GAP TAUBERIAN THEOREM FOR LOGARITHMIC SUMMABILITY (L)

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(In homage to the late Professor C. T. Rajagopal)

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1. We write $\sum_{n=0}^{\infty} a_n = s(L)$ when

$$(1.1) \qquad L(x) = \frac{1}{\log\left(\frac{1}{1-x}\right)} \sum_{n=0}^{\infty} \frac{A_n}{n+1} x^{n+1} \to s \quad \text{as} \quad x \to 1-0,$$

and say that $\sum_{n=0}^{\infty} a_n$ is summable to s by the logarithmic method (L) of summability. The series on the right of (1.1) is tacitly assumed to converge for $0<x<1$ and $\{A_n\}$ is the sequence of partial sums of the series $\sum_{n=0}^{\infty} a_n$, i.e. $A_n = a_0 + \cdots + a_n$. We say that the sequence $\{A_n\}$ is

$$(1.2) \qquad t_n = \frac{1}{P(n)} \sum_{\nu=0}^{n} \frac{A_\nu}{\nu+1} \to s \quad \text{as} \quad n \to \infty,$$

where

$$P(n) = \sum_{\nu=0}^{n} \frac{1}{\nu+1} \sim \log(n+1),\ n \to \infty.$$

It is known ([3], Theorem 3) that

$$(1.3) \qquad \sum_{n=0}^{\infty} a_n = s(l) \Rightarrow \sum_{n=0}^{\infty} a_n = s(L).$$

The main object of the present paper is to prove the following "pure gap Tauberian theorem" for summability (L).

Theorem 1. *If* $\sum_{n=0}^{\infty} a_n = s\ (L)$ *and if the gap condition*

$$(1.4) \qquad a_n = 0,\ n \neq n_k,\ k = 1, 2, \ldots,\ n_k \geqslant 2, \text{ integers},$$
$$\log\log n_{k+1} - \log\log n_k \geqslant \delta > 0,\ k = 1, 2, \ldots,$$

is fulfilled, then $\sum_{n=0}^{\infty} a_n$ *converges to* s.

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We first show that the above result is *best possible* in the following sense. Let $\{n_k\}$ be any increasing sequence of non-negative integers such that

(1.5) $$\liminf_{k\to\infty} [\log\log n_{k+1} - \log\log n_k] = 0.$$

Then there is a series $\sum_{n=0}^{\infty} a_n$ satisfying the gap condition

(1.6) $$a_n = 0 \ (n \neq n_k)$$

which is summable (L) but not convergent. Indeed, there is a series satisfying this gap condition which is summable (l) but not convergent; this is a stronger result in virtue of implication (1.3).

We have, in fact, the following result.

THEOREM A. *Let $\{n_k\}$ be an increasing sequence of non-negative integers. In order that the summability (l) of $\sum_{n=0}^{\infty} a_n$ together with the gap condition* (1.6) *should imply the convergence of $\sum_{n=0}^{\infty} a_n$ it is necessary and sufficient that*

(1.7) $$\liminf_{k\to\infty} [\log\log n_{k+1} - \log\log n_k] > 0.$$

Proof of Theorem A. We write (1.2) in the form

$$t_n = \frac{1}{P(n)} \sum_{\nu=0}^{n} a_\nu [P(n) - P(\nu-1],$$

where we take $P(-1) = 0$. Now

$$\sum_{P(\nu-1)<w} a_\nu [w - P(\nu-1)]$$

is linear in each of the intervals $[P(\nu-1), P(\nu)]$; hence if

(1.8) $$\frac{1}{w} \sum_{P(\nu-1)<w} a_\nu [w - P(\nu-1)]$$

tends to a limit as $w \to \infty$ through the particular sequence $\{P_n\}$, it tends to a limit as $w \to \infty$ through all values. Thus, as is familiar, summability (l) is equivalent to summability $(R, P(\nu-1), 1)$ and so to $(R, \log \overline{n+1}, 1)$ (see also [2], p. 87, Theorem 37). Now consider the special case in which the gap condition (1.6) is satisfied. Write

$$b_k = a_{n_k}, \quad \lambda_k = P(n_k - 1).$$

Then (1.8) becomes

$$\frac{1}{w} \sum_{\lambda_k < w} b_k (w - \lambda_k).$$

Thus summability $(R, P(\nu-1), 1)$ (and hence also summability (l)) of $\sum_{n=0}^{\infty} a_n$ is the same as summability $(R, \lambda_k, 1)$ of $\sum_{k=0}^{\infty} b_k$. Similarly with summability replaced by boundedness.

Now it is known that, in order that $(R, \lambda_k, 1)$ should be equivalent to convergence, it is necessary and sufficient that

$$\frac{\lambda_{k+1}}{\lambda_k} \geqslant c > 1,$$

or, what is equivalent, that

$$\liminf_{k\to\infty} (\log \lambda_{k+1} - \log \lambda_k) > 0.$$

For sufficiency we refer to ([**1**], page 13) and for necessity ([**6**], Corollary II). Theorem A is now evident.

Theorem 1 follows at once from the sufficiency part of Theorem A once the following proposition is proved.

PROPOSITION. *Under the gap condition* (1.4),

$$\sum_{n=0}^{\infty} a_n = s(L) \Rightarrow \sum_{n=0}^{\infty} a_n = s(l).$$

2. The proof of the proposition depends on the lemmas in this section. Of these, Lemmas 3,4 form the core of the proof, and so only these are demonstrated in detail. In what follows H with or without a suffix denotes a (strictly) positive constant possibly different at each occurrence.

LEMMA 1. ([**2**], p. 174, Theorem 116). *If* $0 \leqslant \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty$ *and* $\lambda_{n+1} \geqslant c\lambda_n$ *for some fixed* $c > 1$, $|\sum_{n=0}^{\infty} b_n e^{-\lambda_n y}| \leqslant H$ *(fixed) for* $y > 0$ *implies* $b_n = O(1)$, $n \to \infty$.

LEMMA 2. ([**2**], p. 124, Theorem 67). *If* $0 \leqslant \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty$,

$$b_n = O\left(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n}\right),\ n \to \infty,$$

and

$$\frac{1}{x}\int_0^x \Big(\sum_{\lambda_n \leqslant u} b_n\Big)\, du \to s \quad \text{as} \quad x \to \infty,$$

then $\sum_{n=0}^{\infty} b_n$ *converges to* s.

LEMMA 3. *If* $L(x)$ *defined by* (1.1) *is bounded for* $0<x<1$ *and the gap condition*

(2.1) $a_n=0,\ n\neq n_k,\ a_n = k=1,\ 2, \dots,\ n_k$ *positive integers with*

$$n_{k+1}\geqslant cn_k,\ c>1 \text{ (fixed)}$$

is satisfied, then $A_n=O(n),\ n\to\infty$.

Remark. (2.1) is the Hadamard gap condition and it implies (1.4).

Proof of Lemma 3. We first observe that

$$\sum_{n=0}^{\infty}\frac{A_n}{n+1}x^{n+1}$$

converges for $0<x<1$, if and only if

$$\sum_{n=1}^{\infty}A_{n-1}\frac{x^{n+1}}{n+1}=\sum_{n=0}^{\infty}A_n\frac{x^{n+2}}{n+2}$$

converges for $0<x<1$. Writing

$$\phi(x)=\sum_{n=0}^{\infty}\frac{A_n}{n+1}x^{n+1},$$

we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}&=\sum_{n=0}^{\infty}\frac{A_n-A_{n-1}}{n+1}x^{n+1}(A_{-1}=0)\\&=\phi(x)-\sum_{n=0}^{\infty}\frac{A_n}{n+2}x^{n+2}\\&=\phi(x)-\sum_{n=0}^{\infty}\frac{A_n}{n+1}x^{n+2}+\sum_{n=0}^{\infty}\frac{A_n}{(n+1)(n+2)}x^{n+2}\\&=\phi(x)-x\phi(x)+\int_0^x\sum_{n=0}^{\infty}\frac{A_n}{n+1}u^{n+1}\,du\\&=(1-x)\phi(x)+\int_0^x\phi(u)\,du.\end{aligned}$$

By hypothesis, there is a constant H such that

$$|\phi(x)|<H\log\left(\frac{1}{1-x}\right),\qquad 0<x<1.$$

Hence

$$\left|\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}\right| < H(1-x)\log\left(\frac{1}{1-x}\right) + H\int_0^x \log\left(\frac{1}{1-u}\right) du = Hx,$$

as can be verified easily. An application of Lemma 1 with

$$b_k = \frac{a_{n_k}}{n_k+1},\ \lambda_k = n_k,\ e^{-y} = x$$

gives $a_{n_k} = O(n_k)$, $k \to \infty$. Hence, if

$$n_k \leq n < n_{k+1},$$

$$|A_n| \leq \sum_{r=1}^{k} |a_{n_r}| \leq H\sum_{r=1}^{k} n_r = Hn_k \sum_{r=1}^{k}\left(\frac{n_r}{n_k}\right) < Hn_k \sum_{r=1}^{k}\left(\frac{1}{c}\right)^{k-r} < \frac{Hn_k}{1-\frac{1}{c}} \leq \frac{c}{c-1} Hn,$$

and this proves the lemma.

LEMMA 4. *If* $L(x)$ *defined by* (1.1) *is bounded for* $0<x<1$ *and the gap condition* (1.4) *is satisfied, then* $A_n = O(n)$, $n \to \infty$, *implies* $A_n = O(\log n)$, $n \to \infty$.

Proof. Let N be a fixed integer such that $(N-1)\,\delta \geq 2$, and k be any integer such that $n_k^{\delta/2} \geq 2$. Let $t = t_k = n_k^{-1-\delta/2}$ and

$$S \equiv S_k = \sum_{n=0}^{\infty} \frac{A_n}{n+1}[e^{-nt}(1-e^{-nt})]^N$$

$$= \sum_{n=0}^{\infty} \frac{A_n}{n+1} e^{-ntN} \sum_{r=0}^{N} \binom{N}{r}(-1)^r e^{-ntr}$$

$$= \sum_{r=0}^{N} \binom{N}{r}(-1)^r \sum_{n=0}^{\infty} \frac{A_n}{n+1}[e^{-(N+r)t}]^n$$

$$= \sum_{r=0}^{N} \binom{N}{r}(-1)^r e^{(N+r)t}\phi(e^{-(N+r)t}).$$

By hypothesis,

$$|\phi(x)|<H\log\left(\frac{1}{1-x}\right),\ 0<x<1.$$

Hence

$$|S|\leq\sum_{r=0}^{N}\binom{N}{r}e^{(N+r)t}H\log\left[\frac{1}{1-e^{-(N+r)t}}\right]. \tag{2.2}$$

For relevant values of the parameters, $(N+r)t\leq H$ and hence

$$1-e^{-(N+r)t}\geq H(N+r)t\geq Ht=Hn_k^{-1-\delta/2}.$$

Thus, for sufficiently large k, we have from (2.2)

$$\begin{aligned}|S|&\leq\sum_{r=0}^{N}\binom{N}{r}e^{(N+r)t}H\log(n_k^{1+\delta}) \\ &<H\log(n_k^{1+\delta})\cdot e^N\sum_{r=0}^{N}\binom{N}{r} \\ &=H\log n_k.\end{aligned} \tag{2.3}$$

On the other hand, since $A_n=A_{n_r}$ for $n_r\leq n<n_{r+1}$ and $A_n=0$ for $n<n_1$,

$$S=\sum_{r=1}^{\infty}T_r,$$

where

$$T_r=A_{n_r}\sum_{n=n_r}^{n_{r+1}-1}\frac{1}{n+1}[e^{-nt}(1-e^{-nt})]^N.$$

$A_n=O(n)$, $n\to\infty$, implies $|A_{n_r}|\leq Hn_r$, $r=1,2,\ldots$. For $r\leq k-1$, we get, since $e^{-nt}<1$, $1-e^{-nt}<nt$,

$$\begin{aligned}|T_r|&\leq|A_{n_r}|\sum_{n=n_r}^{n_{r+1}-1}\frac{(nt)^N}{n+1}<Hn_r\sum_{n=n_r}^{n_{r+1}-1}t^Nn^{N-1} \\ &\leq Hn_rt^N(n_{r+1}^N-n_r^N).\end{aligned}$$

For, if $\sigma\geq0$,

$$\begin{aligned}\sum_{n=m}^{M-1}n^\sigma&\leq\sum_{n=m}^{M-1}\int_n^{n+1}x^\sigma\,dx=\int_m^M x^\sigma\,dx \\ &=\frac{M^{\sigma+1}-m^{\sigma+1}}{\sigma+1}.\end{aligned}$$

Hence

$$\begin{aligned}|T_r|&\leq Ht^N[n_r\cdot n_{r+1}^N-n_r^{N+1}] \\ &<Ht^N[n_{r+1}^{N+1}-n_r^{N+1}],\qquad r\leq k-1.\end{aligned} \tag{2.4}$$

For $r \geq k+1$, we have, since $1 - e^{-nt} < 1$,

(2.5)
$$|T_r| \leq |A_{n_r}| \sum_{n=n_r}^{n_{r+1}-1} \frac{1}{n+1} (e^{-nt})^N$$
$$\leq Hn_r \sum_{n=n_r}^{n_{r+1}-1} \frac{1}{n+1} (e^{-Nt})^n$$
$$\leq H \sum_{n=n_r}^{n_{r+1}-1} (e^{-Nt})^n.$$

Combining (2.4) and (2.5) we get

(2.6)
$$\sum_{r \neq k} |T_r| \leq Ht^N \sum_{r=1}^{k-1} (n_{r+1}^{N+1} - n_r^{N+1})$$
$$+ H \sum_{r=k+1}^{\infty} \sum_{n=n_r}^{n_{r+1}-1} (e^{-Nt})^n$$
$$= Ht^N[n_k^{N+1} - n_1^{N+1}] + H \sum_{n=n_{k+1}}^{\infty} (e^{-Nt})^n$$
$$< Ht^N n_k^{N+1} + H \frac{(e^{-Nt})^{n_{k+1}}}{1 - e^{-Nt}}$$
$$< Hn_k (tn_k)^N + \frac{2H}{t} e^{-Ntn_{k+1}},$$

since,
$$1 - e^{-Nt} > \frac{Nt}{2N} = \frac{t}{2},$$

t being at the most 1. Now $tn_k = n_k^{-\delta/2}$. By the gap condition (1.4),
$$\log \frac{\log n_{k+1}}{\log n_k} \geq \delta, \frac{\log n_{k+1}}{\log n_k} \geq e^\delta > 1 + \delta,$$
$$\log n_{k+1} > (1+\delta) \log n_k,$$
$$n_{k+1} > n_k^{1+\delta}.$$

Hence
$$e^{tn_{k+1}} > tn_{k+1} > tn_k^{1+\delta} = n_k^{\delta/2},$$
$$(e^{-tn_{k+1}})^N < (n_k^{-\delta/2})^N.$$

Therefore, from (2.6),

(2.7)
$$\sum_{r \neq k} |T_r| < Hn_k (n_k^{-\delta/2})^N + \frac{2H}{t} (n_k^{-\delta/2})^N$$
$$= Hn_k^{1-N\delta/2} + 2Hn_k^{1+\delta/2-N\delta/2}$$
$$\leq H$$

since $(N-1)\delta \geq 2$, by our choice of N. It is clear that $n_k t \to 0$ and $n_{k+1}t \to \infty$ as $k \to \infty$. Thus for sufficiently large k,

$$n_k t < \tfrac{1}{2}, \qquad n_{k+1} t > 2.$$

Hence

$$|T_k| \geq |A_{n_k}| \sum_{\frac{1}{2}<nt<2} \frac{1}{n+1} [e^{-nt}(1-e^{-nt})]^N$$

$$\geq H |A_{n_k}| \sum_{\frac{1}{2}<nt<2} \frac{1}{n+1} \geq H |A_{n_k}|.$$

Combining this with (2.3) and (2.7) we find that

$$H |A_{n_k}| \leq |T_k| = \left| S - \sum_{r \neq k} T_r \right|$$

$$\leq H \log n_k + H,$$

and hence

$$|A_{n_k}| < H \log n_k,$$

where H may depend on δ. This inequality holds whenever $n_k^{\delta/2} \geq 2$. Since $A_n = A_{n_k}$ for $n_k \leq n < n_{k+1}$, the proof of the lemma is complete.

The final lemma is an O-version of an easily established result ([**1**], p. 13, Corollary 1.62).

LEMMA 5. *If* $0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty$ *and* $\lambda_{n+1} \geq c\lambda_n$ *for some* $c > 1$, *and if*

$$B_\lambda(u) = b_1 + \cdots + b_n \quad \text{for} \quad \lambda_n \leq u < \lambda_{n+1} (n \geq 1),$$

$$B_\lambda(u) = 0 \quad \text{for} \quad u < \lambda_1,$$

then

$$R(\omega) \equiv \frac{1}{\omega} \int_0^\infty B_\lambda(u)\, du$$

$$= O(1)(\omega \to \infty) \Rightarrow B_\lambda(u) = O(1)(u \to \infty).$$

3. *Proof of Proposition.* Suppose first that $L(x) = O(1)$, $x \to 1\text{-}0$. We may choose $x = 1-(1/m)$, $m = 1, 2, \ldots,$ and this we do. Since $\log(1/1-x) > x$ and $\sum_{n=0}^{\infty} (A_n/n+1)x^{n+1} = O(x)$ as $x \to 0$, it follows that $L(x)$ is bounded in $(0, 1)$. The gap condition (1.4) implies the condition (2.1) of Lemma 3 and so, by that lemma $A_n = O(n)$, $n \to \infty$. Lemma 4 then yields $A_n = O(\log n)$, $n \to \infty$.

Now, $1-x^{n+1}=(1-x)(1+x+\cdots+x^n)<(n+1)(1-x)$.

$$\text{(3.1)}\qquad \left|\sum_{n=0}^{m}\frac{A_n}{n+1}-\sum_{n=0}^{m}\frac{A_n}{n+1}x^{n+1}\right|=\left|\sum_{n=0}^{m}\frac{A_n}{n+1}(1-x^{n+1})\right|$$
$$\leqslant(1-x)\sum_{n=0}^{m}|A_n|$$
$$\leqslant\frac{H}{m}\sum_{n=1}^{m}\log n\,(A_0=A_1=0)$$
$$\leqslant H\log(m+1).$$

Since $\log n/n+1$ decreases for $n\geqslant m$, when m is sufficiently large,

$$\text{(3.2)}\qquad \left|\sum_{n=m+1}^{\infty}\frac{A_n}{n+1}x^{n+1}\right|\leqslant H\sum_{n=m+1}^{\infty}\log n\cdot\frac{x^{n+1}}{n+1}$$
$$\leqslant\frac{\log m}{m+1}\sum_{n=m+1}^{\infty}x^{n+1}\leqslant\frac{\log m}{m+1}\cdot\frac{x^{m+1}}{1-x}$$
$$\leqslant\frac{\log m}{(m+1)(1-x)}\leqslant H\log(m+1).$$

From (3.1), (3.2), we obtain

$$\left|\frac{1}{\log(m+1)}\sum_{n=0}^{m}\frac{A_n}{n+1}-\frac{1}{\log(m+1)}\sum_{n=0}^{\infty}\frac{A_n}{n+1}x^{n+1}\right|$$
$$\leqslant\frac{1}{\log(m+1)}\left\{\left|\sum_{n=0}^{m}\frac{A_n}{n+1}-\sum_{n=0}^{m}\frac{A_n}{n+1}x^{n+1}\right|+\left|\sum_{n=m+1}^{\infty}\frac{A_n}{n+1}x^{n+1}\right|\right\}$$
$$<H,$$

for sufficiently large m. The boundedness of $L(x)$ as $x\to1-0$ therefore implies the boundedness of $(1/\log(m+1))\sum_{n=0}^{m}(A_n/(n+1))$ as $m\to\infty$ or what is the same, the boundedness of $R(\omega)$ of Lemma 5 as $\omega\to\infty$ (see the observation made in the proof of Theorem A) where, as earlier, $b_k=a_{n_k}$ and $\lambda_k=\log n_k$. It follows from Lemma 5 that $A_n=O(1)$, $n\to\infty$, because

$$B_\lambda(u)=A_{n_k},\qquad \lambda_k\leqslant u<\lambda_{k+1},$$
$$A_n=A_{n_k},\qquad n_k\leqslant n<n_{k+1}.$$

If $|A_n|\leqslant H$ (to streamline a proof due originally to K. Ishiguro),

$$\left|\sum_{n=0}^{m}\frac{A_n}{n+1}-\sum_{n=0}^{m}\frac{A_n}{n+1}x^{n+1}\right|\leqslant H(1-x)m\leqslant H$$

and

$$\left|\sum_{n=m+1}^{\infty}\frac{A_n}{n+1}x^{n+1}\right|\leq H\sum_{n=m+1}^{\infty}\frac{x^{n+1}}{n+1}$$

$$\leq\frac{H}{m(1-x)}=H.$$

Thus

$$\left|\frac{1}{\log(m+1)}\sum_{n=0}^{m}\frac{A_n}{n+1}-\frac{1}{\log(m+1)}\sum_{n=0}^{\infty}\frac{A_n}{n+1}x^{n+1}\right|$$

$$\leq\left|\frac{1}{\log(m+1)}\sum_{n=0}^{m}\frac{A_n}{n+1}-\frac{1}{\log(m+1)}\sum_{n=0}^{m}\frac{A_n}{n+1}x^{n+1}\right|$$

$$+\frac{1}{\log(m+1)}\left|\sum_{n=m+1}^{\infty}\frac{A_n}{n+1}x^{n+1}\right|$$

$$\leq\frac{H}{\log(m+1)}.$$

Hence for such A_n, $L(x)\to s$ as $x\to 1-0$ implies $\sum_{n=0}^{\infty} a_n = s(l)$. The proof of the proposition is complete.

4. The method (L) of summability is a totally regular method defined by the transform (after a change of the variable x to $e^{-1/y}$ in (1.1))

$$\tau(y)=\sum c_n(y)A_n,\ y>0,$$

where

$$c_n(y)=-\frac{1}{\log(1-e^{-1/y})}\cdot\frac{e^{-(n+1)/y}}{n+1}.$$

With the function $\Phi(y)=\log\log y$ it fulfils all the requirements of Theorem I′ of Rangachari's [7]. The details of checking the fulfilment are, in fact, available in § 1-3 above. Thus the condition (T_δ) for every $\delta>0$ of [7] is a Tauberian condition for the method (L) of summability. In particular, we have

THEOREM 2. *If* $\sum_{n=0}^{\infty} a_n = s(L)$, $\{n_k\}$ *is a sequence of integers satisfying the condition*

$$\log\log n_{k+1}-\log\log n_k\geq c>0\quad (c\textit{ fixed})$$

and $A_n = a_0+\cdots+a_n$, *satisfies the Tauberian condition*

$$\limsup_{k\to\infty}\ \max_{\substack{n_k\leq n<n'<n_{k+1}\\ \log\log n'-\log\log n\leq\epsilon}}|A_{n'}-A_n|\leq\omega(\epsilon)\to 0\quad\text{as}\quad\epsilon\to 0,$$

then

$$\sum_{n=0}^{\infty} a_{n} = s.$$

We note that Theorem 2 also includes the Tauberian theorem for summability (*L*) with a slow oscillation type of Tauberian condition (cf. [5], Theorem A for a condition of slow decrease type). It may be finally remarked that Theorem 1 can also be proved using an unpublished result of C. T. Rajagopal and M. S. Rangachari or a result of Kohanovskii ([4], Theorem 1).

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