Acceptance sampling plan for Marshall-Olkin extended Fréchet distribution

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Applications of Marshall–Olkin Fréchet Distribution

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In this article, we consider the applications of Marshall–Olkin Fréchet distribution. The reliability of a system when both stress and strength follows the new distribution is discussed and related characteristics are computed for simulated data. The model is applied to a real data set on failure times of air-conditioning systems in jet planes and reliability is estimated. We also develop acceptance sampling plan for the acceptance of a lot whose lifetime follows this distribution. Four different autoregressive time series models of order 1 *are developed with minification structure as well as max-min structure having these stationary marginal distributions. Some properties of the models are also established.*

Keywords Acceptance sampling plan; Auto regressive models; Marshall–Olkin Fréchet distribution; Max-min process; Stress-strength analysis; Time series modeling.

Mathematics Subject Classification Primary 60E05; Secondary 62P99.

1. Introduction

The procedure of expanding a family of distributions for added flexibility or to construct covariate models is a well-known technique in the literature. Marshall and Olkin (1997) introduced a new method of adding a parameter into a family of distributions. According to them if $\overline{F}(x)$ denote the survival or reliability function of a continuous random variable X then the method of adding a new parameter results in another survival function $\overline{G}(x)$ given by

$$\overline{G}(x) = \frac{\alpha\overline{F}(x)}{1-\bar{\alpha}\overline{F}(x)}, \quad x \in \mathbb{R}, \quad \alpha > 0, \quad \bar{\alpha} = 1-\alpha. \tag{1.1}$$

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If $g(x)$ is the probability density function (pdf) corresponding to $\overline{G}(x)$, then

$$g(x, \alpha) = \frac{\alpha f(x)}{[1 - \bar{\alpha}\overline{F}(x)]^2}, \quad x \in \mathbb{R}, \quad \alpha > 0. \tag{1.2}$$

Krishna et al. (in press) introduced Marshall–Olkin Fréchet distribution with survival function

$$\overline{G}(x) = \frac{\alpha[1 - e^{-(\delta/x)^{\beta}}]}{\alpha + (1-\alpha)e^{-(\delta/x)^{\beta}}}, \quad x, \delta, \beta, \quad \alpha > 0 \tag{1.3}$$

and studied various properties including estimation of parameters. As a continuation, in this article we discuss the application of the newly developed distribution in reliability contexts, acceptance sampling, and time series analysis. In Sec. 2, we develop stress-strength analysis with respect to a simulated data as well as for a real data. In Sec. 3, we develop a sampling plan for the rejection or acceptance of a lot and the minimum sample size values are computed. In Sec. 4, we develop four types of AR(1) models and derive some properties of these models. Conclusions are given in Sec. 5.

2. Stress-Strength Analysis

In this section, we consider the statistical inference of the stress-strength parameter $R = P(X < Y)$ when X and Y are independent Marshall–Olkin Fréchet random variables. Here, Y represents random strength and X represents the random stress. The system fails if stress exceeds the strength. Thus, this quantity is the reliability of the system. This measure of reliability is widely used in engineering problems. It may be noted that R has more interest than just a reliability measure. It can be used as a general measure of difference between two populations such as treatment group and control group in bio-statistical contexts and clinical trials. Let X and Y be two independent random variables following Marshall–Olkin Fréchet distributions with parameters $(\delta, \beta, \alpha_1)$ and $(\delta, \beta, \alpha_2)$, respectively. Then using Gupta et al. (2010), we obtain

$$\begin{aligned} P(X < Y) &= \int_{-\infty}^{\infty} P(Y > X/X = x) g_X(x) dx \\ &= \int_0^{\infty} \frac{\alpha_2(1 - e^{-(\frac{\delta}{x})^{\beta}})}{\alpha_2 + (1-\alpha_2)e^{-(\frac{\delta}{x})^{\beta}}} \frac{\alpha_1 \beta \delta^{\beta} e^{-(\frac{\delta}{x})^{\beta}}}{x^{\beta+1}(\alpha_1 + (1-\alpha_1)e^{-(\frac{\delta}{x})^{\beta}})^2} dx \\ &= \frac{\alpha}{(\alpha-1)^2}\left[-\log\alpha + \alpha - 1\right], \end{aligned}$$

where $\alpha = \alpha_2/\alpha_1$. Hence, to estimate R it is enough to estimate α_1 and α_2 because R is a function of α_1 and α_2 only.

Now we consider the pdf of the Marshall–Olkin Fréchet distribution given by:

$$g(x; \alpha, \delta, \beta) = \frac{\alpha \beta \delta^{\beta} e^{-(\frac{\delta}{x})^{\beta}}}{x^{\beta+1}(\alpha + (1-\alpha)e^{-(\frac{\delta}{x})^{\beta}})^2}, \quad x, \alpha, \delta, \quad \beta > 0.$$

Let $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ be two independent random samples of sizes m and n taken from Marshall–Olkin Fréchet distributions with tilt parameters α_1 and α_2, respectively, and common unknown parameters δ and β. The log likelihood function is given by

$$\begin{aligned}
L(\alpha_1, \alpha_2, \delta, \beta) &= \sum_{i=1}^{m} \log g(x_i; \alpha_1, \delta, \beta) + \sum_{i=1}^{n} \log g(y_i; \alpha_2, \delta, \beta) \\
&= m \log \alpha_1 + n \log \alpha_2 + (m+n) \log \beta + (m+n)\beta \log \delta - \sum_{i=1}^{m} (\delta/x_i)^{\beta} \\
&\quad - \sum_{j=1}^{m} (\delta/y_j)^{\beta} - (\beta+1) \sum_{i=1}^{m} \log x_i - (\beta+1) \sum_{j=1}^{n} \log y_j \\
&\quad - 2 \sum_{i=1}^{m} \log(\alpha_1 + (1-\alpha_1) e^{-(\delta/x_i)^{\beta}}) - 2 \sum_{j=1}^{n} \log(\alpha_2 + (1-\alpha_2) e^{-(\delta/y_j)^{\beta}}).
\end{aligned}$$

The maximum likelihood estimates (mle) of the unknown parameters α_1, α_2, are the solutions of the nonlinear equations

$$\frac{\partial L}{\partial \alpha_1} = \frac{m}{\alpha_1} - 2 \sum_{i=1}^{m} \frac{1 - e^{-(\frac{\delta}{x_i})^{\beta}}}{\alpha_1 + (1-\alpha_1) e^{-(\frac{\delta}{x_i})^{\beta}}} = 0,$$

$$\frac{\partial L}{\partial \alpha_2} = \frac{n}{\alpha_2} - 2 \sum_{j=1}^{n} \frac{1 - e^{-(\frac{\delta}{y_j})^{\beta}}}{\alpha_2 + (1-\alpha_2) e^{-(\frac{\delta}{y_j})^{\beta}}} = 0.$$

The elements of information matrix are

$$\begin{aligned}
I_{11} &= -E\left(\frac{\partial^2 L}{\partial \alpha_1^2}\right) \\
&= \frac{m}{\alpha_1^2} - 2mE\left(\frac{(1 - e^{-(\frac{\delta}{X})^{\beta}})^2}{[1 - \bar{\alpha}_1(1 - e^{-(\frac{\delta}{X})^{\beta}})]^2}\right) \\
&= \frac{m}{\alpha_1^2} - 2m \int_0^{\infty} \frac{(1 - e^{-(\frac{\delta}{x})^{\beta}})^2 \alpha_1 \beta (\frac{\delta}{x})^{\beta+1} e^{-(\frac{\delta}{x})^{\beta}} dx}{\delta(1 - \bar{\alpha}_1(1 - e^{-(\frac{\delta}{x})^{\beta}}))^4} \\
&= \frac{m}{\alpha_1^2} - 2m\alpha_1 \int_0^1 \frac{t^2}{(1 - \bar{\alpha} t)^4} dt \\
&= m\left(\frac{1}{\alpha_1^2} - \frac{2}{3\alpha_1^2}\right) \\
&= \frac{m}{3\alpha_1^2}.
\end{aligned}$$

Similarly,

$$I_{22} = -E\left(\frac{\partial^2 L}{\partial \alpha_2^2}\right) = \frac{n}{3\alpha_2^2}$$

$$I_{12} = I_{21} = -E\left(\frac{\partial^2 L}{\partial \alpha_1 \partial \alpha_2}\right) = 0.$$

By the property of mle for $m \to \infty$, $n \to \infty$, we obtain that

$$(\sqrt{m}(\hat{\alpha}_1 - \alpha_1), \sqrt{n}(\hat{\alpha}_2 - \alpha_2))^T \xrightarrow{d} N_2\left(\mathbf{0}, \mathrm{diag}\{a_{11}^{-1}, a_{22}^{-1}\}\right),$$

where $a_{11} = \lim_{m,n\to\infty} \frac{1}{m} I_{11} = \frac{1}{3\alpha_1^2}$ and $a_{22} = \lim_{m,n\to\infty} \frac{1}{n} I_{22} = \frac{1}{3\alpha_2^2}$. The 95% confidence interval for R is given by

$$\widehat{R} \mp 1.96\ \hat{\alpha}_1 b_1(\hat{\alpha}_1, \hat{\alpha}_2)\sqrt{\frac{3}{m} + \frac{3}{n}},$$

where $\widehat{R} = R(\hat{\alpha}_1, \hat{\alpha}_2)$ is the estimator of R and

$$b_1(\alpha_1, \alpha_2) = \frac{\partial R}{\partial \alpha_1} = \frac{\alpha_2}{(\alpha_1 - \alpha_2)^3}\left[2(\alpha_1 - \alpha_2) + (\alpha_1 + \alpha_2)\log\frac{\alpha_2}{\alpha_1}\right].$$

2.1. *Simulation Study*

We generate $N = 10,000$ sets of X-samples and Y-samples from Marshall–Olkin Fréchet distribution with parameters α_1, δ, β and α_2, δ, β, respectively. The combinations of samples of sizes $m = 20, 25, 30$ and $n = 20, 25, 30$ are considered. The estimates of α_1 and α_2 are then obtained from each sample to obtain $\widehat{R}$. The validity of the estimate of R is discussed by the following measures.

1. Average bias of the simulated N estimates of R:

$$\frac{1}{N}\sum_{i=1}^{N}(\widehat{R}_i - R)$$

2. Average mean square error of the simulated N estimates of R:

$$\frac{1}{N}\sum_{i=1}^{N}(\widehat{R}_i - R)^2$$

3. Average length of the asymptotic 95% confidence intervals of R:

$$\frac{1}{N}\sum_{i=1}^{N} 2(1.96)\hat{\alpha}_{1i} b_{1i}(\hat{\alpha}\alpha_{1i}, \hat{\alpha}\alpha_{2i})\sqrt{\frac{3}{m} + \frac{3}{n}}$$

4. The coverage probability of the N simulated confidence intervals given by the proportion of such interval that include the parameter R.

The numerical values obtained for the measures listed above are presented in Tables 1 and 2. For $\alpha_1 < \alpha_2$ the average bias is positive and for $\alpha_1 > \alpha_2$ the average bias is negative but in both cases the average bias decreases as the sample size increases. Performance of confidence interval is quite good. The coverage probability is close to 0.95 and approaches to the nominal value as the sample size increases. The simulation study indicates that the average bias, average MSE, average confidence interval and coverage probability do not show much variability for various parameter combinations.

Table 1
Average bias and average MSE of the simulated estimates of R for $\delta = 3$ and $\beta = 2$

	(α_1, α_2)							
	Average bias ($\bar{b}$)				Average Mean Square Error (AMSE)			
(m, n)	(0.5,0.8)	(0.8,1.2)	(0.8,0.5)	(1.2,0.8)	(0.5,0.8)	(0.8,1.2)	(0.8,0.5)	(1.2,0.8)
(20,20)	0.0833	0.0739	−0.0830	−0.0732	0.0083	0.0071	0.0084	0.0070
(20,25)	0.0830	0.0736	−0.0836	−0.0742	0.0085	0.0071	0.0085	0.0072
(20,30)	0.0820	0.0740	−0.0833	−0.0736	0.0084	0.0072	0.0084	0.0071
(25,20)	0.0851	0.0763	−0.0833	−0.0717	0.0086	0.0072	0.0079	0.0066
(25,25)	0.0844	0.0755	−0.0814	−0.0721	0.0085	0.0071	0.0079	0.0067
(25,30)	0.0846	0.0752	−0.0809	−0.0720	0.0085	0.0071	0.0079	0.0067
(30,20)	0.0862	0.0763	−0.0798	−0.0714	0.0087	0.0072	0.0076	0.0065
(30,25)	0.0859	0.0764	−0.7999	−0.0706	0.0087	0.0072	0.0076	0.0063
(30,30)	0.0852	0.0762	−0.0801	−0.0710	0.0085	0.0072	0.0077	0.0064

Let us consider now the data from Gupta et al. (2010). We consider two data sets which represents the times (in hours) of successive failure intervals of the air conditioning system of two jet planes. The data set for X is 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95. The data set for Y is 487, 18, 100, 7, 98, 5, 85, 91, 43, 230, 3, 130. First we consider the Fréchet distribution with parameters δ and β and estimate the unknown parameters by considering each data set separately. For the first data set we obtain the estimates $\hat{\delta} = 14.616$ and $\hat{\beta} = 0.724$ with the estimated log-likelihood as -155.1144. For the second data set we obtain the estimates $\hat{\delta} = 20.796$ and $\hat{\beta} = 0.656$ with the estimated log-likelihood as -69.2517.

Table 2
Average confidence length and coverage probability of the simulated 95 percentage confidence intervals of R for $\delta = 3$ and $\beta = 2$

	(α_1, α_2)							
	Average confidence length				Coverage probability			
(m, n)	(0.5,0.8)	(0.8,1.2)	(0.8,0.5)	(1.2,0.8)	(0.5,0.8)	(0.8,1.2)	(0.8,0.5)	(1.2,0.8)
(20,20)	0.3559	0.3557	0.3558	0.3556	0.989	0.9912	0.9894	0.9925
(20,25)	0.3376	0.334	0.3376	0.3373	0.9797	0.9848	0.9817	0.9865
(20,30)	0.3248	0.3246	0.3215	0.346	0.9755	0.979	0.9763	0.9801
(25,20)	0.3377	0.3376	0.338	0.3348	0.9816	0.9871	0.9889	0.9911
(25,25)	0.3183	0.3182	0.3185	0.3183	0.9715	0.9791	0.9799	0.9856
(25,30)	0.3049	0.3047	0.305	0.3048	0.9574	0.9718	0.9724	0.9774
(30,20)	0.3248	0.3249	0.3252	0.325	0.9789	0.9832	0.988	0.9916
(30,25)	0.305	0.3048	0.3055	0.3049	0.9625	0.9724	0.9799	0.9844
(30,30)	0.2908	0.2906	0.2909	0.2907	0.9512	0.9624	0.9665	0.9767

Table 3
χ^2 and p-values for the data set of successive failure times of the air conditioning system of two jet planes

	χ^2 value		p-Value	
Distribution	Plane-1	Plane-2	Plane-1	Plane-2
Lomax	1.0232	1.0194	0.5995	0.6007
Fréchet	0.5098	0.1405	0.7750	0.9322

Now we consider the values $\hat{\delta} = (14.616 + 20.796)/2 = 17.706$ and $\hat{\beta} = (0.724 + 0.656)/2 = 0.690$. We test the null hypotheses that the true values are $\delta = 17.706$ and $\beta = 0.690$. The log-likelihood for these values and the first data set is -155.3693, which implies that the chi-square statistic and the p-value of the likelihood ratio test are 0.5098 and 0.7750. For the second data set, the chi-square statistic and the p-value of the likelihood ratio test are respectively 0.1405 and 0.9322. We can conclude that we can accept the null hypotheses that the true values are $\delta = 17.706$ and $\beta = 0.690$.

Table 3 gives a comparison between the Fréchet model and the Lomax model given in Gupta et al. (2010). It is clear that the Fréchet model is a better fit than the other.

Now we derive the estimates of the parameters α_1 and α_2 by considering the Marshall–Olkin Fréchet distribution. We obtain $\hat{\alpha}_1 = 0.9501$ and $\hat{\alpha}_2 = 1.1241$, which implies that the estimate of R is $\widehat{R} = 0.5280$ with standard error $SE(\widehat{R}) = 0.0983$. The asymptotic 95% confidence interval of R is (0.3353, 0.7207).

3. Reliability Test Plan

In statistical quality control acceptance sampling plan is concerned with the inspection of a sample of products taken from a lot and decision whether to accept or not to accept the lot based on the quality of product. Kantam et al. (2001) and Srinivasa Rao et al. (2009) discussed the acceptance sampling plan for log-logistic model and Marshall–Olkin extended exponential distribution. Let the quality of the product inspected is the lifetime of the product. In this section, we discuss the reliability test plan for accepting or rejecting a lot where the life time of the product follows Marshall–Olkin Fréchet distribution. In a life testing experiment the procedure is to terminate the test by a pre-determined time "t" and note the number of failures. If the number of failures at the end of time "t" does not exceed a given number "c", called acceptance number then we accept the lot with a given probability of at least p^*. But if the number of failures exceeds "c" before time "t" then the test is terminated and the lot is rejected. For such truncated life test and the associated decision rule we are interested in obtaining the smallest sample size to arrive at a decision. For Marshall–Olkin Fréchet distribution with probability of failure,

$$G(x, \alpha, \beta, \delta) = \frac{e^{-(\frac{\delta}{x})^{\beta}}}{\alpha + \bar{\alpha} e^{-(\frac{\delta}{x})^{\beta}}}, \quad x, \alpha, \beta, \delta > 0, \tag{3.1}$$

the average life time depends only on δ if α and β are known. Let δ_0 be the required minimum average life time. Then

$$G(x, \alpha, \beta, \delta) \leq G(x, \alpha, \beta, \delta_0) \Leftrightarrow \delta \geq \delta_0.$$

A sampling plan is specified by the following quantities:

1. the number of units n on test;
2. the acceptance number c;
3. the maximum test duration t; and
4. the minimum average lifetime represented by δ_0.

The consumers risk, i.e., the probability of accepting a bad lot should not exceed the value $1 - p^*$, where p^* is a lower bound for the probability that a lot of true value δ below δ_0 is rejected by the sampling plan. For fixed p^* the sampling plan is characterized by $(n, c, \delta_0/t)$. By sufficiently large lots we can apply binomial distribution to find acceptance probability. The problem is to determine the smallest positive integer "n" for given value of c and δ/t_0 such that

$$L(p_0) = \sum_{i=0}^{c} \binom{n}{i} p_0^i (1 - p_0)^{n-i} \leq 1 - p^*, \tag{3.2}$$

where $p_0 = G(t, \alpha, \beta, \delta_0)$. The function $L(p)$ is called operating characteristic function of the sampling plan, i.e., the acceptance probability of the lot as a function of the failure probability $p(\delta) = G(t, \alpha, \beta, \delta)$. The average life time of the product is increasing with δ and therefore the failure probability $p(\delta)$ decreases implying that the operating characteristic function is increasing in δ. The minimum values of n satisfying (3.2) are obtained for $\alpha = 2$, $\beta = 2$ and $p* = 0.95$ and 0.99 and for the ratio $\delta_0/t = 0.562, 0.762, 0.953, 1.212$. The results are displayed in Table 4. If $p_0 = G(t, \alpha, \beta, \delta_0)$ is small and n is large, the binomial probability may be approximated by Poisson probability with parameter $\lambda = np_0$ so that (3.2) becomes

$$L_1(p_0) = \sum_{i=0}^{c} \frac{\lambda^i}{i!} e^{-\lambda} \leq 1 - p^*. \tag{3.3}$$

The minimum values of n satisfying (3.3) are obtained for the same combination of values of α, β, and δ_0/t for various values of p^* are presented in Table 5.

4. Application in Time Series Modeling

Time series modeling is finding its application in diversified fields today: economics, social sciences, demography, medical sciences, and actuarial science are just a few of them. Warming trend in global temperature, and levels of pollution causing mortality in a particular region are other major areas in present scenario where time series modeling is found effective. Gaver and Lewis (1980) developed a first-order autoregressive time series model with exponential stationary marginal distribution. They extended it to the case of gamma and mixed exponential processes. Jayakumar and Pillai (1993) extended it to the case of Mittag–Leffler processes. Several authors

Table 4
Minimum sample size for the specified ratio δ_0/t, confidence level p^*, acceptance number c, $\alpha = 2$ and $\beta = 2$ using binomial approximation

		δ_0/t						δ_0/t			
p^*	c	1.212	0.953	0.762	0.562	p^*	c	1.212	0.953	0.762	0.562
0.95	0	22	11	5	4	0.99	0	34	17	10	6
0.95	1	35	18	11	7	0.99	1	49	24	15	9
0.95	2	47	23	14	9	0.99	2	62	31	19	11
0.95	3	58	29	18	11	0.99	3	74	37	23	14
0.95	4	68	34	21	13	0.99	4	86	43	26	16
0.95	5	79	40	25	16	0.99	5	97	49	30	19
0.95	6	89	45	28	18	0.99	6	108	54	33	21
0.95	7	99	50	31	20	0.99	7	119	60	37	23
0.95	8	108	55	34	22	0.99	8	130	65	40	25
0.95	9	118	60	37	24	0.99	9	140	70	44	28
0.95	10	128	65	40	26	0.99	10	150	76	47	30

have developed similar processes with other non Gaussian marginals like Weibull, Laplace, Linnik, etc. Brown et al. (1984), Gibson (1986), Anderson and Arnold (1993), Alice and Jose (2001, 2004), and Naik and Jose (2008) are some of the researchers who worked on this topic. In this context, we are discussing various autoregressive models of order 1 with Marshall–Olkin Fréchet distribution as marginals, namely MIN AR(1) models I and II and MAX-MIN AR(1) models I and II and explore some properties.

Table 5
Minimum sample size for the specified ratio δ_0/t, confidence level p^*, acceptance number c, $\alpha = 2$ and $\beta = 2$ using Poisson approximation.

		δ_0/t						δ_0/t			
p^*	c	1.212	0.953	0.762	0.562	p^*	c	1.212	0.953	0.762	0.562
0.95	0	24	12	8	6	0.99	0	36	19	12	9
0.95	1	37	19	13	10	0.99	1	52	27	18	13
0.95	2	49	26	17	13	0.99	2	65	34	24	17
0.95	3	60	32	20	16	0.99	3	78	41	29	20
0.95	4	71	37	24	18	0.99	4	90	47	33	23
0.95	5	81	43	28	21	0.99	5	101	53	37	26
0.95	6	92	48	31	24	0.99	6	113	59	41	29
0.95	7	102	53	34	26	0.99	7	124	65	42	32
0.95	8	112	58	38	29	0.99	8	134	70	45	34
0.95	9	121	63	41	31	0.99	9	145	76	49	37
0.95	10	131	68	44	34	0.99	10	155	81	52	40

4.1. *MIN AR(1) Model-I with Marshall–Olkin Fréchet Marginal Distribution*

Consider an AR(1) structure given by

$$X_n = \begin{cases} \varepsilon_n & \text{with probability } p \\ \min(X_{n-1}, \varepsilon_n) & \text{with probability } 1-p, \end{cases} \tag{4.1}$$

where $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables independent of $\{X_n\}$ and $p \in (0, 1)$. Then the process is stationary Markovian with Marshall–Olkin distribution as marginal. Thus we have the following theorem.

Theorem 4.1. *In an AR*(1) *process with structure* (4.1), $\{X_n\}$ *is stationary Markovian with Marshall–Olkin Fréchet distribution with parameters p, δ, and β if and only if* $\{\varepsilon_n\}$ *is distributed as Fréchet distribution with parameters δ and β.*

Proof. To prove sufficiency we assume that ε_n follows Fréchet distribution with parameters δ and β. From (4.1) it follows that

$$\overline{F}_{X_n}(x) = p\overline{F}_{\varepsilon_n}(x) + (1-p)\overline{F}_{X_{n-1}}(x)\overline{F}_{\varepsilon_n}(x). \tag{4.2}$$

Under stationarity equilibrium, this gives

$$\overline{F}_X(x) = \frac{p\overline{F}_\varepsilon(x)}{1-(1-p)\overline{F}_\varepsilon(x)},$$

which is of the Marshall–Olkin form. Similarly to establish the necessary part, let us assume that X_n follows Marshall–Olkin Fréchet distribution with parameters p, δ, and β. From (4.2) under stationarity, we have

$$\overline{F}_\varepsilon(x) = \frac{\overline{F}_X(x)}{p + (1-p)\overline{F}_X(x)}.$$

On simplification we get $\overline{F}_\varepsilon(x) = 1 - e^{-(\frac{\delta}{x})^\beta}$, which is the survival function of Fréchet distribution with parameters δ and β. Let us first consider the joint survival function of random variables X_{n+k} and X_n, $k \geq 1$. We have

$$\begin{aligned} S_k(x, y) &\equiv P(X_{n+k} > x, X_n > y) \\ &= p\overline{F}_\varepsilon(x)\overline{F}_X(y) + (1-p)\overline{F}_\varepsilon(x)S_{k-1}(x, y) \\ &= p\overline{F}_\varepsilon(x)\overline{F}_X(y)\sum_{j=0}^{k-1}(1-p)^j\overline{F}_\varepsilon^j(x) + (1-p)^k\overline{F}_\varepsilon^k(x)S_0(x, y) \\ &= p\overline{F}_\varepsilon(x)\overline{F}_X(y)\frac{1-(1-p)^k\overline{F}_\varepsilon^k(x)}{1-(1-p)\overline{F}_\varepsilon(x)} + (1-p)^k\overline{F}_\varepsilon^k(x)S_0(x, y), \end{aligned}$$

where

$$S_0(x, y) = P(X_n > \max(x, y)) = \begin{cases} \overline{F}_X(x), & x \geq y, \\ \overline{F}_X(y), & x < y. \end{cases}$$

Letting $k \to \infty$, we get

$$S_\infty(x, y) = \frac{p\overline{F}_\varepsilon(x)\overline{F}_X(y)}{1-(1-p)\overline{F}_\varepsilon(x)},$$

i.e., we can see that the joint survival function of random variables X_{n+k} and X_n can be represented as a product of two survival function of random variables with parameters p, δ, and β. Now we will show that the joint survival function of random variables X_{n+k} and X_n is not a continuous function so that the probability $P(X_{n+k} = X_n)$ is positive. We have

$$\begin{aligned} P(X_{n+k} = X_n) &= (1-p)P(X_{n+k-1} = X_n, X_{n+k-1} < \varepsilon_{n+k}) \\ &= (1-p)^2 P(X_{n+k-2} = X_n, X_{n+k-2} < \varepsilon_{n+k-1}, X_{n+k-2} < \varepsilon_{n+k}) \\ &= (1-p)^k P(X_n < \min(\varepsilon_{n+1}, \ldots, \varepsilon_{n+k-1}, \varepsilon_{n+k})). \end{aligned} \tag{4.3}$$

Now, since random variables ε_{n+i}, $i = 1, 2, \ldots, k$, have the survival function $\overline{F}_\varepsilon(x)$, it follows that a random variable $\min(\varepsilon_{n+1}, \ldots, \varepsilon_{n+k-1})$ has the survival function $\overline{F}^k_\varepsilon(x)$. Using this, we obtain

$$\begin{aligned} P(X_n < \min(\varepsilon_{n+i}, i = 1, 2, \ldots, k)) &= \int_0^\infty \overline{F}^k_\varepsilon(x) f_X(x)dx \\ &= p\int_0^\infty \overline{F}^k_\varepsilon(x)\frac{f_\varepsilon(x)}{(1-(1-p)\overline{F}_\varepsilon(x))^2}dx \\ &= 1 - \frac{pk}{k+1}\,{}_2F_1(1, 1+k; 2+k; 1-p). \end{aligned} \tag{4.4}$$

Finally, replacing (4.4) in (4.3), we obtain the probability $P(X_{n+k} = X_n)$ is positive. Now we will derive the probability of the event $\{X_{n+k} > X_n\}$, $k \geq 1$. We have

$$\begin{aligned} P(X_{n+k} > X_n) &= pP(\varepsilon_{n+k} > X_n) + (1-p)P(\min(X_{n+k-1}, \varepsilon_{n+k}) > X_n) \\ &= p\sum_{j=0}^{k-1}(1-p)^j P(\min(\varepsilon_{n+k-j}, \ldots, \varepsilon_{n+k}) > X_n), \end{aligned}$$

since the probability of the event $\{\min(X_n, \varepsilon_{n+k-j}, \ldots, \varepsilon_{n+k}) > X_n\}$ is 0. Using (4.4) we obtain

$$P(X_{n+k} > X_n) = p\sum_{j=0}^{k-1}(1-p)^j\left(1 - \frac{p(j+1)}{j+2}\,{}_2F_1(1, j+2; j+3; 1-p)\right).$$

For $k = 1$, we have

$$P(X_{n+1} > X_n) = p\left(1 - \frac{p}{2}\,{}_2F_1(1, 2; 3; 1-p)\right) = \frac{p(1-p+p\log p)}{(1-p)^2}.$$

This probability is an increasing function on p. Also, we can see that it takes values from (0, 1/2). Thus we can conclude that as p increases that we can observed more down runs of the process $\{X_n\}$.

4.2. *MIN AR(1) Model-II with Marshall–Olkin Fréchet Marginal Distribution*

Now we discuss a more general structure which allows probabilistic selection of process values, innovations and combinations of both. Consider an AR(1) structure given by

$$X_n = \begin{cases} X_{n-1} & \text{with probability } p_1 \\ \varepsilon_n & \text{with probability } p_2 \\ \min(X_{n-1}, \varepsilon_n) & \text{with probability } 1 - p_1 - p_2, \end{cases} \tag{4.5}$$

where $p_1,\ p_2,\ p_3 > 0,\ p_1 + p_2 < 1$ and $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables independent of $\{X_n\}$. Then the process is stationary Markovian with Marshall–Olkin distribution as marginal.

Theorem 4.2. *In an AR*(1) *process with structure* (4.5), $\{X_n\}$ *is stationary Markovian with Marshall–Olkin Fréchet distribution with parameters* q, δ, β *if and only if* $\{\varepsilon_n\}$ *is distributed as Fréchet distribution with parameters* δ *and* β, *where* $q = \frac{p_2}{1-p_1}$.

Proof. Similar to the proof of Theorem 4.1.

Let us first consider the joint survival function of random variables X_{n+k} and X_n, $k \geq 1$. We have:

$$\begin{aligned} S_k(x, y) &= p_2\overline{F}_\varepsilon(x)\overline{F}_X(y) + [p_1 + (1 - p_1 - p_2)\overline{F}_\varepsilon(x)]S_{k-1}(x, y) \\ &= p_2\overline{F}_\varepsilon(x)\overline{F}_X(y)\sum_{j=0}^{k-1}[p_1 + (1 - p_1 - p_2)\overline{F}_\varepsilon(x)]^j \\ &\quad + [p_1 + (1 - p_1 - p_2)\overline{F}_\varepsilon(x)]^k S_0(x, y) \\ &= p_2\overline{F}_\varepsilon(x)\overline{F}_X(y)\frac{1 - [p_1 + (1 - p_1 - p_2)\overline{F}_\varepsilon(x)]^k}{1 - p_1 - (1 - p_1 - p_2)\overline{F}_\varepsilon(x)} \\ &\quad + [p_1 + (1 - p_1 - p_2)\overline{F}_\varepsilon(x)]^k S_0(x, y). \end{aligned}$$

As in the case when $p_1 = 0$, letting $k \to \infty$, we can see that the joint survival function of random variables X_{n+k} and X_n can be represented as a product of two survival function of random variables with parameters q, δ, and β. Let us consider now the probability $P(X_{n+k} = X_n)$. To simplify the derivations, we will denote by $A^j_{i_1,\ldots,i_r}$ the event $\{X_{n+j} = X_n, X_{n+j} < \min(\varepsilon_{n+i_1}, \ldots, \varepsilon_{n+i_r})\}$. We have:

$$\begin{aligned} P(X_{n+k} = X_n) &= p_1 P(X_{n+k-1} = X_n) + (1 - p_1 - p_2)P(A^{k-1}_k) \\ &= p_1^2 P(X_{n+k-2} = X_n) + p_1(1 - p_1 - p_2)P(A^{k-2}_{k-1}) \\ &\quad + p_1(1 - p_1 - p_2)P(A^{k-2}_k) + (1 - p_1 - p_2)^2 P(A^{k-2}_{k-1,k}) \\ &= p_1^k + p_1^{k-1}(1 - p_1 - p_2)\sum_{i_1=1}^{k} P(A^0_{i_1}) \\ &\quad + p_1^{k-2}(1 - p_1 - p_2)^2\sum_{i_1<i_2} P(A^0_{i_1,i_2}) + \ldots \\ &\quad + p_1(1 - p_1 - p_2)^{k-1}\sum_{i_1<\cdots<i_{k-1}} P(A^0_{i_1,\ldots,i_{k-1}}) \\ &\quad + (1 - p_1 - p_2)^k P(A^0_{i_1,\ldots,i_k}). \end{aligned} \tag{4.6}$$

From (4.4) we have

$$P(A^0_{i_1,\ldots,i_r}) = 1 - \frac{qr}{1+r}{}_2F_1(1, 1+r; 2+r; 1-q).$$

Replacing this in (4.6), we obtain the probability of the event $\{X_{n+k} = X_n\}$ is

$$P(X_{n+k} = X_n) = \sum_{j=0}^{k} p_1^j (1-p_1-p_2)^{k-j} \binom{k}{j} \left[1 - \frac{qj}{1+j}{}_2F_1(1, 1+j; 2+j; 1-q)\right].$$

Now we will derive the probability of the event $\{X_{n+1} > X_n\}$. From the definition of the process and (4.4), we have

$$P(X_{n+1} > X_n) = p_2 P(\varepsilon_{n+1} > X_n) = \frac{p_2(1-q+q\log q)}{(1-q)^2}. \qquad \square$$

4.3. *MAX-MIN AR(1) Model-I with Marshall–Olkin Fréchet Marginal Distribution*

Consider the AR(1) structure given by

$$X_n = \begin{cases} \max(X_{n-1}, \varepsilon_n) & \text{with probability } p_1 \\ \min(X_{n-1}, \varepsilon_n) & \text{with probability } p_2 \\ X_{n-1} & \text{with probability } 1-p_1-p_2, \end{cases} \tag{4.7}$$

where $0 < p_1, p_2 < 1$, $p_2 < p_1$, $p_1 + p_2 < 1$ and $\{\varepsilon_n\}$ is a sequence of i.i.d. random variables independently distributed of X_n. Then the process is stationary Markovian with Marshall-Olkin distribution as marginal.

Theorem 4.3. *In AR(1) Max-Min process with structure (4.7), $\{X_n\}$ is a stationary Markovian AR(1) Max-Min process with Marshall–Olkin Fréchet distribution with parameters q, δ, and β if and only if $\{\varepsilon_n\}$ follows Fréchet distribution with parameters δ and β, where $q = \frac{p_1}{p_2}$.*

Proof. Similar to the proof of Theorem 4.1.

In many situations of practical interest is the probability of the event $\{X_{n+1} > X_n\}$. After some calculations, we can show that

$$P(X_{n+1} > X_n) = p_1 P(\varepsilon_{n+1} > X_n) = \frac{p_1(1-q+q\log q)}{(1-q)^2}.$$

4.4. *MAX-MIN AR(1) Model-II with Marshall–Olkin Fréchet Marginal Distribution*

Finally, we consider more general Max-Min process which includes maximum, minimum as well as the innovations and the process. The AR(1) structure is given by

$$X_n = \begin{cases} \max(X_{n-1}, \varepsilon_n) & \text{with probability } p_1 \\ \min(X_{n-1}, \varepsilon_n) & \text{with probability } p_2 \\ \varepsilon_n & \text{with probability } p_3 \\ X_{n-1} & \text{with probability } 1-p_1-p_2-p_3, \end{cases} \tag{4.8}$$

where $0 < p_1, p_2, p_3 < 1$, $p_1 + p_2 + p_3 < 1$ and $\{\varepsilon_n\}$ is a sequence of i.i.d. random variables independently distributed of X_n. Then the process is stationary Markovian with Marshall–Olkin distribution as marginal.

Theorem 4.4. *$AR(1)$ Max-Min process $\{X_n\}$ with structure (4.8) is a stationary Markovian $AR(1)$ Max-Min process with Marshall–Olkin Fréchet distribution with parameters q, δ, and β if and only if $\{\varepsilon_n\}$ follows Fréchet distribution with parameters δ and β, where $q = \frac{p_1+p_3}{p_2+p_3}$.*

Proof. Similar to the proof of Theorem 4.1.

Remark. The above model can describe the response to treatment of a patient suffering from B.P. In a normal situation, X_n is same as X_{n-1}. For an acute patient always the innovation ε_n is important. In some cases, we have to keep the minimum as well as maximum at particular levels.

5. Conclusion

In this article, applications of Marshall-Olkin Fréchet distribution in stress-strength reliability analysis, acceptance sampling, and time series modeling are discussed. The reliability with respect to two such distributions with tilt parameters α_1 and α_2 are compared with respect to a real data set. The results obtained have applications in various areas such as clinical trial experiments for comparing efficiency of one medicine over another, efficiency of competing instruments, etc.

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