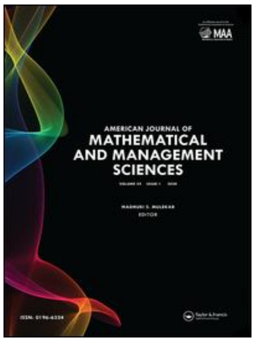

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American Journal of Mathematical and Management Sciences

ISSN: 0196-6324 (Print) 2325-8454 (Online) Journal homepage: https://www.tandfonline.com/loi/umms20

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To cite this article: S. M. Sunoj, N. Unnikrishnan Nair, Asok K. Nanda & R. S. Rasin (2020): Ageing Intensity Function for Conditionally Specified Models, American Journal of Mathematical and Management Sciences, DOI: 10.1080/01966324.2020.1762143

To link to this article: https://doi.org/10.1080/01966324.2020.1762143

Published online: 13 May 2020.

AMERICAN JOURNAL OF MATHEMATICAL AND MANAGEMENT SCIENCES
https://doi.org/10.1080/01966324.2020.1762143

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Ageing Intensity Function for Conditionally Specified Models

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ABSTRACT
Jiang et al. have introduced a quantitative measure known as the ageing intensity function for evaluating the ageing properties of a component/system. The present study extends this ageing intensity function to the conditionally specified and conditional survival models. In a two component system, these two conditional ageing intensity functions provide the ageing patterns of one component when the other component has either failed or survived a specified period of time. The proposed conditionally ageing intensity function models provide simple expressions for some commonly used bivariate lifetime models. Characterization relationships are established for bivariate exponential, bivariate Weibull, conditional proportional hazards and bivariate weighted models.

KEY WORDS AND PHRASES
Bivariate models; characterization; reliability measures; weighted models

MSC 2010
62E10; 62N05

1. Introduction

The notion of ageing plays an important role in reliability and survival analysis as it is an inherent property of all systems and products. The ageing characteristics are generally determined through the failure (hazard) rate, and classified into: (i) positive ageing, if the failure rate is increasing, (ii) non-ageing, if the failure rate is constant and (iii) anti-ageing or negative ageing, if the failure rate is decreasing. Unlike such qualitative ageing characteristics based on the failure rate, Jiang et al. (2003) proposed a new quantitative measure, known as ageing intensity (AI) function, an alternative measure to study the ageing pattern of probability models. It is defined as follows. Let X be a non-negative random variable representing lifetime of a living organism, a component or a system with an absolutely continuous cumulative distribution function $F_X(.)$, survival function $\bar{F}_X(.) = 1 - F_X(.)$ and hazard rate function $h_X(.) = \frac{f_X(.)}{\bar{F}_X(.)}$, where $f_X(.)$ is the corresponding density function. Then AI function of X is defined as the ratio of hazard rate to a baseline hazard rate. When the baseline hazard rate is average hazard rate $\frac{1}{t}\int_0^t h_X(u)du$, the AI function is defined by

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$$L_X(t) = \frac{h_X(t)}{\frac{1}{t}\int_0^t h_X(u)du} = -\frac{tf_X(t)}{\bar{F}_X(t)\log\left(\bar{F}_X(t)\right)}. \tag{1}$$

Jiang et al. (2003) have shown that the failure rate can be viewed as quasi-constant if (1) is close to 1. Also, they have identified the quasi-constancy, quasi-increasing or quasi-decreasing properties of probability models based on the quantitative value of $L_X(.)$. For instance, $L_X(t) = 1$ for all $t > 0$ if and only if the failure rate function $h_X(t)$ is constant (*i.e.*, X is both increasing failure rate (IFR) and decreasing failure rate (DFR) or exponential), $L_X(t) > 1$ if $h_X(t)$ is increasing in t (*i.e.*, X is IFR), and $L_X(t) < 1$ if $h_X(t)$ is decreasing in t (*i.e.*, X is DFR). Nanda et al. (2007) studied more properties of (1) for various probability distributions. The AI ordering, and its closure properties under different reliability operations, viz., formation of k-out-of-n system, and increasing transformations are also given in Nanda et al. (2007). The larger the value of $L_X(.)$, the stronger the tendency of ageing of the random variable X. Also, $L_X(t) = c$, for $x > 0$, c being a constant, characterizes the Weibull distribution with shape parameter c. It is to be noted that the failure rate function uniquely determines the AI function but not conversely. More properties of AI function are available in Nanda et al. (2007), Bhattacharjee et al. (2013), Sunoj and Rasin (2018), Szymkowiak (2018) and Szymkowiak and Iwińska (2019).

It is inherently difficult to visualize bivariate distributions. However, conditional densities can be easily visualized unlike marginal or joint densities. Sometimes one could identify a joint distribution by specifying one of the marginals and a conditional density. Alternatively, one may specify the distribution solely in terms of the features of two families of conditional densities. This approach is called conditional specification of the joint distribution (see Arnold et al. (1999)). Another popular type of conditioning is the conditional survival models, where component survival times on events are conditioned. These two types of models are often useful in two-component reliability systems where the operational status of one component is known in advance. For more recent works on conditionally specified and conditional survival models, we refer to Arnold (1995, 2009), Arnold et al. (1999), Gupta (2008), Navarro and Sarabia (2010, 2013), Navarro et al. (2011), Sunoj and Vipin (2019), Ghosh and Balakrishnan (2017) and the references therein. The conditionally specified and conditional survival models are, respectively, denoted by the random variables, X_i given $X_j = t_j$ or $(X_i|X_j = t_j)$, and X_i given $X_j > t_j$ or $(X_i|X_j > t_j)$ for $i, j = 1, 2; i \neq j$. Motivated with the usefulness of various conditional measures in identifying bivariate lifetime distributions, in the present paper we extend the concept of AI function in (1) to the conditionally specified and conditional survival models. The proposed conditional AI functions provide a tool to characterize certain bivariate models.

The paper is unfolded as follows. In Section 2, we extend AI function for the conditionally specified random variables and prove some characterization results. The AI function for conditional survival models are studied in Section 3.

2. Ageing Intensity Function for Conditionally Specified Models

In this section, we consider AI function based on conditioning of first type for the random variables $(X_1|X_2 = t_2)$ and $(X_2|X_1 = t_1)$ and study their properties.

Let (X_1, X_2) be a non-negative random vector admitting an absolutely continuous distribution function F_{X_1,X_2} with respect to Lebesgue measure in the positive octant $R_2^+ = \{(t_1, t_2)|t_1, t_2 > 0\}$ of the two dimensional Euclidean space R_2. The joint probability density function and survival function of $(X_1,\ X_2)$ are denoted by f_{X_1,X_2} and $\bar{F}_{X_1,X_2}$, respectively. Consider the conditionally specified random variables $(X_i|X_j = t_j)$ for $i, j = 1, 2; i \neq j,$ with survival function, probability density function and hazard rate function as

$$\bar{F}_{X_i|X_j}(t_i|t_j) = P(X_i > t_i|X_j = t_j), \qquad f_{X_i|X_j}(t_i|t_j) = -\frac{\partial}{\partial t_i}\bar{F}_{X_i|X_j}(t_i|t_j)$$

and

$$h_{X_i|X_j}(t_i|t_j) = -\frac{\partial}{\partial t_i}\log\bar{F}_{X_i|X_j}(t_i|t_j),$$

respectively, for $i, j = 1, 2; i \neq j$. Using (1), the AI function for $(X_i|X_j = t_j)$ is defined as a vector

$$\left(L_{X_1|X_2}(t_1|t_2), L_{X_2|X_1}(t_2|t_1)\right) = \left(L_{(X_1|X_2=t_2)}(t_1|t_2), L_{(X_2|X_1=t_1)}(t_2|t_1)\right),$$

where

$$L_{X_i|X_j}(t_i|t_j) = -\frac{t_i h_{X_i|X_j}(t_i|t_j)}{\log\bar{F}_{X_i|X_j}(t_i|t_j)} = -\frac{t_i f_{X_i|X_j}(t_i|t_j)}{\bar{F}_{X_i|X_j}(t_i|t_j)\log\bar{F}_{X_i|X_j}(t_i|t_j)}, i, j = 1, 2; i \neq j, \tag{2}$$

is the conditional AI function of X_i evaluated at the point t_i given that $X_j = t_j$.

Like the univariate measure $L_X(t)$ due to Jiang et al. (2003), the conditional AI function $L_{X_i|X_j}(t_i|t_j) = 1$ if the conditional failure rate $h_{X_i|X_j}(t_i|t_j)$ is quasi-constant. $L_{X_i|X_j}(t_i|t_j) > 1$ if $h_{X_i|X_j}(t_i|t_j)$ is increasing in t_i, and $L_{X_i|X_j}(t_i|t_j) < 1$ if $h_{X_i|X_j}(t_i|t_j)$ is decreasing in t_i. Thus, $L_{X_i|X_j}(t_i|t_j)$ takes larger (smaller) values to indicate a stronger tendency of ageing (anti-ageing).

The following example provides bivariate models with some simple forms of conditional AI function $L_{X_i|X_j}(t_i|t_j)$. From a reliability point of view, the conditional AI function with each of these models gives the ageing behavior of a component X_i in a two-component system (X_1, X_2) that follows a bivarate distribution when the other component X_j has failed at time t_j.

Example 1. Consider a bivariate family of distributions proposed by Navarro and Sarabia (2013) whose conditional distributions follow proportional hazards model, with probability density function given by

$$f(t_1, t_2) = c(\phi)a_1a_2\lambda_1(t_1)\lambda_2(t_2)\,exp\left(-a_1\Lambda_1(t_1) - a_2\Lambda_2(t_2) - \phi a_1a_2\Lambda_1(t_1)\Lambda_2(t_2)\right), \tag{3}$$

$t_1, t_2 \geq 0, a_1, a_2 > 0, \phi \geq 0,$ where $\lambda_i(t_i)$ and $\Lambda_i(t_i) = \int_0^{t_i}\lambda_i(u)du$ denote respectively the baseline hazard and the cumulative hazard functions of the random variable $X_i, i = 1, 2$. The model given in (3) is a re-parametrization of the bivariate conditional proportional hazards model given in Arnold and Kim (1996). The case when $\phi = 0$ corresponds to the case of independence. Then the conditional distributions

$$f_{X_i|X_j}(t_i|t_j) = a_i\lambda_i(t_i)\left(1 + \phi a_j\Lambda_j(t_j)\right)\,exp\left(-a_i\Lambda_i(t_i)\left(1 + \phi a_j\Lambda_j(t_j)\right)\right), i, j = 1, 2; i \neq j,$$

assume the proportional hazards model, with conditional AI function given by

$$L_{X_i|X_j}(t_i|t_j) = L_{X_i}(t_i), i,j = 1,2; i \neq j, \tag{4}$$

where $L_{X_i}(t_i) = \frac{t_i \lambda_i(t_i)}{\Lambda_i(t_i)}$. Some important bivariate distributions, who are members of the family (3), provide the following forms to $L_{X_i|X_j}(t_i|t_j)$.

(i) When $\Lambda_i(t_i) = t_i, i = 1,2$ in (3), we have the bivariate distribution with exponential conditional due to Arnold and Strauss (1988) having density

$$f(t_1,t_2) = c(\phi)a_1a_2 \exp\left(-a_1t_1 - a_2t_2 - \phi a_1a_2t_2t_2\right)$$

Then $L_{X_i|X_j}(t_i|t_j) = 1, i = 1,2$ is a characterizing property of bivariate distribution with exponential conditionals, as proved in Theorem 2.1.

(ii) For $\Lambda_i(t_i) = t_i^{\gamma_i}, \gamma_i > 0, i = 1,2,$ (3) reduces to a bivariate Weibull model,

$$f(t_1,t_2) = c(\phi)a_1a_2\gamma_1\gamma_2 t_1^{\gamma_1-1}t_2^{\gamma_2-1} \exp\left(-a_1t_1^{\gamma_1} - a_2t_2^{\gamma_2} - \phi a_1a_2t_1^{\gamma_1}t_2^{\gamma_2}\right)$$

such that $L_{X_i|X_j}(t_i|t_j) = \gamma_i, i = 1,2,$ another characterization to bivariate Weibull.

(iii) When $\Lambda_i(t_i) = \log\frac{(\beta_i+t_i)}{\beta_i}, \beta_i > 0. i = 1,2,$ (3) turns into a bivariate Pareto model with probability density function given by

$$f(t_1,t_2) = c(\phi)a_1a_2\left(\frac{\beta_1}{\beta_1+t_1}\right)^{a_1+1}\left(\frac{\beta_2}{\beta_2+t_2}\right)^{a_2+1} \exp\left(-\phi a_1a_2 \log\left(\frac{\beta_1+t_1}{\beta_1}\right)\log\left(\frac{\beta_2+t_2}{\beta_2}\right)\right)$$

such that $L_{X_i|X_j}(t_i|t_j) = \frac{\beta_i t_i}{(\beta_i+t_i)(\log(\beta_i+t_i) - \log\beta_i)}, i = 1,2,$

(iv) When $\Lambda_i(t_i) = \log\frac{(\beta_i+t_i^{\gamma_i})}{\beta_i}, \beta_i, \gamma_i > 0. i = 1,2$ in (3), we get a bivariate Burr distribution

$$f(t_1,t_2) = c(\phi)a_1a_2\gamma_1\gamma_2\left(\frac{\beta_1}{\beta_1+t_1^{\gamma_1}}\right)^{a_1+1}\left(\frac{\beta_2}{\beta_2+t_2^{\gamma_2}}\right)^{a_2+1} t_1^{\gamma_1-1}t_2^{\gamma_2-1}$$
$$\exp\left(-\phi a_1a_2 \log\left(\frac{\beta_1+t_1^{\gamma_1}}{\beta_1}\right)\log\left(\frac{\beta_2+t_2^{\gamma_2}}{\beta_2}\right)\right)$$

with $L_{X_i|X_j}(t_i|t_j) = \frac{\beta_i\gamma_i t_i^{\gamma_i}}{(\beta_i+t_i^{\gamma_i})(\log(\beta_i+t_i^{\gamma_i}) - \log\beta_i)}, i = 1,2.$

The situations in which it may be difficult to derive closed form expression for $L_{X_i|X_j}(t_i|t_j)$, obtaining bounds of $L_{X_i|X_j}(t_i|t_j)$ in terms of other measures are of importance. Accordingly, we now obtain a lower (upper) bound for $L_{X_i|X_j}(t_i|t_j)$. From (2), we get

$$-L_{X_i|X_j}(t_i|t_j)\log\bar{F}_{X_i|X_j}(t_i|t_j) = t_i h_{X_i|X_j}(t_i|t_j).$$

Differentiating the above equation with respect to t_i, we get

$$-\frac{\partial L_{X_i|X_j}(t_i|t_j)}{\partial t_i}\log\bar{F}_{X_i|X_j}(t_i|t_j) + L_{X_i|X_j}(t_i|t_j)h_{X_i|X_j}(t_i|t_j) = t_i\frac{\partial h_{X_i|X_j}(t_i|t_j)}{\partial t_i} + h_{X_i|X_j}(t_i|t_j),$$

or equivalently,

$$\frac{\partial L_{X_i|X_j}(t_i|t_j)}{\partial t_i}\log \bar{F}_{X_i|X_j}(t_i|t_j) = \left(L_{X_i|X_j}(t_i|t_j)-1\right)h_{X_i|X_j}(t_i|t_j) - t_i\frac{\partial h_{X_i|X_j}(t_i|t_j)}{\partial t_i}.$$

When $L_{X_i|X_j}(t_i|t_j)$ is increasing (decreasing) partially with respect to t_i for $i = 1, 2,$ that is, $\frac{\partial L_{X_i|X_j}(t_i|t_j)}{\partial t_i} \geq (\leq)0,$ we get

$$L_{X_i|X_j}(t_i|t_j) \geq (\leq) t_i\frac{\partial \log h_{X_i|X_j}(t_i|t_j)}{\partial t_i} + 1,$$

thus providing a lower (upper) bound for $L_{X_i|X_j}(t_i|t_j)$.

Similar to $L_X(.)$ for a univariate random variable, the conditional AI function can also be treated as a quantitative measure. For specified values of t_1 and t_2, $L_{X_i|X_j}(t_i|t_j)$ provides a measure to determine the ageing behavior of a two-component system. In particular, $L_{X_1|X_2}(t_1|t_2)$ measures the ageing pattern of the first component conditioned on the fact that the second component has failed at time t_2. A better interpretation of $L_{X_1|X_2}(t_1|t_2)$ in some applied problems, say in the actuarial studies, is as follows. Let X_1 and X_2 denote respectively the ages of a woman and her husband who bought life insurance at possibly different ages. Then $L_{X_1|X_2}(68|72)$, say, provides the ageing behavior of the woman given that her current age is 68 and her husband died at 72.

In the univariate case $L_X(.) = 1$ when X is both IFR and DFR or exponential. Now $L_{X_i|X_j}(t_i|t_j)$ holds an equivalent property for the bivariate model (5).

Theorem 1. *The conditional AI function* $L_{X_i|X_j}(t_i|t_j) = 1, i, j = 1, 2; i \neq j$ *if and only if* (X_1, X_2) *follows bivariate distribution with exponential conditionals (Arnold and Strauss (1988)) having joint probability density function, given by*

$$f(t_1, t_2) = K_1 \exp(-\lambda_1 t_1 - \lambda_2 t_2 - \theta t_1 t_2), K_1, \lambda_1, \lambda_2, \theta, t_1, t_2 > 0. \tag{5}$$

Proof. When (X_1, X_2) follows (5), we have $f_{X_i|X_j}(t_i|t_j) = (\lambda_i + \theta t_j)e^{-(\lambda_i+\theta t_j)t_i}, i, j = 1, 2; i \neq j,$ and using (2) we obtain $L_{X_i|X_j}(t_i|t_j) = 1, i, j = 1, 2; i \neq j.$ Conversely, assume that $L_{X_i|X_j}(t_i|t_j) = 1, i, j = 1, 2; i \neq j$ holds. Using (2), we get

$$-\log \bar{F}_{X_i|X_j}(t_i|t_j) = t_i h_{X_i|X_j}(t_i|t_j).$$

Differentiating both sides with respect t_i yields

$$\frac{\partial h_{X_i|X_j}(t_i|t_j)}{\partial t_i} = 0,$$

or equivalently,

$$h_{X_i|X_j}(t_i|t_j) = A_i + B_i(t_j), i, j = 1, 2; i \neq j.$$

Applying the proof of Theorem 3.4 of Sunoj and Vipin (2019), we obtain the required model (5). This completes the proof. □

For some applications of (5) in the context of random stress-dependent and accelerated lifetime models, refer to SenGupta (2006). The following theorem extends the univariate characterizing property of Weibull distribution that $L_X(t) = c,$ where c is the shape parameter to the bivariate Weibull model using the conditional AI function.

Theorem 2. *The conditional AI function* $L_{X_i|X_j}(t_i|t_j) = \gamma_i, i,j = 1,2; i \neq j$ *if and only if* (X_1, X_2) *follows bivariate Weibull distribution given in Example 1 with joint probability density function, given by*

$$f(t_1, t_2) = c(\phi)a_1a_2\gamma_1\gamma_2 t_1^{\gamma_1-1}t_2^{\gamma_2-1}\exp\left(-a_1t_1^{\gamma_1} - a_2t_2^{\gamma_2} - \phi a_1a_2t_1^{\gamma_1}t_2^{\gamma_2}\right), \tag{6}$$

$t_1, t_2 \geq 0, a_1, a_2, \gamma_1, \gamma_2 > 0, \phi \geq 0,$ *where* $\gamma_i, i = 1,2$ *represents the shape parameter of (6).*

Proof. When (X_1, X_2) follows a bivariate Weibull distribution given in (6), we have

$$f_{X_i|X_j}(t_i|t_j) = a_i\gamma_i t_i^{\gamma_i-1}\left(1+\phi a_j t_j^{\gamma_j}\right)e^{-\left(1+\phi a_j t_j^{\gamma_j}\right)a_i t_i^{\gamma_i}}$$

and $\bar{F}_{X_i|X_j}(t_i|t_j) = e^{-(1+\phi a_j t_j^{\gamma_j})a_i t_i^{\gamma_i}}$ so that $L_{X_i|X_j}(t_i|t_j) = \gamma_i, i,j = 1,2; i \neq j$. To prove the converse part, we assume that $L_{X_i|X_j}(t_i|t_j) = \gamma_i, i,j = 1,2; i \neq j$ holds. Using (2), we obtain

$$-t_i h_{X_i|X_j}(t_i|t_j) = \gamma_i \log \bar{F}_{X_i|X_j}(t_i|t_j).$$

Differentiating both sides with respect to t_i, we get

$$-t_i\frac{\partial h_{X_i|X_j}(t_i|t_j)}{\partial t_i} - h_{X_i|X_j}(t_i|t_j) = -\gamma_i h_{X_i|X_j}(t_i|t_j),$$

or, equivalently,

$$\frac{\partial}{\partial t_i}\log h_{X_i|X_j}(t_i|t_j) = \frac{(\gamma_i - 1)}{t_i}.$$

Integrating with respect to t_i, we get

$$\log h_{X_i|X_j}(t_i|t_j) = (\gamma_i - 1)\log t_i + \log K_i(t_j),$$

where $K_i(t_j)$ is the constant of integration. This, in turn, results

$$h_{X_i|X_j}(t_i|t_j) = K_i(t_j)t_i^{\gamma_i-1}, i,j = 1,2; i \neq j. \tag{7}$$

From the definition of $h_i(t_i|t_j)$ we have

$$h_{X_i|X_j}(t_i|t_j) = -\frac{\partial}{\partial t_i}\log \bar{F}_{X_i|X_j}(t_i|t_j) = -\frac{f(t_1,t_2)}{\frac{\partial}{\partial t_j}\bar{F}(t_i,t_j)}, i,j = 1,2; i \neq j.$$

Then (7) becomes,

$$\frac{\partial}{\partial t_j}\bar{F}(t_i,t_j) = -\frac{f(t_1,t_2)}{t_i^{\gamma_i-1}K_i(t_j)}.$$

Differentiating with respect to t_i, we get, after some algebra,

$$\frac{\partial}{\partial t_i}\log f(t_i,t_j) = \frac{(\gamma_i - 1)}{t_i} - t_i^{\gamma_i-1}K_i(t_j)$$

Now integrating with respect to t_i, we obtain

$$\log f(t_i,t_j) = (\gamma_i - 1)\log t_i - \frac{K_i(t_j)t^{\gamma_i}}{\gamma_i} + \log m_i(t_j),$$

where $m_i(t_j), i, j = 1, 2; i \neq j$ is the constant of integration, and equivalently we get

$$f(t_i, t_j) = t_i^{\gamma_i - 1} m_i(t_j) e^{-\frac{K_i(t_j) t_i^{\gamma_i}}{\gamma_i}}, i, j = 1, 2; i \neq j. \tag{8}$$

Applying for $i = 1, 2$ and equating we get,

$$t_1^{\gamma_1 - 1} m_1(t_2) e^{-\frac{K_1(t_2) t_1^{\gamma_1}}{\gamma_1}} = t_2^{\gamma_2 - 1} m_2(t_1) e^{-\frac{K_2(t_1) t_2^{\gamma_2}}{\gamma_2}}. \tag{9}$$

As $t_1 \to 1$, (9) becomes

$$m_1(t_2) = m_2(1) t_2^{\gamma_2 - 1} e^{\frac{K_1(t_2)}{\gamma_1} - \frac{K_2(1) t_2^{\gamma_2}}{\gamma_2}},$$

and when $t_2 \to 1$, (9) reduces to

$$m_2(t_1) = m_1(1) t_1^{\gamma_1 - 1} e^{\frac{K_2(t_1)}{\gamma_2} - \frac{K_1(1) t_1^{\gamma_1}}{\gamma_1}}.$$

Substituting $m_1(t_2)$ and $m_2(t_1)$ in (9) provides

$$f(t_1, t_2) = \begin{cases} t_1^{\gamma_1 - 1} t_2^{\gamma_2 - 1} m_2(1) e^{-\frac{K_1(t_2) t_1^{\gamma_1}}{\gamma_1} - \frac{K_2(1) t_2^{\gamma_2}}{\gamma_2} + \frac{K_1(t_2)}{\gamma_1}} \\ t_1^{\gamma_1 - 1} t_2^{\gamma_2 - 1} m_1(1) e^{-\frac{K_1(1) t_1^{\gamma_1}}{\gamma_1} - \frac{K_2(t_1) t_2^{\gamma_2}}{\gamma_2} + \frac{K_2(t_1)}{\gamma_2}}, \end{cases} \tag{10}$$

where $m_1(1), m_2(1) \geq 1$. Taking logarithm, (10) reduces to

$$\log m_2(1) - \frac{K_1(t_2) t_1^{\gamma_1}}{\gamma_1} - \frac{K_2(1) t_2^{\gamma_2}}{\gamma_2} + \frac{K_1(t_2)}{\gamma_1} = \log m_1(1) - \frac{K_1(1) t_1^{\gamma_1}}{\gamma_1} - \frac{K_2(t_1) t_2^{\gamma_2}}{\gamma_2} + \frac{K_2(t_1)}{\gamma_2}. \tag{11}$$

Setting $t_1 = t_2 = 1$, we get

$$\log m_2(1) = \log m_1(1) + \frac{K_2(1)}{\gamma_2} - \frac{K_1(1)}{\gamma_1}$$

and substituting it in (11) gives,

$$\frac{K_2(1)}{\gamma_2}\left(1 - t_2^{\gamma_2}\right) + \frac{K_1(t_2)}{\gamma_1}\left(1 - t_1^{\gamma_1}\right) = \frac{K_1(1)}{\gamma_1}\left(1 - t_1^{\gamma_1}\right) + \frac{K_2(t_1)}{\gamma_2}\left(1 - t_2^{\gamma_2}\right),$$

or, equivalently,

$$\frac{K_1(t_2) - K_1(1)}{\gamma_1}\left(1 - t_1^{\gamma_1}\right) = \frac{K_2(t_1) - K_2(1)}{\gamma_2}\left(1 - t_2^{\gamma_2}\right),$$

or

$$\frac{K_1(t_2) - K_1(1)}{\gamma_1\left(1 - t_2^{\gamma_2}\right)} = \frac{K_2(t_1) - K_2(1)}{\gamma_2\left(1 - t_1^{\gamma_1}\right)}, \tag{12}$$

where the left and the right sides are functions of t_2 and t_1 alone, but (12) holds for all t_1 and t_2 and therefore both sides of (12) must be equal to a constant, say ϕ. This implies that

$$K_1(t_2) = K_1(1) + \phi\gamma_1\left(1 - t_2^{\gamma_2}\right) \tag{13}$$

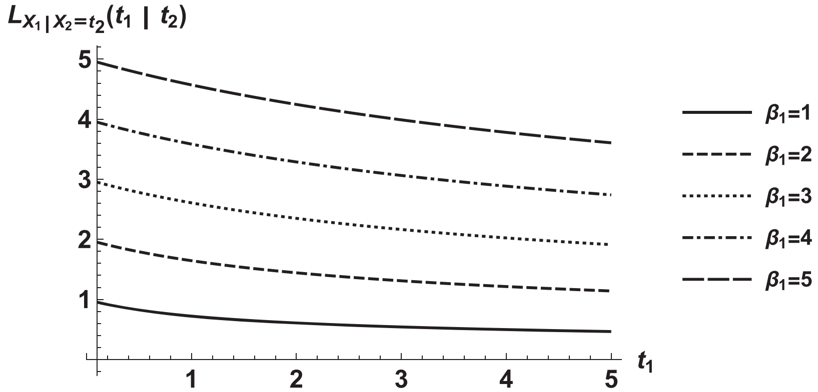

Figure 1. Conditional AI function of $X_1|X_2 = t_2$ for bivariate Pareto.

and

$$K_2(t_1) = K_2(1) + \phi\gamma_1\left(1 - t_1^{\gamma_1}\right). \tag{14}$$

Substituting (13) and (14) in (10) and by choosing $\phi, K_1(1)$ and $K_2(1)$ appropriately, we obtain the required bivariate Weibull model (6). This completes the proof. □

Even if $L_{X_i|X_j}(t_i|t_j), i,j = 1,2; i \neq j$ are univariate functions of $t_i, i = 1,2,$ Theorem 1 and Theorem 2 prove that components of $L_{X_i|X_j}(t_i|t_j)$ unity (resp. constant) characterize bivariate exponential model (5) (resp. bivariate Weibull model (6)). The ageing behavior of bivariate Pareto and Burr distributions given in Example 1 based on $L_{X_i|X_j}(t_i|t_j)$ can be obtained from Figures 1 and 2 respectively.

In survival studies, the most widely used semi-parametric model is the Cox proportional hazard rates (PHR) model. Let (X_1, X_2) and (Y_1, Y_2) be two bivariate random vectors with joint probability density functions f_{X_1,X_2} and g_{Y_1,Y_2} and joint survival functions given by $\bar{F}_{X_1,X_2}(t_1,t_2) = \bar{F}(t_1,t_2) = P(X_1 > t_1, X_2 > t_2)$ and $\bar{G}_{Y_1,Y_2}(t_1,t_2) = \bar{G}(t_1,t_2) = P(Y_1 > t_1, Y_2 > t_2)$, respectively. Let us assume that the common support is $S = (l,\infty) \times (l,\infty)$ for $l \geq 0$. Also let $g_{Y_i|Y_j}(\cdot|t_j), \bar{G}_{Y_i|Y_j}(\cdot|t_j)$, and $k_{Y_i|Y_j}(\cdot|t_j)$ denote respectively the probability density function, the survival function and the hazard rate function of $(Y_i|Y_j = t_j)$ for $i = 1,2, i \neq j$. Analogous to the proportional hazard rates model for univariate random variables, the random vectors (X_1, X_2) and (Y_1, Y_2) satisfy the conditional PHR model (see Sankaran and Sreeja (2007)) when, for $i = 1,2, i \neq j$,

$$k_{Y_i|Y_j}(t_i|t_j) = \theta_i(t_j)h_{X_i|X_j}(t_i|t_j), \tag{15}$$

where $\theta_i(t_j)$ is a nonnegative function of t_j.

In the following theorem, conditional PHR model is characterized in terms of the equality of the conditional AT functions for the conditionally specified model.

Theorem 3. *Let $L_{Y_i|Y_j}(t_i|t_j)$ denote the conditional AI function of the random variable $(Y_i|Y_j = t_j)$. Then $L_{Y_i|Y_j}(t_i|t_j) = L_{X_i|X_j}(t_i|t_j)$ for $i = 1,2, i \neq j$ if and only if (X_1, X_2) and (X_1, X_2) satisfy the conditional PHR model (15).*

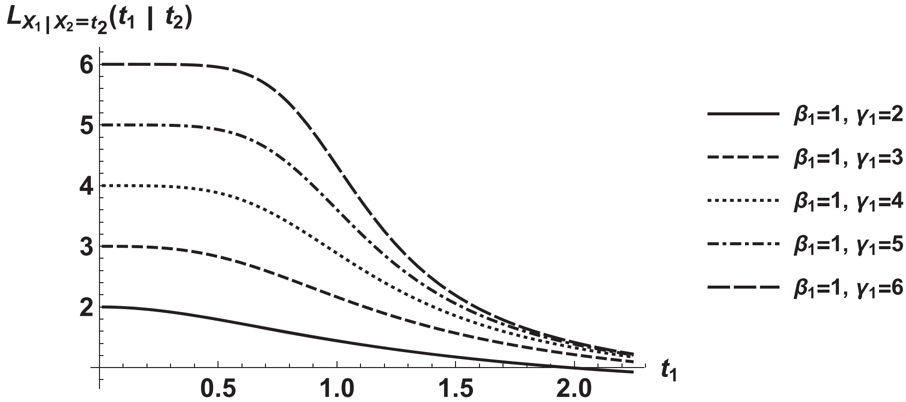

Figure 2. Conditional AI function of $X_1|X_2 = t_2$ for bivariate Burr.

Proof. Assume that (X_1, X_2) and (Y_1, Y_2) satisfy the conditional PHR model (15), which is equivalent to saying that $\bar{G}_{Y_i|Y_j}(t_i|t_j) = (\bar{F}_{X_i|X_j}(t_i|t_j))^{\theta_i(t_j)}$. Then using (2), we get

$$L_{Y_i|Y_j}(t_i|t_j) = -\frac{t_i k_{Y_i|Y_j}(t_i|t_j)}{\log \bar{G}_{Y_i|Y_j}(t_i|t_j)} = -\frac{t_i \theta_i(t_j) h_{X_i|X_j}(t_i|t_j)}{\theta_i(t_j) \log \bar{F}_{X_i|X_j}(t_i|t_j)} = -\frac{t_i h_{X_i|X_j}(t_i|t_j)}{\log \bar{F}_{X_i|X_j}(t_i|t_j)} = L_{X_i|X_j}(t_i|t_j). \tag{16}$$

The converse part is obtained by retracing (16). □

Next, we consider a random vector (X_1^w, X_2^w) that has a bivariate weighted distribution associated to (X_1, X_2) and two nonnegative real functions $w_1(.)$ and $w_2(.)$. Then its joint probability density function is given by

$$f^w(t_1, t_2) = \frac{w_1(t_1) w_2(t_2)}{E(w_1(X_1) w_2(X_2))} f(t_1, t_2),$$

where $0 < E(w_1(X_1)w_2(X_2)) < \infty$. For different properties of a more general weight function $w(\cdot, \cdot)$ one may refer to Nanda and Jain (1999). Using this definition it is easy to see that the marginal random variable X_i^w has a univariate weighted distribution associated with X_i, and weight function $w_i^*(t_i) = w_i(t_i)E(w_j(X_j|X_i = t_i))$ for $i = 1, 2, i \neq j$. Also, $(X_i^w|X_j^w = t_j)$ has a (univariate) weighted distribution associated with $(X_i|X_j = t_j)$ with weight $w_i(t_i)$ for $i = 1, 2, i \neq j$. When $w_i(t_i) = t_i, i = 1, 2,$ the corresponding random vector (X_1^w, X_2^w) is called the length-biased random vector. Navarro et al. (2011) have proved that when (X_1^w, X_2^w) and (X_1, X_2) satisfy the conditional PHR model, (15) characterizes conditional Kullback-Leibler discrimination information and (X_1, X_2) have the joint probability density function given by

$$f(t_1, t_2) = c a_1 a_2 \frac{w_1'(t_1) w_2'(t_2)}{w_1^{a_1+1}(t_1) w_2^{a_2+1}(t_2)} \exp\left(-\phi a_1 a_2 \left(\log \frac{w_1(t_1)}{w_1(l)}\right)\left(\log \frac{w_2(t_2)}{w_2(l)}\right)\right) \tag{17}$$

for $t_1, t_2 \geq l,$ where $c > 0, \phi \geq 0$ and $a_i > 1$ or $a_i < 0$ for $i = 1, 2$. The model in (17) is a truncated version of bivariate model due to Arnold and Strauss (1988) in the support $S = (l, \infty) \times (l, \infty)$ and when $l = 0$ both models coincide. For different weight functions, the model (17) contains many parametric models. For instance, when $l=1$ and $w_1(t_1)=t_1, w_2(t_2)=t_2$ for $t_1,t_2\geq 1$, (17) reduces to bivariate Pareto model,

$$f(t_1,t_2)=\frac{ca_1a_2}{t_1^{a_1+1}t_2^{a_2+1}}\exp\left(-\phi a_1a_2(\log t_1)(\log t_2)\right); t_1,t_2\geq 1, c>0, a_1,a_2>1, \phi\geq 0.$$

Now using Theorem 3 of Navarro et al. (2011) and Theorem 3 given above, the characterization result of Theorem 4 is obtained.

Theorem 4. *Let (X_1^w,X_2^w) be a random vector which has the bivariate weighted distribution associated to (X_1,X_2) with two nonnegative and differentiable weight functions $w_1(.)$ and $w_2(.)$. Let us assume that the support of (X_1,X_2) is $S=(l,\infty)\times(l,\infty)$ for $l\geq 0$. Then the following conditions are equivalent:*

(a) *(X_1^w,X_2^w) and (X_1,X_2) satisfy the conditional PHR model (15) for $i=1,2$.*
(b) *$L_{Y_i|Y_j}(t_i|t_j)=L_{X_i|X_j}(t_i|t_j)$ for $i=1,2, i\neq j$.*
(c) *(X_1,X_2) has the joint probability density function (17).*

3. Ageing Intensity Function for Conditional Survival Models

In this section, we consider AI function based on conditioning of the second type for the random variables $(\tilde{X}_1,\tilde{X}_2)$, where $\tilde{X}_1=(X_1|X_2>t_2)$ and $\tilde{X}_2=(X_2|X_1>t_1)$. Let $\bar{F}_{X_i|X_j>t_j}(t_i)=P(X_i>t_i|X_j>t_j)$, $f_{X_i|X_j>t_j}(t_i)=-\frac{\partial}{\partial t_i}\bar{F}_{X_i|X_j>t_j}(t_i)$ and $h_{X_i|X_j>t_j}(t_i)=-\frac{\partial}{\partial t_i}\log\bar{F}_{X_i|X_j>t_j}(t_i)$ for $i,j=1,2; i\neq j$, respectively denote the survival functions, the probability density functions and the hazard rate functions corresponding to $(\tilde{X}_1,\tilde{X}_2)$. Here, $f_{X_i|X_j>t_j}(t_i)$ is the simple hidden truncation model due to Arnold (2009), and is given by

$$f_{X_i|X_j>t_j}(t_i)=\frac{\bar{F}_{X_j|X_i}(t_j|t_i)}{\bar{F}_{X_j}(t_j)}f_{X_i}(t_i). \tag{18}$$

Here, the marginal density of X_j and the conditional density of X_i given X_j will determine the resulting hidden truncation model. Also, model (18) is a weighted version of the original density for X_i, with weight function $\bar{F}_{X_j|X_i}(t_j|t_i)$ (see Arnold (2009)). Motivated with this, we propose a second measure of AI function based on the conditional random variables $(\tilde{X}_1,\tilde{X}_2)$.

Let (X_1,X_2) be a non-negative random vector admitting an absolutely continuous distribution function F_{X_1,X_2} with respect to Lebesgue measure in the positive octant $R_2^+=\{(t_1,t_2)|t_1,t_2>0\}$ of the two dimensional Euclidean space R_2. Then the AI function for $(\tilde{X}_1,\tilde{X}_2)$ is defined as a vector

$$\left(L_{X_1|X_2>t_2}(t_1), L_{X_2|X_1>t_1}(t_2)\right),$$

where

$$L_{X_i|X_j>t_j}(t_i)=-\frac{t_ih_{X_i|X_j>t_j}(t_i)}{\log\bar{F}_{X_i|X_j>t_j}(t_i)}=-\frac{t_if_{X_i|X_j>t_j}(t_i)}{\bar{F}_{X_i|X_j>t_j}(t_i)\log\bar{F}_{X_i|X_j>t_j}(t_i)},\ i,j=1,2; i\neq j \tag{19}$$

is called the conditional survival AI.

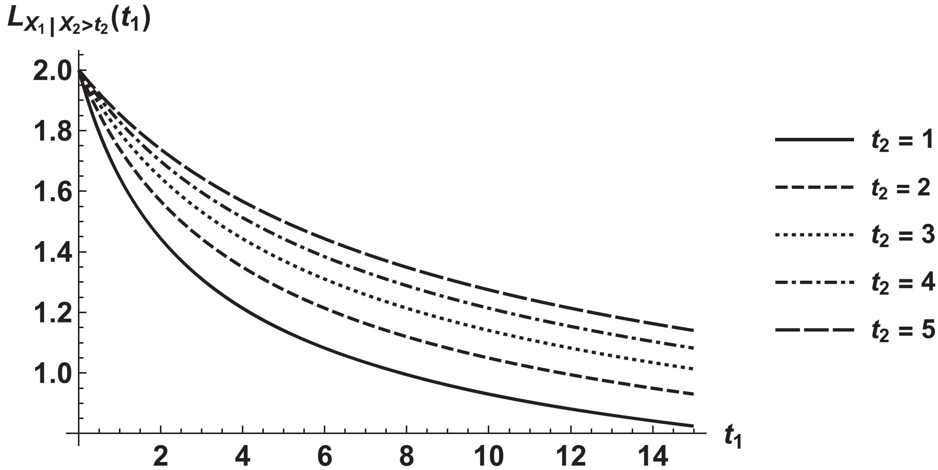

Figure 3. AI function of X_1 for bivariate Pareto II ($a_1 = a_2 = b = 1$) when X_2 has survived different t_2 values.

Eq. (19) is equivalent to

$$L_{X_i|X_j>t_j}(t_i) = -\frac{t_i h_i(t_1, t_2)}{\log \bar{F}(t_1, t_2) - \log \bar{F}_j(t_j)}, i, j = 1, 2; i \neq j, \tag{20}$$

where $h_i(t_1, t_2) = h_{X_i|X_j>t_j}(t_i) = -\frac{\partial}{\partial t_i} \log \bar{F}(t_1, t_2), i, j = 1, 2; i \neq j,$ denotes the ith component of the vector-valued bivariate failure rate $h(t_1, t_2) = (h_1(t_1, t_2), h_2(t_1, t_2))$ due to Johnson and Kotz (1975) and $\bar{F}_j(t_j) = P(X_j > t_j)$ denotes the marginal survival function of the random variable $X_j, j = 1, 2$.

Even if both $L_{X_i|X_j}(t_i|t_j)$ and $L_{X_i|X_j>t_j}(t_i)$ are defined based on the conditional distributions, $L_{X_i|X_j>t_j}(t_i)$ in (20) stands different as it can be computed from the joint distribution of (X_1, X_2) and can be used as an easy tool to determine ageing behavior of a bivariate random vector.

The ageing behavior of $(X_1|X_2 > t_2)$, in terms of te conditional survival AI, for different values of t_2, when the joint distribution of (X_1, X_2) is bivariate Pareto II, is described in the following example.

Example 2. Let (X_1, X_2) follow a bivariate Pareto II distribution with survival function given by

$$\bar{F}(t_1, t_2) = (1 + a_1 t_1 + a_2 t_2)^{-b}, t_1, t_2 > 0, a_1, a_2 > 0, b > 1.$$

Then $L_{X_i|X_j>t_j}(t_i) = \frac{b a_i t_i}{(1+a_1 t_1+a_2 t_2) \log\left(\frac{(1+a_1 t_1+a_2 t_2)}{1+a_j t_j}\right)}, i, j = 1, 2; i \neq j$. The ageing behavior of the above bivariate Pareto II distribution based on $L_{X_i|X_j>t_j}(t_i)$ is given in Figure 3.

Example 3. Let (X_1, X_2) be a bivariate random vector defined on R_2^+. Then its bivariate equilibrium distribution is the distribution of a random vector (X_1^E, X_2^E) such that the probability density functions of $(X_1^E|X_2^E > t_2)$ and $(X_2^E|X_1^E > t_1)$ are given by

$$g_{X_i^E|X_j^E>t_j}(t_i) = \frac{\bar{F}_{X_i|X_j>t_j}(t_i)}{E(X_i|X_j > t_j)}, i,j = 1,2; i \neq j. \tag{21}$$

Further, the corresponding survival function is given by

$$\bar{G}_{X_i^E|X_j^E>t_j}(t_i) = \frac{r_{X_i|X_j>t_j}(t_i)\bar{F}_{X_i|X_j>t_j}(t_i)}{E(X_i|X_j > t_j)} = \frac{r_i(t_1,t_2)\bar{F}(t_1,t_2)}{E(X_i|X_j > t_j)},$$

where

$$r_{X_i|X_j>t_j}(t_i) = E(X_i - t_i|X_1 > t_1, X_2 > t_2) = \frac{1}{\bar{F}(t_1,t_2)}\int_{t_i}^{\infty} \bar{F}(u_i,t_j)du_i = r_i(t_1,t_2)$$

denotes the *i*th component of the vector-valued mean residual life function (Arnold and Zahedi (1988)), and hence the conditional survival AI is derived as

$$L_{X_i^E|X_j^E>t_j}(t_i) = -\frac{t_i}{r_i(t_1,t_2)\left(\log r_i(t_1,t_2)\bar{F}(t_1,t_2) - \log E(X_i|X_j > t_j)\right)}, i,j = 1,2; i \neq j.$$

In the following characterization theorem we prove that $L_{X_i|X_j>t_j}(t_i)$ assumes the value unity for Gumbel's bivariate exponential distribution with survival function given in (22).

Theorem 5. *The conditional survival AI function $L_{X_i|X_j>t_j}(t_i) = 1, i,j = 1,2; i \neq j$ if and only if (X_1, X_2) follows bivariate exponential distribution due to (*Gumbel (1960)*) with joint survival function given by*

$$\bar{F}(t_1,t_2) = \exp(-\lambda_1 t_1 - \lambda_2 t_2 - \theta t_1 t_2), K_1, \lambda_1, \lambda_2, \theta, t_1, t_2 > 0. \tag{22}$$

Proof. Assume that $L_{X_i|X_j>t_j}(t_i) = 1, i,j = 1,2; i \neq j$. Then using (20), we have

$$-\log \bar{F}(t_1,t_2) + \log \bar{F}_j(t_j) = t_i h_i(t_1,t_2), i,j = 1,2; i \neq j.$$

Differentiating both sides with respect to t_i yields,

$$t_i \frac{\partial h_i(t_1,t_2)}{\partial t_i} = 0$$

which gives

$$h_i(t_1,t_2) = C_i + D_i(t_j). \tag{23}$$

It is well-known that the vector-valued hazard rate uniquely determines a bivariate distribution using the following identity (Johnson and Kotz (1975)),

$$\bar{F}(t_1,t_2) = \begin{cases} \exp\left(-\int_0^{t_1} h_1(u,t_2)du - \int_0^{t_2} h_2(0,v)dv\right), t_1,t_2 \geq 0 \\ \exp\left(-\int_0^{t_1} h_1(u,0)du - \int_0^{t_2} h_2(t_1,v)dv\right), t_1,t_2 \geq 0. \end{cases} \tag{24}$$

Substituting (23) in (24) we get

$$\bar{F}(t_1,t_2) = \begin{cases} \exp(-C_1 t_1 - C_2 t_2 - D_1(t_2)t_1 - D_2(0)t_2), t_1,t_2 \geq 0 \\ \exp(-C_1 t_1 - C_2 t_2 - D_1(0)t_1 - D_2(t_1)t_2), t_1,t_2 \geq 0. \end{cases} \tag{25}$$

The pair of identity (25) is equivalent to

$$e^{-C_1t_1-C_2t_2-D_1(t_2)t_1-D_2(0)t_2} = e^{-C_1t_1-C_2t_2-D_1(0)t_1-D_2(t_1)t_2}. \tag{26}$$

Eq. (26) implies that

$$D_1(t_2)t_1 + D_2(0)t_2 = D_1(0)t_1 + D_2(t_1)t_2,$$

which is possible if and only if $D_1(t_2) = \theta t_2$ and $D_2(t_1) = \theta t_1$, where θ is a positive constant, which, in turn, reduces to the required model (22). The converse part is straightforward. □

In the following theorem, the bivariate Weibull distribution is characterized in terms of a given value of the conditional survival AI function.

Theorem 6. *The conditional survival AI function* $L_{X_i|X_j>t_j}(t_i) = \gamma, i,j = 1,2; i \neq j$ *if and only if* (X_1, X_2) *follows bivariate Weibull distribution with joint survival function given by*

$$\bar{F}(t_1,t_2) = \exp\left(-\lambda_1 t_1^{\gamma} - \lambda_2 t_2^{\gamma} - \theta t_1^{\gamma} t_2^{\gamma}\right), \lambda_1, \lambda_2, \theta, \gamma, t_1, t_2 > 0. \tag{27}$$

Proof. The proof of 'only if' part is straightforward. To prove the 'if part', assume that $L_{X_i|X_j>t_j}(t_i) = \gamma, i,j = 1,2; i \neq j$ holds. Using (20), we have

$$-t_i h_i(t_1,t_2) = \gamma \log \bar{F}(t_1,t_2) - \gamma \log \bar{F}_j(t_j), i,j = 1,2; i \neq j$$

Differentiating both sides with respect to t_i and simplifying we get

$$\frac{\partial \log h_i(t_1,t_2)}{\partial t_i} = \frac{\gamma-1}{t_i}, i = 1,2. \tag{28}$$

Integrating (28) with respect to t_i, we have

$$h_i(t_1,t_2) = t_i^{\gamma-1} p_i(t_j), \tag{29}$$

where $p_i(t_j), i,j = 1,2; i \neq j$ denote the constant of integration. Substituting (29) in (24), and equating, we get

$$p_1(t_2)t_1^{\gamma} + p_2(0)t_2^{\gamma} = p_1(0)t_1^{\gamma} + p_2(t_1)t_2^{\gamma},$$

which is possible if and only if $p_1(t_2) = \alpha_1 + \theta t_2^{\gamma}$ and $p_2(t_t) = \alpha_2 + \theta t_1^{\gamma}$ for some constants $\alpha_1, \alpha_2, \theta > 0$. This reduces to the required form (27). □

4. Application to Real Data

In this section, we provide an example of how an empircal estimator can be used for estimating the AI function of conditinal survival model, $\hat{L}_{X_i|X_j>t_j}(t_i), i,j = 1,2, i \neq j$ and examine its performance using a cancer recurrence data, from Kulkarni and Rattihalli (2002). The data for the patients with bladder tumors, given in Table 1, consist of $X =$ time (in months) to the first recurrence of a tumor and $Y =$ time (in months) to the second recurrence of a tumor. Let N number of patients be put into test at the beginning of the study. Further, let the number of patients that survived at ordered times t_j and $t_j + \Delta t_j$ be $N_s(t_j)$ and $N_s(t_j) + \Delta t_j$ respectively. Then an empirical estimator for conditional AI function is given by,

Table 1. Cancer Data from Kulkarni and Rattihalli (2002).

Patients	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
X_i	12	10	3	3	7	3	2	28	2	3	12	9	16	3	9	3	2	5	2
Y_i	16	15	16	9	10	15	26	30	17	6	15	17	19	6	11	15	15	14	8

Table 2. Empirical estimates of $\hat{L}_{X_i\|X_j>t_j}(t_i)$.

t_2	t_1	$N_S(t_2)$	$N_S(t_2)-N_S(t_2+\Delta(t_2))$	$\hat{\bar{F}}_{X_1\|X_2>t_2}(t_1)$	$\hat{h}_{X_1\|X_2>t_2}(t_1)$	$\hat{L}_{X_1\|X_2>t_2}(t_1)$
0	0	19	10	1.00	0.18	–
	3	9	1	0.47	0.04	$0.11t_1$
	6	8	3	0.42	0.13	$0.33t_1$
	9	5	5	0.26	0.33	$0.58t_1$
6	0	17	8	1.00	0.16	–
	3	9	1	0.53	0.04	$0.13t_1$
	6	8	3	0.47	0.13	$0.38t_1$
	9	5	5	0.29	0.33	$0.63t_1$
9	0	15	6	1.00	0.13	–
	3	9	1	0.60	0.04	$0.17t_1$
	6	8	3	0.53	0.13	$0.46t_1$
	9	5	5	0.33	0.33	$0.70t_1$
12	0	13	6	1.00	0.15	–
	3	7	1	0.54	0.05	$0.18t_1$
	6	6	1	0.46	0.06	$0.17t_1$
	9	5	5	0.38	0.33	$0.80t_1$

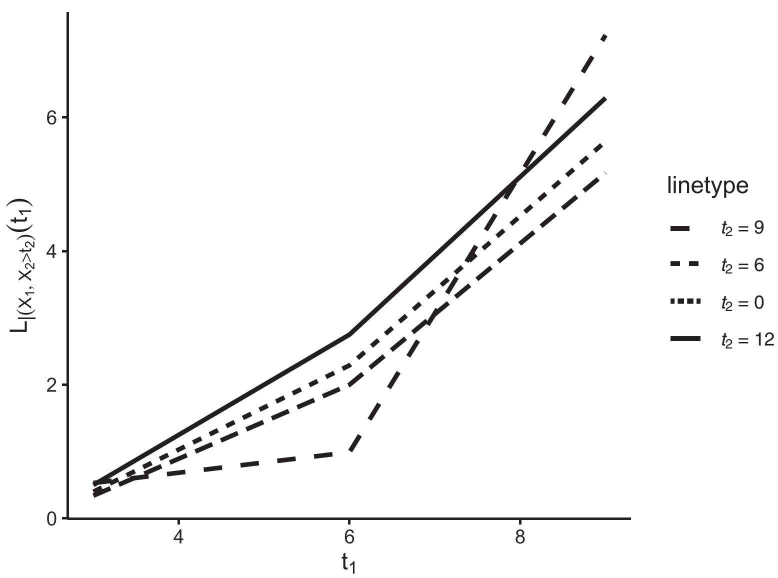

Figure 4. Plot of $\hat{L}_{X_1|X_2>t_2}(t_1)$ for different values of t_2.

$$\hat{L}_{X_i|X_j>t_j}(t_i) = -\frac{t_i\{N_S(t_j) - N_S(t_j + \Delta t_j)\}}{N_S(t_j)\Delta t_j(\log N_S(t_j) - \log N)}, i = 1, 2.$$

We compute now $\hat{L}_{X_1|X_2>t_2}(t_1)$. We arbitrarily fixed the second recurrence time of the tumor (t_2) at 0, 6, 9 and 12 respectively, and then estimated the conditional AI function of first recurrence times (t_1). The computed values of $\hat{L}_{X_1|X_2>t_2}(t_1)$ are displayed in Table 2, which also provides the empirical estimates of conditional survival and hazard rate functions, $\hat{\bar{F}}_{X_1|X_2>t_2}(t_1)$ and $\hat{h}_{X_1|X_2>t_2}(t_1)$ respectively.

To ensure the monotonicity of conditional AI function, we plot the function for different values of t_2. The estimates of conditional AI function for the data are plotted in Figure 4, where the dotted line, the dotdashed line, the dashed line and the longdashed line are plotted for $t_2 = 0, 6, 9, 12$ respectively. It is evident from Figure 4 that irrespective of second occurrence time of the tumor, the conditional AI functions of first occurrence times show an increasing trend, which indicates a faster ageing in the first occurrence times of the tumor.

Acknowledgements

The first author wish to thank the support of the University Grants Commission, Government of India, under the Special Assistance Program. We wish to thank the referees for constructive comments and valuable suggestions.

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