OPSEARCH (Jul–Sep 2011) 48(3):257–278
DOI 10.1007/s12597-011-0048-z

THEORETICAL ARTICLE

Joint importance measures for multistate reliability systems

V. M. Chacko · M. Manoharan

Accepted: 18 July 2011 / Published online: 30 July 2011
© Operational Research Society of India 2011

Abstract Importance and joint importance measures in reliability engineering are used to identify the weak areas of a system and signify the roles of components in either causing or contributing to proper functioning of the system. Existing joint reliability importance measure of a multistate system with n components provides the joint reliability importance of k ($k \leq n$) components. This paper introduces, for two multistate components, joint performance achievement worth, joint performance reduction worth, joint performance Fussell–Vesely measure and joint performance Birnubaum importance measure, using reliability, availability and risk as output performance measures (OPMs) of the multistate system. With reference to a predefined threshold of component performance, the component's reachable states are restricted to those corresponding to performances either larger or not larger than the threshold level. A steady state performance level distribution with restriction to the component's states is used to obtain the introduced measures. Use of universal generating function (UGF) for the evaluation of proposed joint importance measures is given. An illustrative example is provided.

Keywords Multistate system · Reliability · Availability · Risk · Importance measure · Joint importance measure · Universal generating function

V. M. Chacko (✉)
Department of Statistics, St. Thomas College, Thrissur, 680001, Kerala, India
e-mail: chackovm@gmail.com

M. Manoharan
Department of Statistics, University of Calicut, Malappuram, 673635, Kerala, India

1 Introduction

Importance measures (IMs) quantify the criticality of a particular component within a system design. They have been widely used as tools for identifying system weakness, and to prioritize reliability improvement activities. Measures of importance are quantitative criteria for ordering different components in the coherent system whose improvement may result in the greatest improvement for the system based on their critical roles in the functioning or the failure of the system and to provide a checklist for failure diagnosis. They can also provide valuable information for the safety and efficient operation of the system. From the design point of view, it is crucial to identify the weakness of the system and how failure of each individual component affects proper functioning of the system; so that efforts can be spent properly to improve the system reliability [3]. However, the extend to which a group of component and its states affect the system is a major concern to the system designer and system controller. To solve this problem, methods dependent on the information obtained from joint importance measures can be developed for efficient resource allocation. The knowledge about the joint importance measures can be used as a guide to provide redundancy so that system reliability is increased. It is more informative to the system designers about the interaction effect of two or more components in improving system performance. Information about this type of interaction importance of components constituting a system, with respect to its safety, reliability, availability and risk, is of great practical aid to system designers and managers. Measures of joint importance provide the information on the type and degree of interactions between two or more components by identifying the sign and size of it. A little work has been reported in literature on joint importance measures and the existing measures are extensions of Birnbaum importance measures.

In the binary classical reliability theory Birnbaum [6] and Barlow and Proschan [4] proposed some concepts of importance. Although the concept of component importance is very useful one, a few has been systematically generalized it to the multistate case, see Barlow and Wu [5], El-Neweihi et al. [13], Griffith [15], El-Neweihi and Proschan [12] and Bueno [7]. Abouammoh and Al-Khadi [1] reviewed the measures on importance for multistate coherent systems (MCSs). Levitin and Lisnianski [23] proposed importance and sensitivity measures for multistate systems (MSSs) with binary capacited components. These measures account both for MSS performance which is caused by the capacited components and stochastic system demand. Their evaluation method is performed via the universal generating function (UGF) method. These approaches have proven to be valuable to the development of multistate IMs. Wu and Chan [32] proposed IMs for MSSs with respect to performance utility and related their measure to Griffith's IM. Ramirez-Marquez and Coit [26] proposed new importance measures for MSSs from two perspectives: (1) how a specific component affects MSS reliability and (2) how a particular state or set of states affects MSS reliability. IMs are widely used in risk informed applications of the nuclear industry to characterize the importance of basic events, i.e., element failures, human errors, common cause failures, etc, with respect to the risk associated to the system. Vasseur and Llory [30] mentioned reliability achievement worth (RAW), reliability reduction worth (RRW), Fussell–Vesely (FV) measure and Birnubaum measure as the most valuable IMs for binary systems in risk informed applications. Further extensions of these measures to the multistate case can be seen in Ramirez-Marquez and Coit [26] and Zio and Podofillini [33]. Levitin et al. [24] proposed similar measures using performance measures such as availability and risk. Also Zio and Podofillini [33] introduced identical measures in terms of system risk-unavailability or unreliability. The use of IMs to analyze probabilistic risk assessment results is discussed in detail by Cheok et al. [11] and Van der Borst and Shoonakker [29].

However, the joint importance measure provides additional information, which the traditional marginal importance cannot provide, to the system designers, see Hong et al. [17]. Joint importance measures for binary system can be seen in Armstrong [2] and Hong and Lie [16]. Hong et al. [17] investigated joint reliability importance (JRI) of two gate events along with its properties in a fault tree. Wu [31] extended the component IMs to joint importance measures for two multistate components in a MSS with respect to system structure and expected performance. A limitation of the IMs currently used in reliability and risk analysis is that they rank only individual components or basic events whereas they are not directly applicable to combinations or groups of components or basic events. To partially overcome this limitation, recently, the differential importance measure (DIM), has been introduced for use in risk-informed decision making. The DIM is a first-order sensitivity measure that ranks the parameters of the risk model according to the fraction of total change in the risk that is due to a small change in the parameters' values, taken one at a time. However, it does not account for the effects of interactions among components. Zio and Podofillini [34] proposed a second-order extension of the DIM, named DIMII, for accounting of the interactions of pairs of components when evaluating the change in system performance due to changes of the reliability parameters of the components.

We recall the existing importance and joint importance measures followed by introducing new joint importance measures-joint structural and reliability importance measures-for two or more components. We also propose joint importance measures as an extension to RAW, RRW, FV and Birnubaum measure of components and generalize it to other performance measures such as availability, and risk-unavailability or unreliability. We find the distribution of the performance of the system, under constraints on the performance of its elements. Once the system performance is determined, one can focus on specific system performance measures. With reference to the predefined threshold of element performance, the element's reachable states are limited to those corresponding to performance either larger or not larger than the threshold level.

The remaining sections of this paper is arranged as follows. Joint reliability importance measure for more than two components of a MSS is recalled in

Section 2. Joint reliability achievement worth for two components in a MSS is proposed in Section 3. Joint reliability reduction worth for two components in a MSS is proposed in Section 4. In Section 5, the joint Fussell–Vesely measure w.r.t. reliability for two components in a MSS is proposed. The joint reliability Birnubaum measure for more than two components is proposed in Section 6. The joint risk importance measures based on unreliability or unavailability are proposed in Section 7. The evaluation of the proposed joint importance measures using UGF is given in Section 8. An illustrative example is provided in Section 9. Conclusion is given in last section.

2 Joint reliability importance measure

Joint reliability importance (JRI) of two or more components is a quantitative measure of the interactions of two or more components or states of two or more components. The value of JRI represents the degree of interactions between two or more components with respect to system reliability. JRI indicates how components interact in system reliability, see Armstrong [2]. Consider the vector of component states $\mathbf{X} = (X_1, X_2, ..., X_n)$, where X_i is the random variable representing the state of the ith component. Let $\phi(.)$ be the structure function of the system.

In the binary setup, the marginal reliability importance of a component is

$$I(i) = \frac{\partial R}{\partial R_i}$$

and the JRI of two components i and j is

$$JRI(i, j) = \frac{\partial^2 R}{\partial R_i . \partial R_j} \tag{2.1}$$

where $R = E(\phi(\mathbf{X}))$ and R_i and R_j are reliabilities of the components i and j respectively. That is, joint reliability importance of two binary components is

$$JRI(i, j) = R\left(1_i, 1_j, \mathbf{p}\right) - R\left(1_i, 0_j, \mathbf{p}\right) - R\left(0_i, 1_j, \mathbf{p}\right) + R\left(0_i, 0_j, \mathbf{p}\right) \tag{2.2}$$

where $R(._i, ._j, \mathbf{p}) = E(\phi(X_1, ..., ._i, ..., ._j, ..., X_n))$. In order to generalize this equation for more than two components, i.e., to measure the improvement of reliability importance of the system with respect to the interactive effect of more than two components, at first we shall calculate change in the JRI of two components with respect to the change of reliability of third component. If there is any change in the JRI due to change in state of third component we can say that there is an interactive effect of three components for the system reliability improvement. That is, in the binary setup, the change in the JRI is found to be as follows

$$JRI(i, j, k) = JRI(i, j|k = 1, \mathbf{p}) - JRI(i, j|k = 0, \mathbf{p}) \tag{2.3}$$

where $JRI(i, j|k = q, \mathbf{p}) = R(1_i, 1_j, q_k, \mathbf{p}) - R(1_i, 0_j, q_k, \mathbf{p}) - R(0_i, 1_j, q_k, \mathbf{p}) + R(0_i, 0_j, q_k, \mathbf{p})$, $q = 0$ or 1, i.e., change in JRI of two components when third component is improved from its failure state to its functioning state. The value of the $JRI(i, j, k)$ indicates how the JRI of two components changes with the change of the state of third component.

In order to find JRI in MSS, we consider X_i's and system states ϕ take values in the set {0, 1, 2, ..., M}. Let

$$P\left[\phi(\mathbf{X}) \geq j\right] = P\left[\phi\left(0_i, \mathbf{X}_i\right) \geq j\right] + \sum_{m=1}^{M} \left(P\left[\phi\left(m_i, \mathbf{X}_i\right) \geq j\right] - P\left[\phi\left(\hat{m}_i, \mathbf{X}_i\right) \geq j\right]\right) \times P\left[X_i \geq m\right], \quad \text{and}$$

$$E_s = \sum_{j=1}^{M} P[\phi(\mathbf{X}) \geq j], \text{ and } R_i m = P\left[X_i \geq m\right].$$

We shall prove the following lemma. It shows how JRI of three components express in terms of JRI of two components proposed by Wu [31]. The following results are proved in Chacko and Manoharan [8].

Lemma 2.1 *Let ϕ be a structure function of MSS with n components. Then,*

$$\frac{\partial^3 E_s}{\partial R_i m \partial R_l k \partial R_r n} = \left.\frac{\partial^2 E_s}{\partial R_i m \partial R_l k}\right|_{n_r} - \left.\frac{\partial^2 E_s}{\partial R_i m \partial R_l k}\right|_{\hat{n}_r}. \tag{2.4}$$

Let JRIM represent the joint reliability importance for the MSS. Now we define the JRIM for three components. This definition provides a measure for finding JRI of three components in a MSS.

Definition 2.1 The joint reliability importance of three components with respect to state m of component i, state k of component l and state n of component r of a multistate system is

$$JRIM(i, l, r; m, k, n) = \frac{\partial^3 E_s}{\partial R_i m \partial R_l k \partial R_r n}. \tag{2.5}$$

Taking summation over m, k, and n we get the joint reliability importance of three components.

We have expressed the JRI of three components in a MSS using existing JRI measures of Wu [31]. Now we consider a general theorem on interaction importance of k of components of a system having $n(\geq k)$ components.

Theorem 2.1 *Suppose that*

$$JRIM(a_1, ..., a_k; b_1, ..., b_k) = \frac{\partial^k E_s}{\partial R_{a_1} b_1 .... \partial R_{a_k} b_k}, \; k = 2, 3, ..., n \tag{2.6}$$

represents the interaction importance of k components. Then the joint reliability importance of k of components can be derived as, for k = 2, ..., n,

$$\frac{\partial^k E_s}{\partial R_{a_1} b_1 \ldots \partial R_{a_k} b_k} = \left.\frac{\partial^{k-1} E_s}{\partial R_{a_1} b_1 \ldots \partial R_{a_{k-1}} b_{k-1}}\right|_{b_{ka_k}} - \left.\frac{\partial^{k-1} E_s}{\partial R_{a_1} b_1 \ldots \partial R_{a_{k-1}} b_{k-1}}\right|_{\hat{b}_{ka_k}}.$$

Chacko and Manoharan [8] defined the JRI of k components in a MSS.

Definition 2.2 The joint reliability importance of k components with respect to state b_1 of the component a_1, state b_2 of the component a_2, ..., state b_k of the component a_k of the multistate system is

$$JRIM\,(a_1, ..., a_k; b_1, ..., b_k) = \frac{\partial^k E_s}{\partial R_{a_1} b_1 \ldots \partial R_{a_k} b_k}, k = 2, 3, ..., n. \tag{2.7}$$

As a result we get the joint importance of 2, 3, ..., components in the system. We can reach some important conclusions regarding the joint importance such as whether the joint importance is different for different group of components. The size of the joint importance gives information about the degree of interaction. Following the above method we get the module importance.

The joint importance measures of two components for MSS with the OPMs, reliability and availability, with reference to the existing measures of importance, RAW, RRW, FV, and Birnbaum for individual components are introduced in the following sections. For the sake of better narration, we consider reliability as the output performance measure and introduce joint importance measures. This results are also true for the OPM availability. So we can generalize the results to both reliability and availability. In following sections, for time dependent binary and MSS, we propose JRAW, JRRW, JRFV for two components and JRBI measures for any number of components.

3 Joint reliability achievement worth

The RAW measure quantifies the maximum percentage increase in system reliability generated by a particular component. From a binary perspective it is defined as

$$RAW_i = \frac{P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 1]}{P[\phi(\mathbf{X}(t)) = 1]}.$$

For a constant demand w_k corresponding to state k, multistate RAW of component i with respect to performance threshold α and corresponding performance state $k_{i\alpha}$ is,

$$MRAW_i = \frac{P[\phi(\mathbf{X}(t)) \geq k | X_i(t) \geq k_{i\alpha}]}{P[\phi(\mathbf{X}(t) \geq k]}.$$

We propose the joint importance measure, JRAW, of two components *i* and *j* of binary state system,

$$JRAW_{ij} = \frac{P_{11} - P_{10} - P_{01}}{P_{1.} + P_{.1}}$$

where

$$P_{11} = P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 1, X_j(t) = 1],$$

$$P_{10} = P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 1, X_j(t) = 0],$$

$$P_{01} = P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0, X_j(t) = 1], \ P_{1.} = P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 1]$$

and $P_{.1} = P[\phi(\mathbf{X}(t)) = 1 | X_j(t) = 1]$, for measuring the joint reliability achievement worth due to interaction. The $JRAW_{ij}$ measure quantifies the maximum percentage increase in system reliability generated by the interaction of two components *i* and *j*. Note that $JRAW_{ij} = JRAW_{ji}$.

The multistate extension of above measures for constant demand w_k corresponding to state *k* can be defined with respect to performance level α and β for two components *i* and *j*, as,

$$MJRAW_{ij} = \frac{P^{\geq\alpha,\geq\beta} - P^{\geq\alpha,<\beta} - P^{<\alpha,\geq\beta}}{P^{\geq\alpha,\cdot} + P^{\cdot,\geq\beta}}.$$

where

$$\begin{aligned}
P^{\geq\alpha,\geq\beta} &= P\left[\phi(\mathbf{X}(t)) \geq k | X_i(t) \geq k_{i\alpha}, X_j(t) \geq k_{j\beta}\right] \\
P^{\geq\alpha,<\beta} &= P\left[\phi(\mathbf{X}(t)) \geq k | X_i(t) \geq k_{i\alpha}, X_j(t) < k_{j\beta}\right] \\
P^{<\alpha,\geq\beta} &= P\left[\phi(\mathbf{X}(t)) \geq k | X_i(t) < k_{i\alpha}, X_j(t) \geq k_{j\beta}\right] \\
P^{\geq\alpha,\cdot} &= P\left[\phi(\mathbf{X}(t)) \geq k | X_i(t) \geq k_{i\alpha}\right] \text{ and} \\
P^{\cdot,\geq\beta} &= P\left[\phi(\mathbf{X}(t)) \geq k | X_j(t) \geq k_{j\beta}\right],
\end{aligned}$$

for measuring the joint reliability achievement worth due to interaction.

Now we define the joint reliability reduction worth for measuring joint effect of two components in reducing reliability.

4 Joint reliability reduction worth

The RRW is an index measuring the potential damage caused to the system by a particular component. The binary expression of the RRW of component *i* is

$$RRW_i = \frac{P[\phi(\mathbf{X}(t)) = 1]}{P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0]}.$$

Then the extension of RRW to the multistate case for constant demand w_k corresponding to state k can be defined, for the performance level α of component i and corresponding performance state $k_{i\alpha}$, as

$$MRRW_i = \frac{P[\phi(\mathbf{X}(t)) \geq k]}{P\left[\phi(\mathbf{X}(t)) \geq k | X_i(t) < k_{i\alpha}\right]}.$$

We propose the joint importance measure, JRRW, of two components i and j of binary state system,

$$JRRW_{ij} = \frac{P\left[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0\right] + P\left[\phi(\mathbf{X}(t)) = 1 | X_j(t) = 0\right]}{P\left[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0, X_j(t) = 0\right]}$$

for measuring the joint reliability reduction worth with respect to interaction of the components at below specified levels. The $JRRW_{ij}$ measure quantifies the potential damage caused to the system by interaction of two components i and j at below specified levels. Note that $JRRW_{ij} = JRRW_{ji}$.

The multistate extension of JRRW for constant demand w_k corresponding to state k, can be defined for performance levels α and β of components i and j, as

$$MJRRW_{ij} = \frac{P\left[\phi(\mathbf{X}(t)) \geq k | X_i(t) < k_{i\alpha}\right] + P\left[\phi(\mathbf{X}(t)) \geq k | X_j(t) < n_{j\beta}\right]}{P\left[\phi(\mathbf{X}(t)) \geq k | X_i(t) < k_{i\alpha}, X_j(t) < n_{j\beta}\right]}. \tag{4.1}$$

We next define the joint Fussell–Vesely measure for finding the maximum decrement in system reliability caused by joint effect of two components at below specified levels.

5 Joint reliability Fussel–Vesely measures

The FV importance measure quantifies the maximum decrement in system reliability caused by a particular component. The binary expression is

$$FV_i = \frac{P[\phi(\mathbf{X}(t)) = 1] - P\left[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0\right]}{P[\phi(\mathbf{X}(t)) = 1]}.$$

It has extended to multistate case for constant demand w_k corresponding to state k as

$$MFV_i = \frac{P\left[\phi(\mathbf{X}(t)) \geq k\right] - P\left[\phi(\mathbf{X}(t)) \geq k | X_i(t) < x_{ik_{i\alpha}}\right]}{P\left[\phi(\mathbf{X}(t)) \geq k\right]}.$$

We propose the following joint importance measure, JRFV, of two components i and j of the binary state system,

$$JRFV_{ij} = \frac{P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0] + P[\phi(\mathbf{X}(t)) = 1 | X_j(t) = 0] - P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0, X_j(t) = 0]}{P[\phi(\mathbf{X}(t)) = 1 | X_i(t) = 0] + P[\phi((\mathbf{X}(t)) = 1 | X_j(t) = 0]}$$

for measuring the joint reliability Fussell–Vesely importance with respect to interaction. The $JRFV_{ij}$ measure quantifies the maximum decrement in system reliability caused by joint effect of two components i and j at below specified levels. Note that $JRFV_{ij} = JRFV_{ji}$.

The multistate extension of JRFV for constant demand w_k corresponding to state k can be defined, with respect to the performance levels α and β of components i and j, as

$$MJFV_{ij} = \frac{P^{<\alpha,\cdot} + P^{\cdot,<\beta} - P^{<\alpha,<\beta}}{P^{<\alpha,\cdot} + P^{\cdot,<\beta}},$$

where $P^{<\alpha,\cdot} = P[\phi(\mathbf{X}(t)) \geq k | X_i(t) < k_{i\alpha}]$, $P^{\cdot,<\beta} = P[\phi(\mathbf{X}(t)) \geq k | X_j(t) < n_{j\beta}]$, and $P^{<\alpha,<\beta} = P\left[\phi(\mathbf{X}(t)) \geq k | X_i(t) < k_{i\alpha}, X_j(t) < n_{j\beta}\right]$.

Now we define the joint Birnbaum importance measure of any number of components with respect to reliability. The component Birnbaum importance measure is the most widely used importance measure by many reliability researchers, engineers and practitioners.

6 Joint reliability Birnbaum importance measures

The Birnbaum measure represents the maximum loss in the system reliability when element i switches from the condition of perfect functioning to the condition of certain failure. Let the state X_i of the ith binary component is random with probability $P[X_i = 1] = R_i = EX_i, i = 1, 2, ..., n$. The reliability of the binary system with structure function $\phi(\mathbf{X})$, $\mathbf{X} = (X_1, ..., X_n), \forall i,\ X_i, \phi \in \{0, 1\}$ is

$$P[\phi(\mathbf{X}) = 1] = h(\mathbf{p}) = E\phi(\mathbf{X}), \mathbf{p} = (R_1, ..., R_n)\,.$$

Birnbaum [6] proposed the following IM for the binary state system.

$$I(i) = \frac{\partial h}{\partial R_i} = h\left(1_i, \mathbf{p}\right) - h\left(0_i, \mathbf{p}\right) = E\left[\phi(1_i, \mathbf{X}) - \phi(0_i, \mathbf{X})\right], i = 1, 2, ..., n.$$

Clearly $I(i)$ describes the rate of improvement of system performance with respect to the improvement in performance of component i.

As an extension of Birnbaum measure to the multistate case, Griffith [15] defined the reliability importance of level l of the ith component of the MSS with structure function ϕ as

$$I_l(i) = E\left[\phi(l_i, \mathbf{X}) - \phi((l-1)_i, \mathbf{X})\right]$$

where $(l_i, \mathbf{X}) = (X_1, ..., X_{i-1}, l_i, X_{i+1}, ..., X_n)$, and $X_i \in \{0, 1, ..., M\}$, $i = 1, 2, ..., n$.

Multistate joint reliability Birnbaum importance measure, MJRBI, of two components i and j with respect to performance levels α and β for the MSS can be defined as

$$\begin{aligned} MJRBI_{ij} = {} & P\left[\phi(\mathbf{X}(t)) \geq k | X_i(t) \geq k_{i\alpha}, X_j(t) \geq n_{j\beta}\right] \\ & - P\left[\phi(\mathbf{X}(t)) \geq k | X_i(t) \geq k_{i\alpha}, X_j(t) < n_{j\beta}\right] \\ & - P\left[\phi(\mathbf{X}(t)) \geq k | X_i(t) < k_{i\alpha}, X_j(t) \geq n_{j\beta}\right] \\ & + P\left[\phi(\mathbf{X}(t)) \geq k | X_i(t) < k_{i\alpha}, X_j(t) < n_{j\beta}\right]. \end{aligned} \tag{6.1}$$

It measures the improvement of system reliability due to the interaction effect of two components.

Proceeding like this we can introduce joint reliability Birnbaum importance measures for three components, four components etc. $MJRBI$ of three components i, j and l with respect to performance levels α, β and γ for the MSS can be defined as

$$MJRBI_{ijl} = MJRBI_{ij}\left(X_l(t) \geq m_{l\gamma}\right) - MJRBI_{ij}\left(X_l(t) < m_{l\gamma}\right) \tag{6.2}$$

where $MJRBI_{ij}(X_l(t) \geq m_{l\gamma})$ is $MJRBI_{ij}$ when component l is above some predefined threshold γ with corresponding state $m_{l\gamma}$. Similar interpretation holds for $MJRBI_{ij}(X_l(t) < m_{l\gamma})$.

Now we redefine the above joint importance measures with general expression of OPM (reliability or availability)-for the MSS. Let component i be constrained to performance below α, while the rest of components of the MSS are not constrained: we denote by $OM_i^{\leq\alpha}$ the system OPM obtained in this situation. Similarly, we denote by $OM_i^{>\alpha}$ the system OPM resulting from the dual situation in which component i is constrained to performances above α. Also let $OM_{i,j}^{\leq\alpha,\leq\beta}$, $OM_{i,j}^{>\alpha,\leq\beta}$, $OM_{i,j}^{\leq\alpha,>\beta}$ and $OM_{i,j}^{>\alpha,>\beta}$ be the OPMs when both components i and j are restricted in their performance based on performance thresholds α and β respectively. We introduce the following measures for two components in a MSS with respect to performance measure-reliability or availability.

1. Joint Performance Achievement Worth

$$MJPAW_{ij} = \frac{OM_{i,j}^{>\alpha,>\beta} - OM_{i,j}^{>\alpha,\leq\beta} - OM_{i,j}^{\leq\alpha,>\beta}}{OM_i^{>\alpha} + OM_j^{>\beta}} \tag{6.3}$$

2. Joint Performance Reduction Worth

$$MJPRW_{ij} = \frac{OM_i^{>\alpha} + OM_j^{>\beta}}{OM_{i,j}^{\leq\alpha,\leq\beta}} \tag{6.4}$$

3. Joint Performance Fussell–Vesely Measure

$$MJPFV_{ij} = \frac{OM_i^{>\alpha} + OM_j^{>\beta} - OM_{i,j}^{\leq\alpha,\leq\beta}}{OM_i^{>\alpha} + OM_j^{>\beta}} \tag{6.5}$$

4. Joint Performance Birnbaum Importance

$$MJPBI_{ij} = OM_{i,j}^{\alpha,>\beta} - OM_{i,j}^{\alpha,\leq\beta} \tag{6.6}$$

where $OM_{i,j}^{\alpha,>\beta}$ represents the Birnbaum importance of the component i when component j is restricted to the performance above level β. Similarly $OM_{i,j}^{\alpha,\leq\beta}$ represents the Birnbaum importance of the component i when component j is restricted to below level β. Similarly we can find third order multistate joint performance Birnbaum importance measures by taking differences of $MJPBI$ of two components after restricting the performance of third component below and above some pre-specified performance levels.

Thus we defined four main joint importance measures with respect to reliability and availability. But we can define the joint importance measures of above type with respect to risk also. We define joint risk importance measures in the following section.

7 Joint risk importance measures

To compare the joint effect of pair of components with the standardly used risk, one can transform the performance measures into risk measures (unreliability or unavailability). In order to introduce the joint risk importance measures, we define the following indexes in terms of system risk.

$F_i^+(t)$, value of risk metric F when component i has been in state below a specified level throughout the time interval $[0, t]$.

$F_i^-(t)$, value of risk metric F when component i has been in its functioning state (above a specified level) throughout the time interval $[0, t]$.

The definition of the four of the risk importance measures for a system is recalled here with reference to the ith component, see Zio and Podofillini [33] for details.

1. Birnbaum Risk Importance Measure: $rB_i(t) = F_i^+(t) - F_i^-(t)$, it measures the maximum deviation of risk when ith component shifts from its condition of perfect functioning to condition of certain failure.
2. Risk Achievement Worth (rAW): $rAW_i = \frac{F_i^+(t)}{F(t)}$, it is the ratio of risk when component i is considered always failed in $[0, t]$ to the actual value of risk.
3. Risk Reduction Worth (rRW): $rRW_i = \frac{F(t)}{F_i^-(t)}$, it is the ratio of the nominal value of risk to the risk when component i is always available. It measures the potential of component in reducing the risk, by considering the maximum decrease in risk achievable when optimizing the components to perfection.
4. Risk Fussell–Vesely Measure (rFV): $rFV_i(t) = \frac{F(t)-F_i^-(t)}{F(t)}$, it represents the maximum fractional decrement in risk achievable when component i is always available.

In order to introduce joint risk measures, multistate joint Risk Birnbaum Importance measure (MJrBI), multistate joint Risk Achievement Worth (MJrAW), multistate joint Risk Reduction Worth (MJrRW), and multistate joint Risk Fussell–Vesely measure (MJrFV), with reference to two components i and j, we define the following indexes in terms of system risk.

$F_{i,j}^{++}(t)$, value of risk metric F when both components i and j have been in state below some specified levels throughout the time interval $[0, t]$.

$F_{i,j}^{+-}(t)$, value of risk metric F when components, i has been in state below some specified level and j has been in state above some specified level, throughout the time interval $[0, t]$.

$F_{i,j}^{-+}(t)$, value of risk metric F when components, i has been in state above some specified level and j has been in state below some specified level, throughout the time interval $[0, t]$.

$F_{i,j}^{--}(t)$, value of risk metric F when both components i and j have been in state above some specified levels throughout the time interval $[0, t]$.

Now we define the multistate joint risk importance measures to the MSS.

1. Multistate Joint Risk Birnbaum measure:

$$MJrBI_{ij} = F_{i,j}^{++}(t) - F_{i,j}^{+-}(t) - F_{i,j}^{-+}(t) + F_{i,j}^{--}(t). \tag{7.1}$$

   It is the maximum variation in risk due to the interaction of components i and j.
2. Multistate Joint Risk Achievement Worth:

$$MJrAW_{ij} = \frac{F_{i,j}^{++}(t)}{F_i^{+}(t) + F_j^{+}(t)}. \tag{7.2}$$

   It is the ratio of risk when both components i and j is below some specified levels to the risk when either of two components is below some specified levels in $[0, t]$.
3. Multistate Joint Risk Reduction Worth:

$$MJrRW_{i,j} = \frac{F_i^{-}(t) + F_j^{-}(t)}{F_{i,j}^{--}(t) - F_{ij}^{+-} - F_{ij}^{-+}(t)}. \tag{7.3}$$

   It is the ratio of the nominal value of risk when either of two components i and j is available to the risk when both components are always available. It measures the interaction effect of two components in reducing the risk, by considering the maximum decrease in risk achievable with respect to joint effect of two components.
4. Multistate Joint Risk Fussell–Vesely measure:

$$MJrFV_{ij} = \frac{F_i^{-}(t) + F_j^{-}(t) - F_{ij}^{--}(t)}{F_i^{-}(t) + F_j^{-}(t)}. \tag{7.4}$$

   It represents the maximum fractional decrement in risk achievable when both of two components i and j are always available to the availability of either of two components.

8 Evaluation of joint importance measures

In certain MSSs, the performance of different system components can have different physical nature whose performance is measured in terms of productivity or capacity etc. Therefore it is important to measure performance rates of these components by their contribution into the entire MSS output performance. Examples of such MSSs are continuous materials or energy transmission systems, power generation systems [14]. In power generation applications the performance measure is usually defined as productivity or capacity (eg. production capacity of 100 MW). The main task of these systems is to provide the desired throughput or transmission capacity for constant energy, material or information flow. The evaluation of system reliability and joint importance measure of such systems are complicated because of multistate behavior and complexity of configuration of system components. Also we cannot use usual generating function to find out the performance and probability distribution of such systems, since for parallel components the performance will not be usual maximum of individual performances but sum (eg. two parallel power generator with productivity 100 MW provide 200 MW as total output). UGF is found to be a fine tool for such systems in evaluation of reliability and importance measure, see Ushakov [27, 28] and Lisnianski and Levitin [25]. Some application of UGF can be seen in Levitin [18–20] and Chacko and Manoharan [8, 10]. So we will use the UGF for the evaluation of performance measure and joint importance measure of systems whose performance is in terms of productivity or capacity.

8.1 Universal generating function

For a MSS which has a finite number of states, there can be $M+1$ different output performance at each time t,

$$G(t) \in \mathbf{G} = \{G_k, 0 \leq k \leq M\}$$

and the system output performance distribution can be defined by two finite vectors $\mathbf{G}$ and $\mathbf{P} = \{p_k(t) = P[G(t) = G_k], 0 \leq k \leq M\}$.

The UGF, represented by a polynomial $U(z)$ can define MSS OPD, i.e., it represents all the possible states of the system (or component) by relating the probabilities of each state, p_k, to performance G_k of the MSS of that state in the form:

$$U_{MSS}(t, z) = \sum_{k=0}^{M} p_k(t) z^{G_k}, \; z \in R. \tag{8.1}$$

Now we discuss the UGF of complex MSS.

8.2 UGF for complex MSS

Real world MSSs are often complex and consist of large number of components composing different types of structures. This is a technique to obtain the entire MSS output performance distribution. This technique uses composition operators for determination of the UGF of a subsystem (component) containing a number of components. These operators determine the subsystem UGF expressed as polynomial $U(z)$ for a group of components using simple algebraic operations over individual UGFs of components. All the composition operators for two different components takes the form

$$\Omega\left(U_1(z), U_2(z)\right) = \Omega\left(\sum_{i=0}^{M} p_{1i} z^{g_{1i}}, \sum_{j=0}^{M} p_{2j} z^{g_{2j}}\right) = \sum_{i=0}^{M}\sum_{j=0}^{M} p_{1i} p_{2j} z^{w\left(g_{1i}, g_{2j}\right)} \quad (8.2)$$

where $U_1(z)$ and $U_2(z)$ are individual UGF of components 1 and 2 with performance distributions $\{g_{1i}, p_{1i}, i \in \{0, 1, ..., M\}\}$ and $\{g_{2j}, p_{2j}, j \in \{0, 1, ..., M\}\}$ respectively. The function $w(.,.)$ in composition operators expresses the entire performance rate of subsystem consisting of different components in terms of the individual performance rates of the components. The definition of the function $w(.,.)$ strictly depends on the type of connection between the components in the reliability logic diagram sense. Here we define composition operators $\Omega\sigma$, $\Omega\pi$ for subsystems with components connected in series and in parallel respectively. In MSS where the performance measure is defined in terms of capacity or productivity, the total capacity of a pair of components connected in parallel is equal to the sum of the capacities of the components. Therefore the function $w(.,.)$ in composition operator takes the form:

$$w\left(g_1, g_2\right) = \pi\left(g_1, g_2\right) = g_1 + g_2.$$

For a pair of components connected in series, the component with the least capacity becomes the bottleneck of the system. In this case the function $w(.,.)$ takes the form:

$$w\left(g_1, g_2\right) = \sigma\left(g_1, g_2\right) = \min\left(g_1, g_2\right).$$

Note that the composition operators for components connected in parallel and in series satisfies the conditions:

$$\Omega\left(U_1(z), ..., U_k(z), U_{k+1}(z), ..., U_n(z)\right) = \Omega\left(U_1(z), ..., U_{k+1}(z), U_k(z), ..., U_n(z)\right)$$

and

$$\Omega\left(U_1(z), ..., U_k(z), U_{k+1}(z), ..., U_n(z)\right) = \Omega\left(\Omega(U_1(z), ..., U_k(z)), U_{k+1}(z), ..., U_n(z)\right)$$

for arbitrary k. Consecutively applying the Ω operators with corresponding functions σ or π to the components, one can obtain the UGF for an arbitrary number of components connected in series or in parallel. Combining the two operators one can obtain UGF representing performance distribution of an arbitrary series-parallel system, [9].

Here we propose the component's performance restriction approach (or state space restriction approach) for evaluation of joint importance measures using UGF. Let OM_{ik} be the OPM of the MSS when component i is in a fixed state k while the rest of components evolve stochastically among their corresponding states with steady state performance distributions $\{x_{jl}, p_{jl}\}$, $1 \leq j \leq n$, $0 \leq l \leq M_j$.

The conditional probability of the component i being in a generic state k characterized by a performance x_{ik} not greater than a pre-specified level threshold α (or equivalently $k \leq k_{i\alpha}$) is

$$P\left[X_i = k | k \leq k_{i\alpha}\right] = \frac{p_{ik}}{\sum_{r=0}^{k_{i\alpha}} p_{ir}} = \frac{p_{ik}}{p_i^{\leq\alpha}} = p_{1ik}^{*}\,(say).$$

Similarly, the conditional probability of component i being in a state k when it is known that $k > k_{i\alpha}$ is

$$P\left[X_i = k | k > k_{i\alpha}\right] = \frac{p_{ik}}{\sum_{r=k_{i\alpha}+1}^{M_i} p_{ir}} = \frac{p_{ik}}{p_i^{>\alpha}} = p_{2ik}^{*}\,(say).$$

Consider the following joint probability distribution of two independent components i and j given four additional restrictions, (1) $k > k_{i\alpha}, h > h_{j\beta}$, (2) $k \leq k_{i\alpha}, h > h_{j\beta}$, (3) $k > k_{i\alpha}, h \leq h_{j\beta}$ and (4) $k \leq k_{i\alpha}, h \leq h_{j\beta}$.

$$\begin{aligned}
P\left[X_i = k, X_j = h | k \leq k_{i\alpha}, h \leq h_{j\beta}\right] &= \frac{p_{ik}p_{jh}}{\sum_{r=0}^{k_{i\alpha}} p_{ir} \sum_{m=0}^{h_{j\beta}} p_{jm}} \\
&= p_1^{**}{}_{kh}\,(say) \\
P\left[X_i = k, X_j = h | k \leq k_{i\alpha}, h > h_{j\beta}\right] &= \frac{p_{ik}p_{jh}}{\sum_{r=0}^{k_{i\alpha}} p_{ir} \sum_{h_{j\beta}+1}^{M_j} p_{jm}} \\
&= p_2^{**}{}_{kh}\,(say) \\
P\left[X_i = k, X_j = h | k > k_{i\alpha}, h \leq h_{j\beta}\right] &= \frac{p_{ik}p_{jh}}{\sum_{r=k_{i\alpha}+1}^{M_i} p_{ir} \sum_{m=0}^{m=h_{j\beta}} p_{jm}} \\
&= p_3^{**}{}_{kh}\,(say) \quad \text{and} \\
P\left[X_i = k, X_j = h | k > k_{i\alpha}, h > h_{j\beta}\right] &= \frac{p_{ik}p_{jh}}{\sum_{r=k_{i\alpha}+1}^{M_i} p_{ir} \sum_{m=h_{j\beta}+1}^{M_j} p_{jm}} \\
&= p_4^{**}{}_{kh}\,(say).
\end{aligned}$$

Using the above conditional probability distributions, we obtain the following OPMs:

$$OM_i^{\le\alpha} = \sum_{k=0}^{k_{i\alpha}} \frac{p_{ik}}{p_i^{\le\alpha}}.OM_{ik}, \tag{8.3}$$

$$OM_i^{>\alpha} = \sum_{k=k_{i\alpha}+1}^{M_i} \frac{p_{ik}}{p_i^{>\alpha}}.OM_{ik}, \tag{8.4}$$

$$OM_{i,j}^{\le\alpha,\le\beta} = \sum_{k=0}^{k_{i\alpha}} \sum_{h=0}^{h_{j\beta}} p_1^{**}{}_{hk}.OM_{ik,jh}, \tag{8.5}$$

$$OM_{i,j}^{>\alpha,\le\beta} = \sum_{k=k_{i\alpha}+1}^{M_i} \sum_{h=0}^{h_{j\beta}} p_3^{**}{}_{hk}.OM_{ik,jh}, \tag{8.6}$$

$$OM_{i,j}^{\le\alpha,>\beta} = \sum_{k=0}^{k_{i\alpha}} \sum_{h=h_{j\beta}+1}^{M_j} p_2^{**}{}_{hk}.OM_{ik,jh} \quad \text{and} \tag{8.7}$$

$$OM_{i,j}^{>\alpha,>\beta} = \sum_{k=k_{i\alpha}+1}^{M_i} \sum_{h=h_{j\beta}+1}^{M_j} p_4^{**}{}_{hk}.OM_{ik,jh}, \tag{8.8}$$

where $OM_{ik,jh}$ be the system steady state OPM when component i is in state k and component j is in state h. Substituting Eq. 8.3–8.8 in Eqs. 6.3–6.6 we get the generalized importance measures using steady state probability distribution of components.

In the same way we can express the joint risk importance measures. At steady state, let F_{ik} be the risk associated to the system when component i is in state k. Similarly, let $F_{ik,jh}$ represents the risk associated with the system when component i is in state k and component j is in state h. Then the joint risk importance measures are,

$$\begin{aligned} MJrBI_{ij} = & \sum_{r=0}^{k_{i\alpha}} \sum_{m=0}^{k_{j\beta}} p_1^{**}{}_{rm} F_{ir,jm} - \sum_{r=0}^{k_{i\alpha}} \sum_{m=k_{j\beta}+1}^{M_j} p_2^{**}{}_{rm} F_{ir,jm} \\ & - \sum_{r=k_{i\alpha}+1}^{M_i} \sum_{m=0}^{k_{j\beta}} p_3^{**}{}_{rm} F_{ir,jm} + \sum_{r=k_{i\alpha}+1}^{M_i} \sum_{m=k_{j\beta}+1}^{M_j} p_4^{**}{}_{rm} F_{ir,jm}, \end{aligned} \tag{8.9}$$

$$MJrAW_{ij} = \frac{\sum_{r=0}^{k_{i\alpha}} \sum_{m=0}^{k_{j\beta}} p_1^{**}{}_{rm} F_{ir,jm}}{\sum_{r=0}^{k_{i\alpha}} p_{1ir}^{*} F_{ir} + \sum_{m=0}^{k_{j\beta}} p_{1\,jm}^{*} F_{jm}}, \tag{8.10}$$

$$MJrRW_{ij} = \frac{F_1}{F_2 - F_3 - F_4} \tag{8.11}$$

where

$$F_1 = \sum_{r=k_{i\alpha}+1}^{M_i} p_{2ir}^{*} F_{ir} + \sum_{m=k_{j\beta}+1}^{M_j} p_{2\,jm}^{*} F_{jm}$$

$$F_2 = \sum_{r=k_{i\alpha}+1}^{M_i} \sum_{m=k_{j\beta}+1}^{M_j} p_{4\ rm}^{**} F_{ir,\,jm}$$

$$F_3 = \sum_{r=0}^{k_{i\alpha}} \sum_{m=k_{j\beta}+1}^{M_j} p_{2\ rm}^{**} F_{ir,\,jm}$$

$$F_4 = \sum_{r=k_{i\alpha}+1}^{M_i} \sum_{m=0}^{k_{j\beta}} p_{3\ rm}^{**} F_{ir,\,jm}$$

$$MJrFV_{ij} = \frac{\sum_{r=k_{i\alpha}+1}^{M_i} p_{2ir}^{*} F_{ir} + \sum_{m=k_{j\beta}+1}^{M_j} p_{2\,jm}^{*} F_{jm} - \sum_{r=k_{i\alpha}+1}^{M_i} \sum_{m=k_{j\beta}+1}^{M_j} p_{4\ rm}^{**} F_{ir,\,jm}}{\sum_{r=k_{i\alpha}+1}^{M_i} p_{2ir}^{*} F_{ir} + \sum_{m=k_{j\beta}+1}^{M_j} p_{2\,jm}^{*} F_{jm}}. \tag{8.12}$$

The following recursive algorithm allows to compute the system OPM, see Lisnianski and Levitin [25].

1. Obtain the UGFs of all of the system components.
2. If the system contains a pair of components connected in parallel or in series, replace this pair with an equivalent macro-component with UGF.
3. If the system contains more than one component, return to step 2.
4. Determine the UGF of the entire series-parallel system as the UGF of the single equivalent macro-component. The system probability and performance distributions are represented by the coefficients and exponents of this UGF, corresponding to the state probabilities and performance, respectively.
5. Compute the system OPM by applying the with the given vectors of probability distribution and output performances.

In order to obtain the state-space restricted OPMs $OM_i^{\leq\alpha}$, $OM_i^{>\alpha}$, $OM_{ij}^{\leq\alpha,\leq\beta}$, $OM_{ij}^{>\alpha,\leq\beta}$, $OM_{ij}^{\leq\alpha,>\beta}$, and $OM_{ij}^{>\alpha,>\beta}$, one has to modify the UGF of components i and j as follows:

$$U_i^{\leq\alpha}(z) = \sum_{r=0}^{k_{i\alpha}} \frac{p_{ir}}{p_i^{\leq\alpha}} z^{x_{ir}}$$

for $OM_i^{\leq\alpha}$,

$$U_i^{>\alpha}(z) = \sum_{r=k_{i\alpha}+1}^{M_i} \frac{p_{ir}}{p_i^{>\alpha}} z^{x_{ir}}$$

for $OM_i^{>\alpha}$,

$$U_{i,j}^{\le\alpha,\le\beta}(z)=\sum_{r=0}^{k_{i\alpha}}\frac{p_{jr}}{p_i^{\le\alpha}}z^{x_{ir}}*\sum_{m=0}^{k_{j\beta}}\frac{p_{jm}}{p_j^{\le\beta}}z^{x_{jm}}=\sum_{r=0}^{k_{i\alpha}}\sum_{m=0}^{k_{j\beta}}\frac{p_{ir}p_{jm}}{p_i^{\le\alpha}p_j^{\le\beta}}z^{\omega\left(x_{ir},x_{jm}\right)}$$

for $OM_{ij}^{\le\alpha,\le\beta}$,

$$U_{i,j}^{>\alpha,\le\beta}(z)=\sum_{r=k_{i\alpha}+1}^{M_i}\frac{p_{ir}}{p_i^{>\alpha}}z^{x_{ir}}*\sum_{m=0}^{k_{j\beta}}\frac{p_{jm}}{p_j^{\le\beta}}z^{x_{jm}}=\sum_{r=k_{i\alpha}+1}^{M_i}\sum_{m=0}^{k_{j\beta}}\frac{p_{ir}p_{jm}}{p_i^{>\alpha}p_j^{\le\beta}}z^{\omega\left(x_{ir},x_{jm}\right)}$$

for $OM_{ij}^{>\alpha,\le\beta}$,

$$U_{i,j}^{\le\alpha,>\beta}(z)=\sum_{r=0}^{k_{i\alpha}}\frac{p_{ir}}{p_i^{\le\alpha}}z^{x_{ir}}*\sum_{m=k_{j\beta}+1}^{M_j}\frac{p_{jm}}{p_j^{>\beta}}z^{x_{jm}}=\sum_{r=0}^{k_{i\alpha}}\sum_{m=k_{j\beta}+1}^{M_j}\frac{p_{ir}p_{jm}}{p_i^{\le\alpha}p_j^{>\beta}}z^{\omega\left(x_{ir},x_{jm}\right)}$$

for $OM_{ij}^{\le\alpha,>\beta}$, and

$$U_{i,j}^{>\alpha,>\beta}(z)=\sum_{r=k_{i\alpha}+1}^{M_i}\frac{p_{ir}}{p_i^{>\alpha}}z^{x_{ir}}*\sum_{m=k_{j\beta}+1}^{M_j}\frac{p_{jm}}{p_j^{>\beta}}z^{x_{jm}}=\sum_{r=k_{i\alpha}+1}^{M_i}\sum_{m=k_{j\beta}+1}^{M_j}\frac{p_{ir}p_{jm}}{p_i^{\le\alpha}p_j^{>\beta}}z^{\omega\left(x_{ir},x_{jm}\right)}$$

for $OM_{ij}^{>\alpha,>\beta}$, then apply the algorithm given above. We use the coefficients of above UGFs for the evaluation of joint importance measures in Eqs. 6.3–6.6 using Eqs. 8.3–8.8 and, Eqs. 8.9–8.12.

Some network problems which can be modelled as MSS can be seen in Levitin [18, 19, 21, 22]. We consider the following network example.

9 Sliding window system

Consider a multistate multiple sliding window system (MSWS) with $n=5$, number of multistate components, see Levitin [22]. It generalizes the linear consecutive $k-out-of-r-from-n:F$ system consists of n linearly ordered multistate components. Each multistate component can have several different states: from complete failure up to perfect functioning. A performance rate is associated with each state. A set of integer numbers is defined such that any $r=3$, or $r=4$ corresponds to the number of consecutive multistate components (length of sliding window). For each r the function, $f_3(x_1,x_2,x_3)=\sum_{i=1}^{3}x_i-5$ and $f_4(x_1,x_2,x_3,x_4)=\sum_{i=1}^{4}x_i-6$, where x_i is the performance rate of ith component, (named the acceptability function), is defined in such a manner that $f_r<0$ constitutes the system failure. The MSWS fails if at least one of the functions f_r over the performance rates of any r consecutive components is negative. Each multistate component has a total

Table 1 Probability distributions of components

Multistate component i	$p_{i,0}$	$p_{i,1}$	$p_{i,2}$	$p_{i,3}$
1	.2	0	.8	0
2	.3	0	.7	0
3	.39	.61	0	0
4	.24	0	0	.76
5	.01	0	.99	0

Table 2 MSWS OPM-reliability with restriction to components performance, $\alpha = .8$, $\beta = .9$

Multistate components	OPM-reliability
$i = 1$	$OM_1^{\geq\alpha} = 2.p_{2,2}p_{3,2}p_{4,1}p_{5,2} = 0.4108104$
$i = 2$	$OM_2^{\geq\alpha} = p_{1,2}p_{3,2}p_{4,1}p_{5,2} + p_{3,2}p_{4,2}p_{5,2} = 0.6937128$
$i = 3$	$OM_3^{\geq\alpha} = p_{1,2}p_{2,2}p_{4,1}p_{5,2} + p_{2,2}p_{4,2}p_{5,2} = 0.760914$
$i = 4$	$OM_4^{\geq\alpha} = p_{2,2}p_{3,2}p_{5,2} = 0.52668$
$i = 5$	$OM_5^{\geq\alpha} = p_{1,2}p_{2,2}p_{3,2}p_{4,1} + p_{2,2}p_{3,2}p_{4,2} = 0.490504$
$i = 1$	$OM_1^{<\alpha} = p_{1,2}p_{2,2}p_{3,2}p_{5,2} = 0.421344$
$i = 1,\ j = 2$	$OM_{1,2}^{\geq\alpha,\geq\beta} = p_{3,2}p_{4,1}p_{5,2} + p_{3,2}p_{4,2}p_{5,2} = 0.897336$
$i = 2,\ j = 3$	$OM_{2,3}^{\geq\alpha,\geq\beta} = p_{1,2}p_{4,1}p_{5,2} + (p_{1,1} + p_{1,2})p_{4,2}p_{5,2} = 0.91278$
$i = 3,\ j = 4$	$OM_{3,4}^{\geq\alpha,\geq\beta} = (p_{1,1} + p_{1,2})p_{2,2}p_{3,2}p_{4,2}p_{5,2} = 0.3212748$
$i = 1,\ j = 2$	$OM_{1,2}^{<\alpha,\geq\beta} = p_{2,2}p_{3,2}p_{4,2}p_{5,2} = 0.3212748$
$i = 4,\ j = 5$	$OM_{4,5}^{<\alpha,\geq\beta} = p_{1,2}p_{2,2}p_{3,2} = 0.4256$
$i = 3,\ j = 4$	$OM_{3,4}^{\geq\alpha,<\beta} = p_{1,2}p_{2,2}p_{5,2} = 0.5544$

Table 3 MJRBI, MJRAW, MJRRW, MJRFV

Multistate components	MJRBI	MJRAW	MJRRW	MJRFV
$i = 1,\ j = 2$	0.5760612	0.5216	0	0
$i = 2,\ j = 3$	0.91278	0.8264	0	0
$i = 3,\ j = 4$	−0.2331252	0.2909	0	0
$i = 4,\ j = 5$	−0.1043252	0.2909	0	0

Table 4 MJrBI, MJrAW, MJrRW, MJrFV

Multistate components	MJrBI	MJrAW	MJrRW	MJrFV
$i = 1,\ j = 2$	−0.5760612	0	8.7224	0.8854
$i = 2,\ j = 3$	0.08722	0	6.2585	0.8401
$i = 3,\ j = 4$	0.2331252	0	1.04962	0.0473
$i = 4,\ j = 5$	0.1043252	0	1.4480	0.3094

failure (corresponding to performance rate 0) and functioning with nominal performance rates 2, 2, 3, 1, and 2, respectively. The UGF of the individual multistate components are

$$U_1(Z) = p_{1,1}Z^0 + p_{1,2}Z^2,$$
$$U_2(Z) = p_{2,1}Z^0 + p_{2,2}Z^2,$$
$$U_3(Z) = p_{3,1}Z^0 + p_{3,2}Z^3,$$
$$U_4(Z) = p_{4,1}Z^0 + p_{4,2}Z^1 \quad \text{and}$$
$$U_5(Z) = p_{5,1}Z^0 + p_{5,2}Z^2.$$

The UGF technique is used for evaluating the MSWS OPM, reliability R, with out restriction to components. The system reliability is

$$R = p_{1,2}p_{2,2}p_{3,2}p_{4,1}p_{5,2} + (p_{1,1} + p_{1,2})p_{2,2}p_{3,2}p_{4,2}p_{5,2} \tag{9.1}$$

We consider the following probability distribution of five multistate components in Table 1. The non-zero OPM with component restriction are computed in Table 2. The proposed joint importance measures for pairs $(i, i+1), i = 1, ..., 4$ are evaluated in Tables 3 and 4.

A numerical comparison can be made for pair of components using the size of the value of relevant measure with regard to their impact on system reliability and unreliability. It is also clear that the greatest joint importance is assigned to the pair $(2, 3)$ based on MRJBI and MJRAW. Again pair $(1, 2)$ have greatest MJrAW and MJrFV. This pairs needs more safety and redundancy operations.

In any statistical problem where the complexity involved, one has to get some simple evaluation method. We gave a method for evaluation of joint importance measures based on UGF. The method is illustrated in a network system (signal transmission system). The UGF has wide application in optimization problems.

10 Conclusion

The information about the interaction effect of two or more components in improving system performance can be drawn from the proposed joint importance measures in various different ways. Information about this type of interaction importance of components constituting a system, with respect to its safety, reliability, availability and risk, can be made useful in safety and redundancy operations. The degree of interactions between two or more components provide some guidelines to preference in safety operations to some groups of components. We cannot say one measure is better than the other, each of the measure has specific use, which will depends on the system engineers objective and use.

References

1. Abouammoh, A.M., Al-kadi.: On measures of importance for components in multi-state coherent system. Microelectron. Reliab. **31**(1), 109–122 (1991)
2. Armstrong, M.J.: Joint reliability importance of components. IEEE Trans. Reliab. **44**(3), 408–412 (1995)
3. Aven, T.: On performance measures for multi-state monotone systems. Reliab. Eng. Sys. Saf. **41**, 259–266 (1993)
4. Barlow, R.E., Proschan, F.: Statistical Theory of Reliability and Life Testing. Holt, Rinehart and Winston, New York (1975)
5. Barlow, R.E., Wu, A.S.: Coherent system with multi-state components. Math. Oper. Res. **3**, 275–281 (1978)
6. Birnbaum, Z.W.: On the importance of different components in a multicomponent system, In: Krishnaiah, P.R. (ed.) Multivariate Analysis II, pp. 591–592. Academic, New York (1969)
7. Bueno, V.C.: On the importance of components for multi-state monotone systems. Stat. Probab. Lett. **7**, 51–59 (1989)
8. Chacko, V.M., Manoharan, M.: Joint importance measures for the multi-state system. In: Verma, A.K., Kapur, A.K., Ghadge, S.G. (eds.) Advances in Performance and Safety of Complex Systems, pp. 308–314. Macmillan, New Delhi (2008)
9. Chacko, V.M.: Joint importance measures of the multi-state *k-out-of-r-from-n: F* system. J. Kerala Stat. Assoc. **19**, 19–30 (2008)
10. Chacko, V.M., Manoharan, M.: Joint criticality importance measures for components in MEMS devices. In: Proc. ICMEMS (2009)
11. Cheok, M.C., Parry, G.W., Sherry, R.R.: Use of importance measures in risk informed applications. Reliab. Eng. Sys. Saf. **60**, 213–226 (1998)
12. El-Neweihi, E., Proschan, F.: Degradable systems: a survey of multi-state reliability theory. Commun. Stat., Theory Methods **13**, 403–432 (1984)
13. El-Neweihi, E., Proschan, F., Sethuraman, J.: Multi-state coherent systems. J. Appl. Probab. **15**, 675–688 (1978)
14. Gandini, A.: Importance and sensitivity analysis in assessing system reliability. IEEE Trans. Reliab. **39**, 61–69 (1990)
15. Griffith, W.S.: Multi-state reliability models. J. Appl. Probab. **17**, 735–744 (1980)
16. Hong, J.S., Lie, C.H.: Joint reliability importance of two edges in an undirected network. IEEE Trans. Reliab. **47**, 97–101 (1993)
17. Hong, J.S., Koo, H.Y., Lie, C.H.: Joint reliability importance of *k-out-n* systems. Eur. J. Oper. Res. **42**, 539–547 (2002)
18. Levitin, G.: Optimal reliability enhancement for multi-state transmission networks with fixed transmission times. Reliab. Eng. Syst. Saf. **76**, 287–299 (2002)
19. Levitin, G.: Optimal allocation of multi-state elements in linear consecutively connected system with delays. Int. J. Reliab. Qual. Saf. Eng. **9**, 89–108 (2002)
20. Levitin, G.: Common supplay failures in linear multi-state sliding window system. Reliab. Eng. Syst. Saf. **82**, 55–62 (2003)
21. Levitin, G.: A universal generating function approach for analysis of multi-state systems with dependent elements. Reliab. Eng. Syst. Saf. **84**, 285–292 (2004)
22. Levitin, G.: Reliability of linear multi-state multiple sliding window system. Reliab. Eng. Syst. Saf. **85**, 385–395 (2004)
23. Levitin, G., Lisnianski, A.: Importance and sensitivity analysis of multi-state systems using universal generating functions method. Reliab. Eng. Syst. Saf. **65**, 271–282 (1999)
24. Levitin, G., Podofillini, L., Zio, E.: Generalized importance measures for multi-state elements based on performance level restrictions. Reliab. Eng. Syst. Saf. **82**, 287–298 (2003)
25. Lisnianski, A., Levitin, G.: Multi-state System Reliability-Assessment, Optimization and Applications. World Scientific, Singapore (2003)
26. Ramirez-Marquez, J.E, Coit, D.W.: Composite importance measures for multi-state systems with multi-state components. IEEE Trans. Reliab. **54**(3), 517–528 (2005)
27. Ushakov, I.: A universal generating function. Sov. J. Comput. Syst. Sci. **24**, 37–49 (1987)
28. Ushakov, I.: The method of generalized generating sequences. Eur. J. Oper. Res. **125**, 316–323 (2000)

29. Van der Borst, M., Shoonakker, H.: An overview of PSA importance measures. Reliab. Eng. Syst. Saf. **72**, 241–245 (2001)
30. Vasseur, D., Llory, M.: International survey on PSA figures of merit. Reliab. Eng. Syst. Saf. **66**, 261–274 (1999)
31. Wu, S.: Joint importance of multi-state systems. Comput. Ind. Eng. **49**, 63–75 (2005)
32. Wu, S., Chan, L.Y.: Performance utility-analysis of multi-state systems. IEEE Trans. Reliab. **50**(1), 14–20 (2005)
33. Zio, E., Podofillini, L.: Importance measures of multi-state components in multi-state systems. Int. J. Reliab. Qual. Saf. Eng. **10**(3), 289–310 (2003)
34. Zio, E., Podofillini, L.: Accounting for components interactions in the differential importance measure. Reliab. Eng. Syst. Saf. **91**, 1163–1174 (2006)