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On the relation of generalized Valiron summability to Cesàro summability

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Abstract. A family (V_a^k) of summability methods, called generalized Valiron summability, is defined. The well-known summability methods $(B_{a,\gamma})$, (E_p), (T_a), (S_β) and (V_a) are members of this family. In §3 some properties of the $(B_{a,\gamma})$ and (V_a^k) transforms are established. Following Satz II of Faulhaber (1956) it is proved that the members of the (V_a^k) family are all equivalent for sequences of finite order. This paper is a good illustration of the use of generalized Boral summability. The following theorem is established:

Theorem. *If s_n $(n \geqslant 0)$ is a real sequence satisfying*

$$\lim_{\epsilon\to 0+} \liminf_{m\to\infty} \min_{m\leqslant n\leqslant m+\epsilon\sqrt{m}} \left(\frac{s_n - s_m}{m^\rho}\right) \geqslant 0 \ (\rho \geqslant 0),$$

and if $s_n \to s(V_a^k)$ then $s_n \to s(C, 2\rho)$.

Keywords. Generalized Valiron summability; Boral summability; Rajagopal's theorem.

1. Introduction

Rajagopal ([4], Theorem 2) proved the following theorem connecting Borel and Cesàro summabilities; and, after him, Sitaraman ([5], Theorem II) proved the theorem with Borel summability replaced by summability (S_β) defined as usual in § 5 :

Theorem A. *If s_n $(n \geqslant 0)$ is a real sequence satisfying*

$$\lim_{\epsilon\to 0+} \liminf_{m\to\infty} \min_{m\leqslant n\leqslant m+\epsilon\sqrt{m}} \left(\frac{s_n - s_m}{m^\rho}\right) \geqslant 0 \ (\rho \geqslant 0), \tag{1}$$

and if $s_n \to s(B)$, then $s_n \to s\ (C, 2\rho)$.

In this paper we prove (Theorem 4) that Theorem A is extensible to a family (V_a^k) of summability methods which include as special cases generalized Borel summability $(B_{a,\gamma})$ defined in § 2 and the well-known summabilities (E_p), (T_α), (S_β) defined in the usual notation in § 5. Of course Theorem A itself obviously

In homage to the late Professor C T Rajagopal.

includes the similar theorem for summability (E_p) instead of summability (B), since $(E_p) \subset (B)$. Valiron summability (V_a) is also a special case of summability (V_a^k), as pointed out in § 5, and the latter is the generalized Valiron summability of the title.

The Tauberian condition (1) reduces to a classic special case when $\rho = 0$. A different special case of (1) and its further specialization are respectively

$$s_n - s_{n-1} = O_L(n^{\rho-1/2}), \tag{1 a}$$

$$s_n - s_{n-1} = o(n^{\rho-1/2}). \tag{1 b}$$

Hardy and Littlewood originally proved the special case of Theorem A with (1 *b*) instead of (1), as stated by Hardy ([3], note on §§9·6–7). Their result was extended by Borwein [1] to generalized Borel summability, and an idea of his (Lemma 7) is used in the sequel.

2. Definitions

The V_a^k transform of a (generally complex) sequence s_n $(n \geqslant 0)$ is the function defined by

$$V_a^k(x) = \sum_{n=0}^{\infty} c_n(x)\, s_n, \quad x > 0,$$

where $c_n(x)$ satisfies the following three conditions :

(i) $c_n(x) \geqslant 0$ *for* $n = 0, 1, 2, \cdots, x > 0$;

(ii) *there exist* $a > 0$ and δ *with* $\frac{1}{2} < \delta < \frac{2}{3}$ *such that, for every positive integer* k, $c_n(x)$ *can be expressed as*

$$c_n(x) = \left(\frac{a}{\pi x}\right)^{1/2} \exp\{-ax^{-1}(n-x)^2 + g_k + R_k\}$$

whenever x *is sufficiently large and* $|n - x| \leqslant x^{\delta}$, *and where*

$$g_k = \sum_{i=0}^{2k-1} \sum_{j=0}^{i+1} l_{ij} \frac{(n-x)^j}{x^i}, \quad l_{12} = 0,$$

l_{ij} *being independent of* n *and bounded as* $x \to \infty$,

$$R_k = O\left(\frac{|n-x|^{2k+1} + 1}{x^{2k}}\right) \text{ as } x \to \infty$$

uniformly in n *for* $|n - x| \leqslant x^{\delta}$;

(iii) *for every* $\sigma \geqslant 0$

$$\sum_{|n-x| > x^{\delta}} (n+1)^{\sigma} c_n(x) = o(1) \text{ as } x \to \infty.$$

We say that s_n is summable (V_a^k) to s (finite), and write $s_n \to s\,(V_a^k)$ if $V_a^k(x) \to s$ as $x \to \infty$.

The $(B_{\alpha,\gamma})$ transform ($\alpha > 0$, γ real) of s_n is the function defined by

$$B(x) = \alpha \exp(-\alpha x) \sum_{n=N}^{\infty} \frac{(\alpha x)^{n\alpha+\gamma-1}}{\Gamma(n\alpha + \gamma)} s_n, \quad x > 0,$$

N being the least positive integer such that $N\alpha + \gamma \geqslant 1$. We say that s_n is summable $(B_{\alpha,\gamma})$ to s (finite), and write $s_n \to s\,(B_{\alpha,\gamma})$ if $B(x) \to s$ as $x \to \infty$.

The nth Cesàro sum and the nth Cesàro mean of s_n, each of order $r > -1$, are denoted by S_n^r and s_n^r respectively. Thus

$$S_n^0 = s_n^0 = s_n; \quad S_n^r = \sum_{\nu=0}^{n} \binom{n-\nu+r-1}{n-\nu} s_\nu = s_n^r \binom{n+r}{n}.$$

We say that s_n is summable (C, r) to s (finite), and write $s_n \to s\,(C, r)$ if $s_n^r \to s$ as $n \to \infty$.

3. Preliminary results

In this section we study some properties of the $(B_{\alpha,\gamma})$ and V_a^k transforms.

Lemma 1. *The $(B_{\alpha,\gamma})$ transform is a V_a^k transform with $a = \alpha/2$.*

Proof. Borwein ([1], Lemma 2 (d)) has proved that $c_n(x)$ defined as below satisfies condition (iii) :

$$c_n(x) = \alpha \exp(-\alpha x) \frac{(\alpha x)^{n\alpha+\gamma-1}}{\Gamma(n\alpha + \gamma)} \text{ for } n \geqslant N \text{ and } c_n(x) = 0 \text{ for } n < N.$$

To verify condition (ii), let $\frac{1}{2} < \delta < \frac{2}{3}$, x be large, $|n - x| \leqslant x^\delta$, and k be any positive integer. Writing $h = n - x + (\gamma - 1)/\alpha$ and using the formula

$$\log \Gamma(y+1) = \frac{1}{2}\log(2\pi) + \left(y + \frac{1}{2}\right)\log y - y + \sum_{r=1}^{k} \frac{(-1)^{r-1} B_r}{(2r-1)\, 2r} y^{-2r+1}$$
$$+ O(y^{-2k-1}) \text{ as } y \to \infty,$$

with $y = \alpha x + \alpha h$ we see that

$$\log\left\{c\ (x)\left(\frac{2\pi x}{\alpha}\right)^{1/2}\right\} = \frac{1}{2}\log(2\pi) - \alpha x + \left(\alpha x + \alpha h + \frac{1}{2}\right)\log(\alpha x)$$
$$- \log \Gamma(\alpha x + \alpha h + 1)$$
$$= A_1 + A_2 + A_3$$

where

$$A_1 = \alpha h + (\alpha x + \alpha h + \tfrac{1}{2})\log(\alpha x) - (\alpha x + \alpha h + \tfrac{1}{2})\log(\alpha x + \alpha h), \tag{2}$$

$$A_2 = -\sum_{r=1}^{k} \frac{(-1)^{r-1} B_r}{(2r-1)\, 2r} (\alpha x + \alpha h)^{-2r+1}, \tag{3}$$

$$|A_3| \leqslant M(\alpha x + \alpha h)^{-2k-1} \tag{4}$$

for some constant M. By Taylor's theorem,

$$\log(1+y) = \sum_{r=1}^{2k} \frac{(-1)^{r-1}}{r} y^r + \frac{y^{2k+1}}{2k+1}(1+\theta y)^{-2k-1},$$

where $0 \leqslant \theta = \theta(k, y) \leqslant 1$. Therefore, from (2),

$$A_1 = \alpha h - \left(\alpha x + \alpha h + \frac{1}{2}\right)\log\left(1 + \frac{h}{x}\right)$$

$$= \alpha h - \alpha x\left\{\frac{h}{x} - \frac{1}{2}\left(\frac{h}{x}\right)^2 + \sum_{r=3}^{2k} \frac{(-1)^{r-1}}{r}\left(\frac{h}{x}\right)^r\right\}$$

$$- \alpha h\left\{\frac{h}{x} + \sum_{r=2}^{2k-1} \frac{(-1)^{r-1}}{r}\left(\frac{h}{x}\right)^r\right\} - \frac{1}{2}\sum_{r=1}^{2k-1} \frac{(-1)^{r-1}}{r}\left(\frac{h}{x}\right)^r + A_4$$

$$= -\frac{\alpha}{2}\frac{h^2}{x} + \sum_{r=2}^{2k-1} u_r \frac{h^{r+1}}{x^r} + \sum_{r=1}^{2k-1} v_r \frac{h^r}{x^r} + A_4$$

where u_r, v_r are independent of h, x and

$$A_4 = \frac{\alpha}{2k}\frac{h^{2k+1}}{x^{2k}} + \frac{1}{2}\frac{1}{2k}\frac{h^{2k}}{x^{2k}} - \frac{(\alpha x + \alpha h + \frac{1}{2})}{2k+1}\left(\frac{h}{x}\right)^{2k+1}\left(1 + \theta\frac{h}{x}\right)^{-2k-1}. \qquad (5)$$

Again, Taylor's theorem gives

$$(1+y)^{-\mu} = \sum_{\nu=0}^{m} (-1)^\nu \binom{\mu+\nu-1}{\nu} y^\nu + (-1)^{m+1} y^{m+1}\binom{\mu+m}{m+1}$$

$$\times (1+\theta y)^{-\mu-m-1},$$

where $0 \leqslant \theta = \theta(\mu, m, y) \leqslant 1$. Using this with $y = h/x$, $\mu = 2r-1$, $m = 2k-2r$, $r = 1, \cdots, k$, we get from (3),

$$A_2 = \sum_{r=1}^{k} \frac{(-1)^r B_r}{(2r-1)2r}\alpha^{-2r+1} x^{-2r+1}\left\{\sum_{\nu=0}^{2k-2r} (-1)^\nu \binom{2r-1+\nu-1}{\nu}\left(\frac{h}{x}\right)^\nu\right.$$

$$\left. + (-1)^{2k-2r+1}\left(\frac{h}{x}\right)^{2k-2r+1}\binom{2k-1}{2k-2r+1}\left(1+\theta_r\frac{h}{x}\right)^{-2k}\right\}$$

$$= \sum_{r=1}^{k}\sum_{\nu=0}^{2k-2r} w_{r,\nu}\frac{h^\nu}{x^{2r+\nu-1}} + A_5,$$

where $w_{r,\nu}$ are independent of h, x and

$$A_5 = \sum_{r=1}^{k} \frac{(-1)^{r+1} B_r}{(2r-1)2r}\alpha^{-2r+1}\frac{h^{2k-2r+1}}{x^{2k}}\binom{2k-1}{2k-2r+1}\left(1+\theta_r\frac{h}{x}\right)^{-2k}. \qquad (6)$$

Thus we have proved that

$$\log\left\{c_n(x)\left(\frac{2\pi x}{\alpha}\right)^{1/2}\right\} = -\frac{\alpha}{2}\frac{h^2}{x} + \sum_{i=1}^{2k-1}\sum_{j=0}^{i+1} \bar{l}_{ij}\frac{h^j}{x^i} + A_3 + A_4 + A_5,$$

where $\bar{l}_{ij}$ are independent of h, x and $\bar{l}_{12} = 0$. Noting that $h = n - x + (\gamma - 1)/\alpha$ and writing $\alpha = 2a$ we get

$$\log c_n(x) = \log\left(\frac{a}{\pi x}\right)^{1/2} - \frac{a}{x}\left\{(n-x)^2 + 2(n-x)\left(\frac{\gamma-1}{\alpha}\right) + \left(\frac{\gamma-1}{\alpha}\right)^2\right\}$$

$$+ \sum_{i=1}^{2k-1}\sum_{j=0}^{i+1} \bar{l}_{ij}\frac{1}{x^i}\sum_{\nu=0}^{j}\binom{j}{\nu}(n-x)^\nu\left(\frac{\gamma-1}{\alpha}\right)^{j-\nu} + A_3 + A_4 + A_5$$

$$= \log\left(\frac{a}{\pi x}\right)^{1/2} - ax^{-1}(n-x)^2 + \sum_{i=1}^{2k-1}\sum_{j=0}^{i+1} l_{ij}\frac{(n-x)^j}{x^i} + A_3 + A_4 + A_5, \tag{7}$$

where l_{ij} are independent of n, x and $l_{12} = \bar{l}_{12} = 0$.

Since $h = n - x + (\gamma - 1)/\alpha$, we have $|h|/x < \frac{1}{2}$ and $1 + \theta h/x > \frac{1}{2}$ whenever $0 \leqslant \theta \leqslant 1$, x is large and $|n - x| \leqslant x^\delta$. Moreover,

$$|h|^\nu \leqslant \{|n-x|^{2k+1} + 1\}\left\{1 + \frac{|\gamma - 1|}{\alpha}\right\}^{2k+1}, \quad \nu = 0, 1, \cdots, 2k+1.$$

Supplying these estimates in (4), (5) and (6), we find that

$$x^{2k}[|A_3| + |A_4| + |A_5|] \leqslant M'[|n-x|^{2k+1} + 1]$$

if $|n - x| \leqslant x^\delta$ and x is large, M' being a constant. This, in view of (7), completes the proof of the lemma.

Lemma 2. *If $c_n(x)$ satisfies the conditions of a V_a^k transform, then for $\sigma \geqslant 0$,*

$$(a)\ c_n(x) = \{1 + \epsilon_1(n, x)\}\left(\frac{a}{\pi x}\right)^{1/2}\exp[-ax^{-1}(n-x)^2],$$

$$(b)\ \left(\frac{n}{x}\right)^\sigma c_n(x) = \{1 + \epsilon_2(n, x)\}\int_n^{n+1}\left(\frac{a}{\pi x}\right)^{1/2}\exp[-ax^{-1}(t-x)^2]\,dt,$$

where $\epsilon_1(n, x)$, $\epsilon_2(n, x) \to 0$ as $x \to \infty$ uniformly in n for $|n - x| \leqslant x^\delta$;

$$(c)\ \sum_{n=1}^{\infty}\left(\frac{n}{x}\right)^\sigma c_n(x) \to 1 \text{ as } x \to \infty;$$

$$(d)\ \theta_1(N, x) \equiv \sum_{n=N}^{\infty}\left(\frac{n}{x}\right)^\sigma c_n(x)(\sqrt{n} - \sqrt{N}) \to 0,$$

$$(e)\ \theta_2(N, x) \equiv \sum_{n=N}^{\infty}\left(\frac{n}{x}\right)^\sigma c_n(x) \to 0,$$

$$(f)\ \theta_3(M, x) \equiv \sum_{n=1}^{M} \left(\frac{n}{x}\right)^{\sigma} c_n(x) \to 0,$$

as x, N, $\sqrt{N} - \sqrt{x}$, M, $\sqrt{x} - \sqrt{M} \to \infty$. *(c) for* $\sigma = 0$ *shows that* V_a^k *is a positive regular transform of* s_n.

Proof. (a) Taking $k = 1$ in condition (ii) on $c_n(x)$, we see that

$$\exp(g_1 + R_1) = \exp\left\{\frac{l_{10}}{x} + l_{11}\frac{(n-x)}{x} + O\left(\frac{|n-x|^3 + 1}{x^2}\right)\right\}$$

$$= 1 + O\left(\frac{1}{x} + \frac{|n-x|}{x} + \frac{|n-x|^3}{x^2}\right)$$

which proves (a), since $\delta < 2/3$.

(b) Noting that $(n/x)^{\sigma} = 1 + \epsilon'(n, x)$ and that

$$\int_n^{n+1} \exp[-ax^{-1}(t-x)^2]\, dt = \exp[-ax^{-1}(n-x)^2]\{1 + \epsilon''(n, x)\}$$

we deduce (b) from (a).

(c) Write the sum in (c) as

$$\left(\sum_{|n-x|\leqslant x^{\delta}} + \sum_{|n-x|>x^{\delta}}\right)\left(\frac{n}{x}\right)^{\sigma} c_n(x) = S_1 + S_2.$$

From (b) it follows that

$$S_1 = \int_{x-x^{\delta}}^{x+x^{\delta}} \left(\frac{a}{\pi x}\right)^{1/2} \exp\{-ax^{-1}(t-x)^2\}\, dt + o(1)$$

$$= \int_{-x^{\delta-\frac{1}{2}}}^{x^{\delta-\frac{1}{2}}} \left(\frac{a}{\pi}\right)^{1/2} \exp(-au^2)\, du + o(1),$$

which tends to 1 as $x \to \infty$ since $\delta > \frac{1}{2}$. On the other hand, $S_2 \to 0$ as $x \to \infty$ by our condition (iii) on $c_n(x)$.

(d) Write $\sqrt{N} - \sqrt{x} = u$ so that $N - x = u(\sqrt{N} + \sqrt{x}) > u\sqrt{x}$.

In view of condition (iii) on $c_n(x)$ it suffices to prove that

$$S = \sum_{N\leqslant n\leqslant x+x^{\delta}} \left(\frac{n}{x}\right)^{\sigma} c_n(x)(\sqrt{n} - \sqrt{N}) \to 0,$$

as x, $u \to \infty$. Since $\sqrt{n} - \sqrt{N} < (n-x)/\sqrt{x}$ for $n \geqslant N > x$, it follows from (b) that, for all large x,

$$S \leqslant \sum_{N\leqslant n\leqslant x+x^{\delta}} \frac{n-x}{\sqrt{x}}\, 2\int_n^{n+1} \left(\frac{a}{\pi x}\right)^{1/2} \exp[-ax^{-1}(t-x)^2]\, dt$$

$$\leqslant 2\left(\frac{a}{\pi x}\right)^{1/2} \sum_{N \leqslant n \leqslant x+x^{\delta}} \int_{n}^{n+1} \frac{t-x}{\sqrt{x}} \exp\left[-ax^{-1}(t-x)^2\right] dt$$

$$\leqslant 2\left(\frac{a}{\pi x}\right)^{1/2} \int_{N}^{\infty} \frac{t-x}{\sqrt{x}} \exp\left[-ax^{-1}(t-x)^2\right] dt$$

$$= 2\left(\frac{a}{\pi}\right)^{1/2} \int_{\frac{N-x}{\sqrt{x}}}^{\infty} v \exp(-av^2)\, dv$$

$$\leqslant 2\left(\frac{a}{\pi}\right)^{1/2} \int_{u}^{\infty} v \exp(-av^2)\, dv$$

which tends to 0 as $u \to \infty$.

Proofs of (*e*) and (*f*) are similar to that of (*d*).

Lemma 3. *If* s_n *is a real sequence satisfying* (1) *then there exist positive constants* K_1, K_2 *such that, for* $n \geqslant m \geqslant 1$,

$$s_n - s_m > -K_1 n^{\rho} (\sqrt{n} - \sqrt{m}) - K_2 m^{\rho}.$$

This is proved exactly like Theorem 239 of Hardy [3] (see Rajagopal [4], Lemma 1).

Lemma 4. *If* s_n *is a real sequence satisfying* (1), *and if*

$$V_a^k(x) = O(x^{\rho})\,(x \to \infty), \tag{8}$$

then $s_n = O(n^{\rho})\,(n \to \infty)$.

Proof. It may be remarked that the proof is applicable to any positive regular transform, in place of $V_a^k(x)$, for which the $c_n(x)$ satisfy (c)–(f) of Lemma 2. We proceed on the lines of the proofs of Theorem 1 of Rajagopal [4] and Theorem 238 of Hardy [3]. Write, for $n \geqslant 1$

$$\sigma_n = \frac{s_n}{n^{\rho}}, \quad \sigma_1(n) = \max_{1 \leqslant r \leqslant n} \sigma_r \quad \text{and} \quad \sigma_2(n) = \max_{1 \leqslant r \leqslant n} (-\sigma_r),$$

and assume, for convenience, that $s_0 = 0$. Since

$$x^{-\rho} V_a^k(x) = \sum_{n=1}^{\infty} (n/x)^{\rho} c_n(x) \sigma_n,$$

it follows from (8) and Lemma 2 (c) that neither $\sigma_n \to \infty$ nor $\sigma_n \to -\infty$ is possible. The lemma is proved by showing that each of the following two cases contradicts (8) :

(I) $\sigma_1(n) \geqslant \sigma_2(n)$ *for infinitely many* n *and* $\sigma_1(n) \to \infty$;

(II) $\sigma_1(n) < \sigma_2(n)$ *for all but a finite number of values of* n *and* $\sigma_2(n) \to \infty$.

Case I. Corresponding to a large positive number H, choose the least $M = M(H)$ for which $\sigma_M = \sigma_1(M) > 2H$ and $\sigma_1(M) \geqslant \sigma_2(M)$, and then the least $N = N(H) > M$ for which $\sigma_N \leqslant \frac{1}{2}\sigma_M$. Define $x = x(H)$ by $2\sqrt{x} = \sqrt{M} + \sqrt{N}$ and write

$$x^{-\rho} V_a^k(x) = \Big(\sum_{n=1}^{M-1} + \sum_{n=M}^{N-1} + \sum_{n=N}^{\infty}\Big)(n/x)^\rho c_n(x)\sigma_n = S_1 + S_2 + S_3. \quad (9)$$

If $(M/N)^\rho \leqslant 2/3$ then

$$\sqrt{N} - \sqrt{M} \geqslant \{1 - (2/3)^{1/2\rho}\}\sqrt{N}. \quad (10)$$

If $(M/N)^\rho > 2/3$ it follows from Lemma 3 that

$$\sigma_N - \sigma_M (M/N)^\rho > -K_1(\sqrt{N} - \sqrt{M}) - K_2(M/N)^\rho,$$

$$K_1(\sqrt{N} - \sqrt{M}) > -\sigma_N + \sigma_M(M/N)^\rho - K_2$$

$$> -\tfrac{1}{2}\sigma_M + \tfrac{2}{3}\sigma_M - K_2$$

$$> \tfrac{1}{3}H - K_2,$$

by the choice of M, N. This, together with (10), shows that $\sqrt{N} - \sqrt{M} \to \infty$ as $H \to \infty$. Hence $\sqrt{N} - \sqrt{x}$, $\sqrt{x} - \sqrt{M} \to \infty$ as $H \to \infty$.

The estimates which follow are when $H \to \infty$ and so x, M, N, $\sqrt{x} - \sqrt{M}$, $\sqrt{N} - \sqrt{x} \to \infty$ as in Lemma 2. By the choice of N, M,

$$s_{N-1} = (N-1)^\rho \sigma_{N-1} > (N-1)^\rho \tfrac{1}{2}\sigma_M > (N-1)^\rho H.$$

Hence, by Lemma 3, we have for $n \geqslant N$,

$$s_n > -K_1 n^\rho \{\sqrt{n} - \sqrt{(N-1)}\} - K_2(N-1)^\rho + s_{N-1}$$

$$> -K_1 n^\rho \{\sqrt{n} - \sqrt{(N-1)}\},$$

and, therefore, by Lemma 2 (d), (e)

$$S_3 > -K_1 \sum_{n=N}^{\infty} (n/x)^\rho c_n(x)\{\sqrt{n} - \sqrt{(N-1)}\}$$

$$= -K_1\theta_1(N-1, x).$$

On the other hand, by Lemma 2 (c), (e), (f) and the choice of M, N,

$$S_2 \geqslant \tfrac{1}{2}\sigma_M \sum_{n=M}^{N-1} (n/x)^\rho c_n(x) = \tfrac{1}{2}\sigma_M\{1 + o(1) - \theta_2(N, x) - \theta_3(M-1, x)\},$$

$$S_1 \geqslant -\sigma_2(M) \sum_{n=1}^{M-1} (n/x)^\rho c_n(x) \geqslant -\sigma_1(M)\theta_3(M-1, x).$$

Combining these estimates for S_1, S_2, S_3 we find from (9) that $x^{-\rho} V_a^k(x) \to \infty$ as $H \to \infty$ contradicting (8).

Case II. Corresponding to a large positive number H choose the least $N = N(H)$ such that $\sigma_2(n) > \sigma_1(n)$ for $n \geqslant N$ and $\sigma_N = -\sigma_2(N) < -2H$; and then the last $M = M(H) < N$ for which $\sigma_M \geqslant \frac{1}{2}\sigma_N = -\frac{1}{2}\sigma_2(N)$. Define x as in case I, and write

$$x^{-\rho} V_a^k(x) = \Big(\sum_{n=1}^{M} + \sum_{n=M+1}^{N} + \sum_{n=N+1}^{\infty}\Big)(n/x)^\rho c_n(x)\sigma_n = S_1 + S_2 + S_3. \quad (11)$$

Using Lemma 3 we find, as in case I, that

$$\begin{aligned} K_1(\sqrt{N}-\sqrt{M}) &> -\sigma_N+\sigma_M(M/N)^\rho - K_2(M/N)^\rho \\ &\geqslant -\sigma_N\{1-\tfrac{1}{2}(M/N)^\rho\} - K_2 \\ &> H - K_2, \end{aligned}$$

and that $\sqrt{N}-\sqrt{x}$, $\sqrt{x}-\sqrt{M}\to\infty$ as $H\to\infty$.

From Lemma 3, it follows that, for $n\geqslant N$,

$$\begin{aligned} \sigma_n &> -\{-\sigma_N+K_2\}(N/n)^\rho - K_1(\sqrt{n}-\sqrt{N}) \\ &\geqslant -\{\sigma_2(N)+K_2\} - K_1(\sqrt{n}-\sqrt{N}) \\ &= -t_n, \end{aligned}$$

say. Thus $-\sigma_n < t_n$ for $n\geqslant N$; while $-\sigma_n\leqslant -\sigma_N < t_N$ for $n < N$ by our choice of N and definition of $\sigma_2(n)$. Since t_n is an increasing function of n, we thus have $-\sigma_m\leqslant t_m\leqslant t_n$ for $n\geqslant m\geqslant N$, and $-\sigma_m < t_N\leqslant t_n$ for $n\geqslant N > m$. This implies that $t_n\geqslant\sigma_2(n)$ for $n\geqslant N$ by the definition of $\sigma_2(n)$. Hence, by the choice of N and Lemma 2 (d), (e)

$$\sigma_n\leqslant\sigma_1(n) < \sigma_2(n)\leqslant t_n = \sigma_2(N)+K_2+K_1(\sqrt{n}-\sqrt{N}) \quad (n\geqslant N),$$

$$\begin{aligned} S_3 &\leqslant \sum_{n=N+1}^{\infty}(n/x)^\rho c_n(x)\{\sigma_2(N)+K_2+K_1(\sqrt{n}-\sqrt{N})\} \\ &\leqslant \{\sigma_2(N)+K_2\}\,\theta_2(N,x)+K_1\theta_1(N,x). \end{aligned}$$

On the other hand, by Lemma 2 (c), (e), (f) and by the choice of M, N,

$$\begin{aligned} S_2 &\leqslant -\tfrac{1}{2}\sigma_2(N)\sum_{n=M+1}^{N}(n/x)^\rho c_n(x) \\ &= -\tfrac{1}{2}\sigma_2(N)\{1+o(1)-\theta_2(N+1,x)-\theta_3(M,x)\}, \end{aligned}$$

$$S_1\leqslant\sigma_1(M)\sum_{n=1}^{M}(n/x)^\rho c_n(x)\leqslant\sigma_2(N)\,\theta_3(M,x),$$

since $\sigma_1(M)\leqslant\sigma_1(N) < \sigma_2(N)$. Combining these estimates for S_1, S_2, S_3 we find from (11) that $x^{-\rho}V_a^k(x)\to-\infty$ as $H\to\infty$ contradicting (8). This completes the proof.

Lemma 5. *If* $s(t)=s_n$ *for* $n\leqslant t<n+1$ *and* $s_n=O(1)$ $(n\to\infty)$, *then*

$$V_a^k(x) - \int_0^\infty (a/\pi x)^{1/2}\exp\{-ax^{-1}(t-x)^2\}\,s(t)\,dt\to 0 \ (x\to\infty).$$

Proof. It suffices to prove that

$$\sum_{n=0}^{\infty}\left|\, c_n(x) - \int_n^{n+1}(a/\pi x)^{1/2}\exp\{-ax^{-1}(t-x)^2\}\,dt\,\right|\to 0 \quad (x\to\infty). \tag{12}$$

Denoting the summand in (12) by $d_n(x)$, we find from the condition (iii) on $c_n(x)$ with $\sigma = 0$ that, as $x \to \infty$

$$\sum_{|n-x|>x^\delta} d_n(x) \leqslant o(1) + \Big(\int_0^{x-x^\delta} + \int_{x+x^\delta}^{\infty} \Big) (a/\pi x)^{1/2} \exp\{-ax^{-1}(t-x)^2\}\, dt$$

$$= o(1) + \Big(\int_{-x^{\frac12}}^{-x^{\delta-\frac12}} + \int_{x^{\delta-\frac12}}^{\infty} \Big) (a/\pi)^{1/2} \exp(-au^2)\, du = o(1)\,;$$

on the other hand, using Lemma 2 (b) with $\sigma = 0$,

$$\sum_{|n-x|\leqslant x^\delta} d_n(x) \leqslant \sum_{|n-x|\leqslant x^\delta} \epsilon \int_n^{n+1} (a/\pi x)^{1/2} \exp\{-ax^{-1}(t-x)^2\}\, dt < \epsilon,$$

for $x > x_0(\epsilon)$. This proves (12).

Lemma 6. *If s_n is a real sequence satisfying*

$$\lim_{\epsilon\to 0+} \liminf_{m\to\infty} \min_{m\leqslant n\leqslant m+\epsilon\sqrt{m}} (s_n - s_m) \geqslant 0, \tag{13}$$

$$(a/\pi x)^{1/2} \int_0^{\infty} \exp\{-ax^{-1}(t-x)^2\}\, s(t)\, dt \to s\,(x \to \infty),$$

where $s(t) = s_n$ for $n \leqslant t < n+1$ and if $s_n = O(1)\,(n\to\infty)$, then $s_n \to s\,(n \to \infty)$.

Taking $s = 0$ this lemma is proved exactly like its case $a = \frac{1}{2}$ ([3], pp. 313–314).

Lemma 7. *If $s_n \to s\,(B_{\alpha,\gamma})$ then, for $r \geqslant 0$,*

$$a^{r+1}\Gamma(r+1)\exp(-\alpha x) \sum_{n=N}^{\infty} \frac{(\alpha x)^{n\alpha+\gamma-1}}{\Gamma(n\alpha+\gamma+r)} S_n^r \to s\,(x\to\infty), \tag{14}$$

$$t_r(n) \equiv \alpha^r \frac{\Gamma(r+1)\,\Gamma(n\alpha+\gamma)}{\Gamma(n\alpha+\gamma+r)} S_n^r \to s\,(B_{\alpha,\gamma}). \tag{15}$$

(14) is known ([1], Lemma 4) and (15) follows readily from (14).

Lemma 8. *If $s_n \to s\,(B_{\alpha,\gamma})$ and $s_n^r = O(n^\sigma)$ for some $r \geqslant 0$, $\sigma \geqslant \frac{1}{2}$, then $s_n^{r+1} = O(n^{\sigma-1/2})$, and, more generally, for an integer q such that $1 \leqslant q \leqslant 2\sigma$, $s_n^{r+q} = O(n^{\sigma-q/2})$.*

Proof. We have identically, for $m < n$,

$$s_n^{r+1} - s_m^{r+1} = \sum_{\nu=m+1}^{n} (s_\nu^{r+1} - s_{\nu-1}^{r+1}) = \sum_{\nu=m+1}^{n} \frac{r+1}{\nu}(s_\nu^r - s_\nu^{r+1}),$$

(see [3], p. 122). By hypothesis, $s_\nu^r = O(\nu^\sigma)$ as $\nu\to\infty$ and so $s_\nu^{r+1} = O(\nu^\sigma)$. Hence as $m \to \infty$, we have uniformly for $m \leqslant n \leqslant m + \epsilon\sqrt{m}$, $0 < \epsilon < 1$,

$$s_n^{r+1} - s_m^{r+1} = \sum_{\nu=m+1}^{n} O(\nu^{\sigma-1}) = \epsilon O(m^{\sigma-1/2}). \tag{16}$$

By the definition of $t_r(n)$ in (15) and Stirling's approximation,

$$t_r(n) = s_n^r \left\{1 + \frac{1}{n} w_r(n)\right\},\ w_r(n) = O(1), \tag{17}$$

so that we can write

$$t_{r+1}(n) - t_{r+1}(m) = s_n^{r+1} - s_m^{r+1} + \frac{1}{n} s_n^{r+1} w_{r+1}(n) - \frac{1}{m} s_m^{r+1} w_{r+1}(m). \quad (18)$$

Using (16) in (18), we get uniformly for $m \leqslant n \leqslant m + \epsilon\sqrt{m}$, $0 < \epsilon < 1$, as $m \to \infty$

$$t_{r+1}(n) - t_{r+1}(m) = \epsilon\, O(m^{\sigma-1/2}) + O(m^{\sigma-1});$$

$$\lim_{\epsilon\to 0} \limsup_{m\to\infty} \max_{m\leqslant n\leqslant m+\epsilon\sqrt{m}} |t_{r+1}(n) - t_{r+1}(m)|\, m^{-\sigma+1/2} = 0.$$

This and $t_r(n) \to s(B_{a,\gamma})$ which is a consequence of the hypothesis $s_n \to s(B_{a,\gamma})$ by Lemma 7, together imply $t_r(n) = O(n^{\sigma-1/2})$ on appeal to Lemma 4 with $t_r(n)$, $\sigma - \frac{1}{2}$ instead of s_n, ρ respectively and with the V_a^k transform specialized to the $(B_{a,\gamma})$ transform. Finally, we have also $s_n^{r+1} = O(n^{\sigma-1/2})$ because of (17).

The more general conclusion of the lemma follows from successive repetitions q times of the above argument.

4. Theorems

Theorem 1. *Suppose that* $s_n = O(n^\sigma)$ $(n \to \infty)$ *for some* $\sigma \geqslant 0$. *Then* $s_n \to s(V_a^k)$ *if and only if*

$$\left(\frac{a}{\pi x}\right)^{1/2} \sum_{n=0}^{\infty} \exp[-ax^{-1}(n-x)^2]\, s_n \to s\, (x \to \infty).$$

The proof is exactly like that of Satz II of Faulhaber [2]. Combining Theorem 1 with Lemma 1 we have

Theorem 2. *Suppose that* $s_n = O(n^\sigma)$ $(n \to \infty)$ *for some* $\sigma \geqslant 0$. *Then* $s_n \to s(V_a^k)$ *if and only if* $s_n \to s(B_{2a,\gamma})$.

Theorem 3. *If* s_n *is a real sequence satisfying* (13) *and* $s_n \to s\ (V_a^k)$ *then* $s_n \to s$ $(n \to \infty)$.

Proof. Since (13) is the special case $\rho = 0$ of (1), it follows from Lemma 4 that $s_n = O(1)$ and thereafter the desired conclusion is obvious from Lemmas 5 and 6.

Theorem 4. *If* s_n *is a real sequence satisfying* (1) *and* $s_n \to s(V_a^k)$, *then* $s_n \to s(C, 2\rho)$.

Proof. After Theorem 3, we need prove only the case $\rho > 0$. Also, $s_n = O(n^\rho)$ by lemma 4, so that we may appeal to Theorem 2 and replace hypothesis $s_n \to s(V_a^k)$ by $s_n \to s(B_{2a,\gamma})$.

If p is the greatest integer less than $2\rho + 1$ then

$$0 < \mu = 2\rho - (p - 1) \leqslant 1$$

and it follows from the more general conclusion of Lemma 8 with $r = 0$ that

$$s_n^{p-1} = O(n^{\rho-(p-1)/2}) = O(n^{\mu/2}).$$

Hence, exactly as elsewhere ([4], proof of Theorem 2, 439–40) we obtain, for $m \leqslant n \leqslant m + \epsilon\sqrt{m}, 0 < \epsilon < 1$, uniformly as $m \to \infty$

$$s_n^{2\rho} - s_m^{2\rho} = \epsilon^{\mu} O(1) + \epsilon\, O(m^{(\mu-1)/2}),\; s_m^{2\rho} = O(m^{\mu/2}). \tag{19}$$

Further, the definition of $t_r(n)$ in (15) gives, by (17),

$$t_{2\rho}(n) - t_{2\rho}(m) = s_n^{2\rho} - s_m^{2\rho} + \frac{1}{n} s_n^{2\rho} w_{2\rho}(n) - \frac{1}{m} s_m^{2\rho} w_{2\rho}(m)$$

$$= \epsilon^{\mu} O(1) + \epsilon O(m^{(\mu-1)/2}) + O(m^{\mu/2-1})$$

on account of (19). Thus

$$\lim_{\epsilon\to 0+} \limsup_{m\to\infty} \max_{m\leqslant n\leqslant m+\epsilon\sqrt{m}} | t_{2\rho}(n) - t_{2\rho}(m) | = 0,$$

while our assumption $s_n \to s\,(B_{2a,\,\gamma})$ implies $t_{2\rho}(n) \to s\,(B_{2a,\,\gamma})$ by Lemma 7. Now appealing to Theorem 3, with s_n replaced by $t_{2\rho}(n)$ and summability (V_a^k) by summability $(B_{2a,\,\gamma})$, we see that

$$t_{2\rho}(n) \to s \quad \text{whence} \quad s_n^{2\rho} \to s$$

as required, by (17).

5. Some standard cases of summability method (V_a^k)

In addition to the general Borel method $(B_{a,\,\gamma})$ and Borel method which is $(B_{1,\,1})$, the following are special cases of the method (V_a^k).

I. The Valiron method (V_a) which corresponds to the transforms of s_n given by

$$V_a(x) = \left(\frac{a}{\pi x}\right)^{1/2} \sum_{n=0}^{\infty} \exp[-ax^{-1}(n-x)^2]\, s_n,\; x > 0,$$

is a special case. For, $V_a(x)$ is $V_a^k(x)$ of § 2, with $g_k + R_k \equiv 0$ in condition (ii) imposed on $c_n(x)$, and with condition (iii) on $c_n(x)$ satisfied in consequence of a result stated by Faulhaber (Hilfssatz 1 with $\rho = 0$).

II. The Euler method (E_p), $p > 0$, the Hardy-Littlewood method (T_α), $0 < \alpha < 1$, and the method (S_β), $0 < \beta < 1$, due to Meyer-Konig and Vermes, correspond to transforms of s_n which may be written

$$\sum_{n=0}^{\infty} c_{n,\,m}\, s_n, \quad m = 1, 2, 3, \cdots,$$

where for (E_p) : $c_{n,\,m} = 2^{-pm}\binom{m}{n}(2^p - 1)^{m-n}$,

for (T_α) : $c_{n,\,m} = (1-\alpha)^{m+1}\binom{n}{m}\alpha^{n-m}$,

for (S_β) : $c_{n,\,m} = (1-\beta)^{m+1}\binom{m+n}{n}\beta^n$, with the convention that $\binom{\nu}{r} = 0$ for $\nu < r$. In all these cases we may define

$$c_n(x) = \begin{cases} c_{n,m} & \text{for } x_m \leqslant x < x_{m+1} \\ 0 & \text{for } 0 < x < x_1, \end{cases} \tag{20}$$

where $x_m = \lambda m$ with a suitable constant λ in each case. We can verify, using arguments of Faulhaber (proof of Hilfssatz 5) with trivial modifications, that the $c_n(x)$ of (20) satisfy for $x = x_m$ the conditions required of the $c_n(x)$ in the definition of $V_a^k(x)$ in § 2, provided we choose

for (E_p) : $a = 2^p/(2^{p+1} - 2)$, $\lambda = 2^{-p}$,

for (T_α) : $a = (1 - \alpha)/2\alpha$, $\lambda = (1 - \alpha)^{-1}$,

for (S_β) : $a = (1 - \beta)/2$, $\lambda = \beta(1 - \beta)^{-1}$.

Finally we can verify that, if the $c_n(x)$ of (20) satisfy the required conditions for $x = x_m$, where x_m is any increasing unbounded sequence with $x_{m+1} - x_m = O(1)$ then they satisfy the conditions for all $x > 0$. In particular, taking $x_m = \lambda m$ we see that the transforms E_p, T_α, S_β are all special cases of the V_a^k transform.

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