shows that $\Psi(t)$ exists for all $t > 0$ and that

$$\left| \left(\frac{1-e^{-t}}{t}\right)^\alpha \Psi(t) - (1-e^{-t})^{\alpha+1} A(t) \right| = \left| (1-e^{-t})^{\alpha+1} \sum_{n=1}^{\infty} A_n \binom{n+\alpha}{\alpha} e^{-nt} \frac{\delta_n(t)}{n} \right|$$

$$\leqslant M''(1-e^{-t})^{\alpha+1} \sum_{n=1}^{\infty} A_n n^{-1} \binom{n+\alpha}{\alpha} e^{-nt} \to 0 \quad \text{as} \quad t \to 0+,$$

since $A_n n^{-1} \to 0$ and the (A_α) method is regular. This proves (i).

It can be directly verified that $\Psi(t) = O(1)\ (t \to \infty)$; and hence (ii) holds.

LEMMA 5. *If* $A_n = O(n^{-\alpha})$, $\Psi(t)$, *defined by* (2), *is bounded in* $(0, \infty)$ *and* (1) *is satisfied, then* $A_n = O(1)\ (n \to \infty)$.

Proof. Writing $u = e^y$ and $t = e^{-x}$ we have

$$\Psi(e^{-x}) = \frac{1}{\Gamma(\alpha+1)} \int_{-\infty}^{\infty} e^{(y-x)(\alpha+1)} \exp\{-e^{y-x}\} \bar{A}(e^y)\, dy.$$

Let
$$k(x) = \frac{1}{\Gamma(\alpha+1)} e^{-(\alpha+1)x} \exp\{-e^{-x}\}, \quad \bar{A}(e^y) = s(y)$$

and $g(x) = \Psi(e^{-x})$. Then

$$g(x) = \int_{-\infty}^{\infty} k(x-y)\, s(y)\, dy.$$

Now the gap condition (1) implies that $s(y)$ is constant in some interval of length not less than $2\delta = \log c$ (closed on the left, open on the right) which contains x. Hence we can choose ξ with $|\xi - x| \leqslant \delta$ such that $s(y) = s(x)$ for $|y-\xi| \leqslant \delta$. This means that $s(y)$ belongs to the Tauberian class T_σ defined earlier. The Fourier transform of $k(x)$ is $K(t) = \Gamma(\alpha+1+it)/\Gamma(\alpha+1)$ which does not vanish for any t. Further, the boundedness of $\Psi(t)$ implies that of $g(x)$ and $A_n = O(n^{-\alpha})$ implies that $s(y) = O(e^{-\alpha y})$. Thus the hypotheses of the lemma imply the hypotheses of Lemma 2 with $\sigma_1 = -\alpha$.

LEMMA 6. *If Σa_n satisfies the hypotheses of our theorem and if $A_n = O(1)$ $(n \to \infty)$ then Σa_n converges to A.*

Proof. It follows from Lemma 4 that

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} k(x-y)\, s(y)\, dy = A \int_{-\infty}^{\infty} k(y)\, dy$$

in the notation used in the proof of Lemma 5. Taking $k_1(x) = e^{-x}$ for $x \geqslant 0$, $k_1(x) = 0$, for $x < 0$, we deduce from Lemma 1 that $\bar{A}(u) \to A(C, 1)$. The lemma now follows from the fact that (1) is a gap Tauberian condition for $(C, 1)$ summability.

The theorem is obviously obtained by combining Lemmas 3, 4, 5 and 6.

I am indebted to the Referee for simplifying the use of Pitt's Tauberian classes and thereby correcting a mistake in my original proof of Lemma 5. I thank Professor Rajagopal and Dr M. S. Rangachari for their help in the preparation of this note.

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