On Exponential-Weibull Distribution Useful in Reliability and Survival Analysis

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Abstract

In this paper, mixture of Exponential and Weibull distributions is considered for modelling real lifetime data. The basic mathematical properties including moments, generating functions, order statistics etc are derived. We obtain the reliability of stress-strength model. The maximum likelihood method is performed to estimate the parameters and a simulation study is conducted to validate the maximum likelihood estimators. The model is fitted to a real data set.

Keywords: Failure rate function, Reliability, Survival Analysis, Stress-Strength

I. Introduction

Lifetime distributions have a significant role in Reliability theory and survival analysis. Exponential, Gamma, Weibull and log-Normal distributions are some of the distributions commonly used for modeling lifetime data. Exponential and Weibull distributions are more popular than Gamma and log-Normal because the survival function of Gamma and log-Normal distributions doesn't have a closed form.

Mixture distributions have been used widely in reliability and survival analysis studies recently. It has been getting great attention, since mixture models are more appropriate and multiple causes of failure can be simultaneously modeled. Due to high flexibility, survival mixture models are better choice to analyze the reliability or survival data in situations when the data are believed to be heterogeneous and a single parametric distribution may not be sufficient to analyze the data.

There are several mixture distributions and generalizations of existing distributions, available in literature. A study on Generalized Lindley distribution useful in reliability study is given by Nadarajah et.al (2011). Nadarajah and Gupta (2007) studied on Exponentiated Gamma distribution with application to drought data. Mustafa et.al (2016) proposed Weibull Generalized Exponential Distribution. Gupta and Kundu (2001) studied on Exponentiated Exponential family as an alternative to gamma and Weibull. A detailed study on Statistical Models and Methods for Lifetime Data can be seen in Lawless (2003). Chacko et. al (2018) proposed Weibull-Lindly Distribution for modeling a bathtub shaped failure rate data.

In section 2, the mixture of Exponential and Weibull distributions is considered. The failure rate or hazard rate function is given in section 3. Moments are given in section 4 and generating functions are given in section 5. Conditional moments are given in section 6 and Quantile function is given in section 7. Mean deviation is given in section 8 and distributions of order statistics are given in section 9. Bonferroni and Lorenz Curves are given in section 10. Reliability in stress-strength model is given in section 11. Estimation of parameters using maximum likelihood estimation method is described in section 12. Simulation study and real data analysis are given in section 13 and 14 respectively. Conclusions are given in last section.

2. Exponential-Weibull distribution

Here we consider the mixture of two lifetime distributions, namely Exponential and Weibull distributions. The cumulative distribution function (cdf) of mixture of Exponential and Weibull distribution can be represented as

$$F(x) = \theta F_E(x) + (1-\theta)F_W(x),$$

where $\theta = \lambda/(1+\lambda)$, $\lambda > 0$, $F_E(x) = 1 - e^{-\lambda x}, \lambda > 0, x > 0$, the cdf of Exponential distribution with scale parameter λ and $F_W(x) = 1 - e^{-(\lambda x)^\alpha}, \lambda > 0,$, $\alpha > 0$, x>0, the cdf of Weibull distribution with scale parameter λ and shape parameter α. The mixture of Exponential and Weibull distributions can be written as,

$$F(x) = 1 - \frac{\lambda e^{-\lambda x} + e^{-(\lambda x)^\alpha}}{1+\lambda}. \tag{1}$$

The corresponding probability density function(pdf) is

$$f(x) = \frac{\lambda^2 e^{-\lambda x} + \alpha \lambda^\alpha x^{\alpha-1} e^{-(\lambda x)^\alpha}}{1+\lambda}. \tag{2}$$

Let X be a random variable, then we say that X has a 'Exponential-Weibull distribution' (EW(α, λ)) with scale parameter α and shape parameter λ, if it has the pdf (2).

The reliability function of the EW(α, λ) distribution is

$$\overline{F}(x) = \frac{\lambda e^{-\lambda x} + e^{-(\lambda x)^\alpha}}{1+\lambda},$$

and corresponding failure rate or hazard rate function is

$$h(x) = \frac{\lambda^2 e^{-\lambda x} + \alpha \lambda^\alpha x^{\alpha-1} e^{-(\lambda x)^\alpha}}{\lambda e^{-\lambda x} + e^{-(\lambda x)^\alpha}}. \tag{3}$$

The mode for the mixture model EW(α, λ) can be found by solving the derivative of the (2)

$$f'(x) = \frac{-\lambda^3 e^{-\lambda x} - \alpha^2 \lambda^{2\alpha} x^{2\alpha-2} e^{-(\lambda x)^\alpha} + \alpha(\alpha-1)\lambda^\alpha x^{\alpha-2} e^{-(\lambda x)\alpha}}{1+\lambda} \tag{4}$$

By solving (4), we observe that the mixture model EW(α, λ) is unimodal. Figure 1 shows the pdf of EW(α, λ) for various choices of parameters.

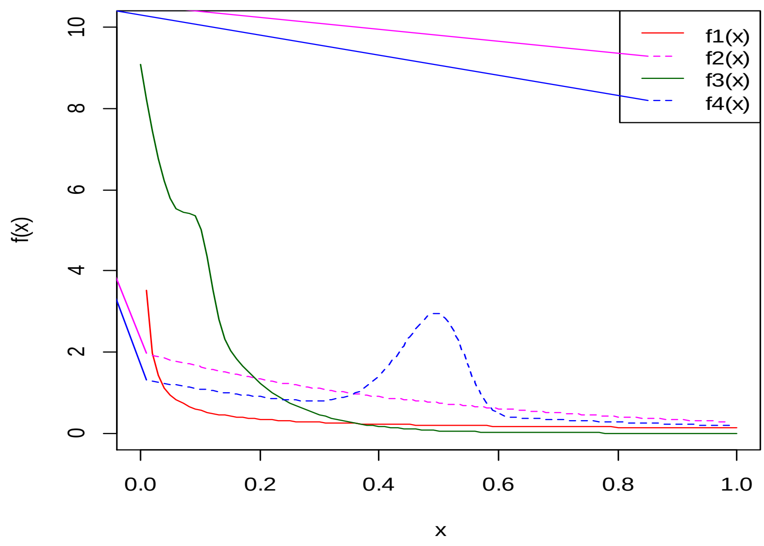

Figure 1: pdf f(α, λ) of EW(α, λ) for values of parameters

$f_1(x) = f(0.2, 0.4)$, $f_2(x) = f(1, 2)$, $f_3(x) = f(5, 10)$ and $f_4(x) = f(10, 2)$

3. Failure Rate Function

The failure rate or hazard rate function of the mixture EW(α, λ) distribution is given as follows:

$$h(x) = \frac{\lambda^2 e^{-\lambda x} + \alpha\lambda^{\alpha} x^{\alpha-1} e^{-(\lambda x)^{\alpha}}}{\lambda e^{-\lambda x} + e^{-(\lambda x)^{\alpha}}}, \lambda > 0, \alpha > 0, x > 0.$$

EW(α, λ) distribution has increasing, decreasing, upside-down bathtub shape behaviors for its failure rate. When α = 1; h(x) = λ, a constant, i.e. it has Exponential distribution with lack of memory property. From (3) $\lim_{x\to 0} h(x) = \frac{\lambda^2}{\lambda+1}$, a constant and $\lim_{x\to\infty} h(x) = 0$. Figure 2 shows the failure function of EW(α, λ) distribution with various choice of parameters. These shapes of failure function show that EW(α, λ) distribution fit in with both monotonic and non-monotonic behaviors which are more likely to be come across when dealing with lifetime data.

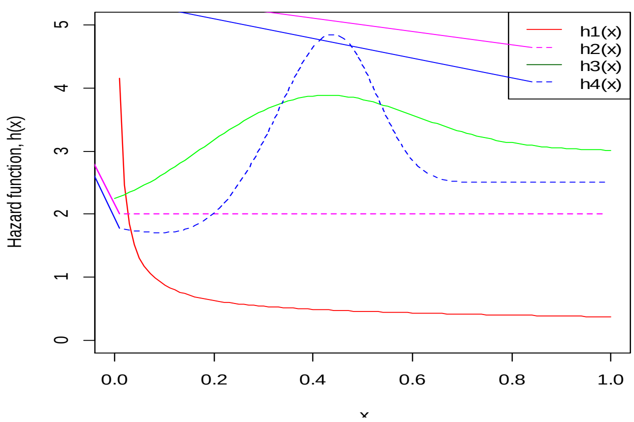

Figure 2: Failure function of EW(α, λ) distribution for various choice of parameters

h1(x) = h(0.2, 0.6), h2(x) = h(1, 2), h3(x) = (2, 3) and h4(x) = (4, 2.5) for h(α, λ)

4. Moments

The r[th] raw moment of the EW(α, λ) distribution with pdf in (2) is given by,

$$\mu_r = E(X^r) = \int_0^\infty x^r f(x)dx$$

$$= \int_0^\infty x^r \left(\frac{\lambda^2 e^{-\lambda x} + \alpha\lambda^\alpha x^{\alpha-1} e^{-(\lambda x)^\alpha}}{1+\lambda} \right) dx$$

$$= \frac{\lambda r! + \Gamma\left(1+\frac{r}{\alpha}\right)}{(1+\lambda)\lambda^r}.$$

The first four raw moments are,

$$E(X) = \frac{\lambda + \Gamma\left(1+\frac{1}{\alpha}\right)}{(1+\lambda)\lambda},\ E(X^2) = \frac{2\lambda + \Gamma\left(1+\frac{2}{\alpha}\right)}{(1+\lambda)\lambda^2},\ E(X^3) = \frac{6\lambda + \Gamma\left(1+\frac{3}{\alpha}\right)}{(1+\lambda)\lambda^3}$$ and

$$E(X^4) = \frac{24\lambda + \Gamma\left(1+\frac{4}{\alpha}\right)}{(1+\lambda)\lambda^4}.$$

The variances of EW(α, λ) distribution is given by;

$$Var(X) = \frac{(1+\lambda)\left(2\lambda + \Gamma\left(1+\frac{2}{\alpha}\right)\right) - \left(\lambda + \Gamma\left(1+\frac{1}{\alpha}\right)\right)^2}{(1+\lambda)^2 \lambda^2}.$$

Central moments can be obtained using raw moments.

5. Generating Functions

Let X be a random variable with probability density function (2). Its moment generating function (mgf) is given by,

$$E(e^{tx}) = \int_0^\infty e^{itx} \left(\frac{\lambda^2 e^{-\lambda x} + \alpha\lambda^\alpha x^{\alpha-1} e^{-(\lambda x)^\alpha}}{1+\lambda} \right) dx$$

$$= \frac{\lambda^2}{(1+\lambda)(\lambda - t)} + \frac{1}{1+\lambda} \sum_{n=0}^{\infty} \frac{t^n}{\lambda^n n!} \Gamma\left(1+\frac{n}{\alpha}\right), \alpha \geq 1,\ t < \lambda.$$

The characteristic function (cf) of X is $\phi(t) = E\left(e^{itx}\right)$, which is

$$E(e^{itx}) = \int_0^\infty e^{itx} \left(\frac{\lambda^2 e^{-\lambda x} + \alpha\lambda^\alpha x^{\alpha-1} e^{-(\lambda x)^\alpha}}{1+\lambda} \right) dx$$

$$= \frac{\lambda^2}{(1+\lambda)(\lambda - it)} + \frac{1}{1+\lambda} \sum_{n=0}^{\infty} \frac{(it)^n}{\lambda^n n!} \Gamma\left(1+\frac{n}{\alpha}\right).$$

The cumulant generating function (cgf) of X is given by,

$$K_X(t) = \log\phi_X(t)$$
$$= \log\left(\frac{1}{1+\lambda}\right) + \log\left(\frac{\lambda^2}{\lambda - it} + \sum_{n=0}^{\infty}\frac{(it)^n}{\lambda^n n!}\Gamma\left(1+\frac{n}{\alpha}\right)\right).$$

6. Conditional Moments

The conditional expectation for the EW(α, λ) distribution is given by,

$$E(X^n / X > x) = \frac{\int_x^{\infty} x^n \left\{\frac{\lambda^2 e^{-\lambda x} + \alpha\lambda^\alpha x^{\alpha-1} e^{-(\lambda x)^\alpha}}{1+\lambda}\right\} dx}{\overline{F}(X)}$$
$$= \frac{1+\lambda}{\lambda e^{-\lambda x} + e^{-(\lambda x)^\alpha}}\left\{\frac{\lambda^2\Gamma(n+1, \lambda x)}{(1+\lambda)\lambda^{n+1}} + \frac{\Gamma\left(\frac{n}{\alpha}+1, (\lambda x)^\alpha\right)}{(1+\lambda)\lambda^n}\right\}$$
$$= \frac{1}{\lambda e^{-\lambda x} + e^{-(\lambda x)^\alpha}}\left\{\frac{\Gamma(n+1, \lambda x)}{\lambda^{n-1}} + \frac{\Gamma\left(\frac{n}{\alpha}+1, (\lambda x)^\alpha\right)}{\lambda^n}\right\}.$$

In particular,

$$E(X / X > x) = \frac{1}{\lambda e^{-\lambda x} + e^{-(\lambda x)^\alpha}}\left\{\Gamma(2, \lambda x) + \frac{\Gamma\left(\frac{1}{\alpha}+1, (\lambda x)^\alpha\right)}{\lambda}\right\},$$

$$E(X^2 / X > x) = \frac{1}{\lambda e^{-\lambda x} + e^{-(\lambda x)^\alpha}}\left\{\frac{\Gamma(3, \lambda x)}{\lambda} + \frac{\Gamma\left(\frac{2}{\alpha}+1, (\lambda x)^\alpha\right)}{\lambda^2}\right\},$$

$$E(X^3 / X > x) = \frac{1}{\lambda e^{-\lambda x} + e^{-(\lambda x)^\alpha}}\left\{\frac{\Gamma(4, \lambda x)}{\lambda^2} + \frac{\Gamma\left(\frac{3}{\alpha}+1, (\lambda x)^\alpha\right)}{\lambda^3}\right\}, \quad and$$

$$E(X^4 / X > x) = \frac{1}{\lambda e^{-\lambda x} + e^{-(\lambda x)^\alpha}}\left\{\frac{\Gamma(5, \lambda x)}{\lambda^3} + \frac{\Gamma\left(\frac{4}{\alpha}+1, (\lambda x)^\alpha\right)}{\lambda^4}\right\}.$$

7. Quantile Function

The p[th] quantile say Q(p), p ϵ (0,1) is defined by Q(p) = p. Let X be a EW random variable with pdf (2), then its quantile function is the root of the equation,

$$1-\frac{\lambda e^{-\lambda Q(p)}+e^{-(\lambda Q(p))^{\alpha}}}{1+\lambda}=p$$

$$(1-p)(1+\lambda)=\lambda e^{-\lambda Q(p)}+e^{-(\lambda Q(p))^{\alpha}}$$

8. Mean Deviation

The average amount of scatter in a population from either the mean or the median is termed as mean deviation. The mean deviation about mean and, mean deviation about median are defined by,

$$\delta_1(X)=\int_0^{\infty}|x-\mu|\,f(x)dx \quad \text{and}$$

$$\delta_2(X)=\int_0^{\infty}|x-M|\,f(x)dx,$$

respectively, where μ and M are mean and median respectively. $\delta_1(X)$ and $\delta_2(X)$ can be calculated using the formulae,

$$\delta_1(X)=2\mu F(\mu)-2\mu+2\int_{\mu}^{\infty}xf(x)dx$$

$$\text{and}\quad \delta_2(X)=-\mu+2\int_{M}^{\infty}xf(x)dx \text{ respectively.}$$

The mean deviation about mean of the EW(α, λ) distribution is,

$$\delta_1(X)=2\mu\left(1-\frac{\lambda e^{-\lambda\mu}+e^{-(\lambda\mu)^{\alpha}}}{1+\lambda}\right)-2\mu+2\int_{\mu}^{\infty}x\left(\frac{\lambda^2e^{-\lambda x}+\alpha\lambda^{\alpha}x^{\alpha-1}e^{-(\lambda x)^{\alpha}}}{1+\lambda}\right)dx$$

$$=-\frac{2\left(\lambda^2\mu e^{-\lambda\mu}+\lambda\mu e^{-(\lambda\mu)^{\alpha}}\right)}{\lambda(1+\lambda)}+\frac{2\left(\lambda\mu e^{-\lambda\mu}+e^{-(\lambda\mu)}\right)}{1+\lambda}+\frac{2\Gamma\left(\frac{1}{\alpha}+1,(\lambda\mu)^{\alpha}\right)}{\lambda(1+\lambda)}$$

$$=\frac{2\left(e^{-\lambda\mu}-\mu e^{-(\lambda\mu)^{\alpha}}\right)}{1+\lambda}+\frac{2\Gamma\left(\frac{1}{\alpha}+1,(\lambda\mu)^{\alpha}\right)}{\lambda(1+\lambda)},$$

where $\gamma(a,x)=\int_0^x t^{a-1}e^{-t}dt$ and $\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt$ are denotes the incomplete gamma functions. The mean deviation about median of the EW(α, λ) distribution is,

$$\delta_2(X)=-\mu+2\int_M^\infty x\left(\frac{\lambda^2 e^{-\lambda x}+\alpha\lambda^\alpha x^{\alpha-1}e^{-(\lambda z)^\alpha}}{1+\lambda}\right)$$

$$=-\mu+\frac{2}{\lambda(1+\lambda)}\left(\lambda^2 Me^{-\lambda M}+\lambda e^{-\lambda M}+\Gamma\left(\frac{1}{\alpha}+1,(M\lambda)^\alpha\right)\right).$$

9. Order Statistics

Let X_1, X_2,.....X_n be a random sample of EW(α, λ) distribution with cdf and pdf as in (1) and (2) respectively. Their corresponding order statistics is denoted by $X_{(1)} < X_{(2)} < X_{(3)} \cdots\cdots < X_{(n)}$. The pdf and cdf of the r[th] order statistic are,

$$f_{X_{(r)}}(x;\alpha,\lambda)=\frac{n!}{(r-1)!(n-r)!}f(x;\alpha,\lambda)F^{r-1}(x;\alpha,\lambda)\overline{F}^{n-r}(x;\alpha,\lambda)$$

$$=\frac{n!}{(r-1)!(n-r)!}\sum_{l=0}^{n-r}\binom{n-r}{l}(-1)^l F^{r-1+l}(x;\alpha,\lambda)f(x;\alpha,\lambda)$$

and

$$F_{X_{(r)}}=\sum_{j=r}^{n}\binom{n}{j}F^j(x;\alpha,\lambda)\{1-F(x;\alpha,\lambda)\}^{n-j}$$

$$=\sum_{j=r}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l F^{j+l}(x;\alpha,\lambda),$$ respectively for r = 1,2,3..n.

The pdf of $X_{(r)}$ of EW(α, λ) distribution is ,

$$f_{X_{(r)}}=\frac{n!\left(\lambda^2 e^{-\lambda x}+\alpha\lambda^\alpha x^{\alpha-1}e^{-(\lambda x)^\alpha}\right)^{n-r}}{(r-1)!(1+\lambda)}\sum_{l=0}^{n-r}\frac{(-1)^l}{l!(n-r-l)!}\left\{1-\frac{\lambda e^{-\lambda x}+e^{-(\lambda x)^\alpha}}{1+\lambda}\right\}^{r-1+l}$$

and corresponding cdf is

$$F_{X_{(r)}}(x;\alpha,\lambda)=\sum_{j=r}^{n}\sum_{l=o}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l\left\{1-\frac{\lambda e^{-\lambda x}+e^{-(\lambda x)^\alpha}}{1+\lambda}\right\}^{j+l}.$$

The pdf of 1[st] order statistic is

$$f_{X_{(1)}}=\frac{n!}{(1+\lambda)}\left(\lambda^2 e^{-\lambda x}+\alpha\lambda^\alpha x^{\alpha-1}e^{-(\lambda x)^\alpha}\right)\sum_{l=0}^{n-1}\frac{(-1)^l}{l!(n-1-r)!}\left\{1-\frac{\lambda e^{-\lambda x}+e^{-(\lambda x)^\alpha}}{1+\lambda}\right\}^l$$

and the pdf of n[th] order statistic is

$$f_{X_{(n)}}=\frac{n}{(1+\lambda)^n}\left(\lambda^2 e^{-\lambda x}+\alpha\lambda^\alpha x^{\alpha-1}e^{-(\lambda x)^\alpha}\right)\left(1+\lambda-\lambda e^{-\lambda x}-e^{-(\lambda x)^\alpha}\right).$$

The distribution of order statistics can be used for obtaining reliability of series or parallel system.

10. Bonferroni & Lorenz Curves

Bonferroni and Lorenz curves are the popular tools for analyzing data emerging in Economics and Reliability. They are the fundamental tool for income analysis. The Bonferroni and Lorenz curves are defined by

$$B(p)=\frac{1}{p\mu}\int_0^q xf(x)dx$$

and

$$L(p)=\frac{1}{\mu}\int_0^q xf(x)dx$$

respectively, or correspondingly by

$$B(p)=\frac{1}{p\mu}\int_0^p F^{-1}(x)dx$$

and

$$L(p)=\frac{1}{\mu}\int_0^p F^{-1}(x)dx$$

respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$. For a random variable X with pdf (2), the Bonferroni and Lorenz curves are

$$B(p)=\frac{1}{p\mu(1+\lambda)}\left\{\left(1-(\lambda q+1)e^{-\lambda q}\right)+\frac{\Gamma\left(\frac{1}{\alpha}+1,(\lambda q)^{\alpha}\right)}{\lambda}\right\}$$

and

$$L(p)=\frac{1}{\mu(1+\lambda)}\left\{\left(1-(\lambda q+1)e^{-\lambda q}\right)+\frac{\Gamma\left(\frac{1}{\alpha}+1,(\lambda q)^{\alpha}\right)}{\lambda}\right\}$$

respectively.

11. Stress-Strength Reliability

Let X_1 and X_2 be two independent random variables, where X_1 represents the strength and X_2 represents the stress. Suppose X_1 and X_2 follows EW distribution, with parameters (α_1, λ_1) and (α_2, λ_2) respectively. Then the system reliability is

$$R = P[X_1 > X_2] = \int_0^\infty f_1(x)F_2(x)dx$$

$$= \int_0^\infty \left(\frac{\lambda_1^2 e^{-\lambda_1 x} + \alpha_1 \lambda_1^{\alpha_1} x^{\alpha_1 - 1} e^{-(\lambda_1 x)^{\alpha_1}}}{1+\lambda_1} \right)\left(1 - \frac{\lambda_2 e^{-\lambda_2 x} + e^{-(\lambda_2 x)^{\alpha_2}}}{1+\lambda_2} \right) dx$$

$$= \frac{1}{(1+\lambda_1)(1+\lambda_2)} \int_0^\infty \left(\lambda_1^2 e^{-\lambda_1 x} + \alpha_1 \lambda_1^{\alpha_1} x^{\alpha_1 - 1} e^{-(\lambda_1 x)^{\alpha_1}} \right)\left(1 + \lambda_2 - \lambda_2 e^{-\lambda_2 x} - e^{-(\lambda_2 x)^{\alpha_2}} \right) dx$$

$$= \frac{1}{(1+\lambda_1)(1+\lambda_2)} \left\{ \begin{array}{l} \lambda_1(1+\lambda_2) - \dfrac{\lambda_1^2 \lambda_2}{(\lambda_1 + \lambda_2)} + \displaystyle\sum_{i=0}^{\infty} \frac{(-1)^{i+1} \lambda_1^2 \lambda_2^{\alpha_2 i}}{i!} \frac{\Gamma(\alpha_2 i + 1)}{(\lambda_1)^{\alpha_2 i + 1}} + (1+\lambda_2) + \\ \displaystyle\sum_{i=0}^{\infty} \frac{(-1)^{i+1} \alpha_1 \lambda_1^{\alpha_1 (i+1)}}{i! \, \lambda_2^{\alpha_1 i + \alpha_2 - 1}} \Gamma(\alpha_1 i + \alpha_2) + \sum_{i=0}^{\infty} \frac{(-1)^{i+1} \left(\dfrac{\lambda_2}{\lambda_1} \right)^{\alpha_2 i}}{i!} \Gamma\left(\frac{\alpha_2}{\alpha_1} i + 1 \right) \end{array} \right\}.$$

12. Estimation of Parameters

If $X_1, X_2, \ldots X_n$ is a random sample from EW(α, λ), then its likelihood function is

$$L(\bar{x}; \alpha, \lambda) = \prod_{i=1}^{n} \frac{\lambda^2 e^{-\lambda x_i} + \alpha \lambda^\alpha x_i^{\alpha-1} e^{-(\lambda x)^\alpha}}{1+\lambda}.$$

It is familiar that maximum likelihood estimate of the parameters is the value of the parameter which maximizes the likelihood function. The partial derivatives of log L with respect to unknown parameters α and λ are,

$$\frac{\partial \log L(\bar{x}; \alpha, \lambda)}{\partial \alpha} = \sum_{i=1}^{n} \frac{\lambda^\alpha x_i^{\alpha-1} e^{-(\lambda x_i)^\alpha} \left\{ \alpha \log \alpha (\alpha - 1) - \alpha \lambda^\alpha x_i^\alpha \log(\lambda x_i) + 1 \right\}}{\lambda^2 e^{-\lambda x_i} + \alpha \lambda^\alpha x_i^{\alpha-1} e^{-(\lambda x_i)^\alpha}}$$

, and

$$\frac{\partial \log L(\bar{x}.; \alpha, \lambda)}{\partial \lambda} = \sum_{i=1}^{n} \frac{\lambda e^{-\lambda x_i} [(1+\lambda)(2 - x_i \lambda) - \lambda] + \alpha (\lambda x_i)^{\alpha-1} e^{-(\lambda x_i)^\alpha} [\alpha (1+\lambda)(1 - (\lambda x_i)^\alpha) - \lambda]}{(1+\lambda)\left(\lambda^2 e^{-\lambda x_i} + \alpha \lambda^\alpha x_i^{\alpha-1} e^{-(\lambda x_i)^\alpha}\right)}$$

By equating the above equations to zero we get two non-linear equations.

$$\sum_{i=1}^{n} \frac{\lambda^\alpha x_i^{\alpha-1} e^{-(\lambda x_i)^\alpha} \left\{ \alpha \log \alpha (\alpha - 1) - \alpha \lambda^\alpha x_i^\alpha \log(\lambda x_i) + 1 \right\}}{\lambda^2 e^{-\lambda x_i} + \alpha \lambda^\alpha x_i^{\alpha-1} e^{-(\lambda x_i)^\alpha}} = 0 \quad \text{and}$$

$$\sum_{i=1}^{n} \frac{\lambda e^{-\lambda x_i} [(1+\lambda)(2 - x_i \lambda) - \lambda] + \alpha (\lambda x_i)^{\alpha-1} e^{-(\lambda x_i)^\alpha} [\alpha (1+\lambda)(1 - (\lambda x_i)^\alpha) - \lambda]}{(1+\lambda)\left(\lambda^2 e^{-\lambda x_i} + \alpha \lambda^\alpha x_i^{\alpha-1} e^{-(\lambda x_i)^\alpha}\right)} = 0.$$

A solution of the non-linear equation gives the maximum likelihood estimates of α and λ. The normal equations cannot be solved by analytically, so that Newton-Raphson's iteration method or any other numerical approximation methods are required. Since the maximum likelihood estimator (MLE)s cannot be expressed in explicit forms, we consider their asymptotic distribution and confidence interval for $\alpha > 0$ and $\lambda > 0$. For large samples, the MLE $\left(\hat{\alpha}, \hat{\lambda}\right)$ of (α, λ) is asymptotically normal with mean zero and variance covariance matrix I^{-1}, where

$$I=\begin{bmatrix} E\left(-\frac{\partial^2 \log L}{\partial \alpha^2}\right) & E\left(-\frac{\partial^2 \log L}{\partial \alpha \partial \lambda}\right) \\ E\left(-\frac{\partial^2 \log L}{\partial \lambda \partial \alpha}\right) & E\left(-\frac{\partial^2 \log L}{\partial \lambda^2}\right) \end{bmatrix}.$$

We can derive the approximate (1- δ)100% confidence interval of the parameters α and λ. By using variance covariance matrix, the confidence intervals are $\hat{\alpha} \pm Z_{\frac{n}{2}} \sqrt{\operatorname{var}\left(\hat{\alpha}\right)}$ and $\hat{\lambda} \pm Z_{\frac{n}{2}} \sqrt{\operatorname{var}\left(\hat{\lambda}\right)}$ where $Z_{\frac{n}{2}}$ is the upper $100\left(\frac{\delta}{2}\right)^{th}$ percentile of the standard normal distribution.

13. Simulation Study

Here we performed a simulation study to validate the maximum likelihood estimation procedure for EW(α, λ) distribution using Newton-Raphson method. For this purpose, we generated samples of sizes 25, 50, 100, 500, 1000 for different combinations of α and λ. We computed the maximum likelihood estimates for each sample and repeated this process thousand times then computed the bias and mean square error(MSE)s of the parameter estimates.

The simulation is conducted for the selected values of α and λ. Here we considered $\alpha = 1$, $\lambda = 1$; $\alpha = 1$, $\lambda = 1.2$; $\alpha = 2$, $\lambda = 1.5$ and $\alpha = 3$, $\lambda = 1.5$ as initial parameter values. The Tables 1, 2, 3 and 4 gives the values of the estimates, bias and MSEs of the corresponding parameters. From the tables, it can be seen that, as sample size increases the bias and MSE of the estimates decreases.

Table 1: *Estimates, Bias and MSE for α = 1 and λ = 1*

N	Estimates	Bias	MSE
25	α = 1.293439 λ = 1.038358	0.2934385 0.03835775	0.81239 0.05436978
50	α = 1.087168 λ = 1.019775	0.08716756 0.0197749	0.0942621 0.0244342
100	α = 1.043303 λ = 1.009724	0.08716756 0.0197749	0.0942621 0.0244342
500	α = 1.011771 λ = 1.000085	0.01177124 8.461249e-05	0.005638945 0.002088029
1000	α = 1.006076 λ = 1.000816	0.006076378 0.0008162859	0.002621145 0.001182026

Table 2 *Estimates, Bias and MSE for $\alpha = 1$ and $\lambda = 1.2$*

N	Estimates	Bias	MSE
25	$\alpha =$ 1.275423 $\lambda =$ 1.250335	0.2754229 0.05033506	0.7921809 0.07960997
50	$\alpha =$ 1.123425 $\lambda =$ 1.227627	0.1234254 0.02762709	0.1598412 0.03496351
100	$\alpha =$ 1.054752 $\lambda =$ 1.203634	0.1234254 0.02762709	0.1598412 0.03496351
500	$\alpha =$ 1.008541 $\lambda =$ 1.198176	0.008540759 -0.001823803	0.00660528 0.003205042
1000	$\alpha =$ 1.006074 $\lambda =$ 1.202171	0.006073959 0.002171475	0.003285917 0.001605966

Table 3: *Estimates, Bias and MSE for $\alpha = 2$ and $\lambda = 1.5$*

N	Estimation	Bias	MSE
25	$\alpha =$ 2.737266 $\lambda =$ 1.53967	0.7372658 0.03966983	6.287694 0.07501019
50	$\alpha =$ 2.304472 $\lambda =$ 1.512797	0.3044722 0.01279734	1.261294 0.03309165
100	$\alpha =$ 2.122918 $\lambda =$ 1.508089	0.122918 0.008089073	0.3660359 0.01584789
500	$\alpha =$ 2.029113 $\lambda =$ 1.50079	0.0291133 0.0007895803	0.04937387 0.003182324
1000	$\alpha =$ 2.012721 $\lambda =$ 1.500226	0.01272087 0.0002263662	0.02528183 0.001405722

Table 4: *Estimates, Bias and MSE for $\alpha = 3$ and $\lambda = 1.5$*

N	Estimates	Bias	MSE
25	$\alpha =$ 4.016253 $\lambda =$ 1.514296	1.016253 0.01429607	10.68158 0.04879727
50	$\alpha =$ 3.388975 $\lambda =$ 1.508105	0.3889748 0.008105423	1.99424 0.02228412
100	$\alpha =$ 3.177544 $\lambda =$ 1.504892	0.1775441 0.004891745	0.7521783 0.01055644
500	$\alpha =$ 3.019977 $\lambda =$ 1.498705	0.01997658 -0.00129502	0.09545959 0.00199238
1000	$\alpha =$ 3.007664 $\lambda =$ 1.500396	0.007664266 0.0003962086	0.04595611 0.001004574

14. Data Analysis

In this section, we considered a real data set of survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). This data set is used to explain the supremacy of the EW(α, λ) distribution. The data are presented in Table 5:

Table 5: *Data set of survival times of 72 guinea pigs infected with virulent tubercle bacilli*

12 48 60 75 109 258 15 52 60 76 110 258 22 53 61 76 121 263 24 54 62 81 127 297 24 54 63 83 129 341 32 55 65 84 131 341 32 56 65 85 143 376 33 57 67 87 146 34 58 68 91 146 38 58 70 95 175 38 59 70 96 175 43 60 72 98 211 44 60 73 99 233

We fitted EW(α,λ) distribution to the given data set and compared the results with Exponential distribution and Weibull distribution. The comparison is carried out based on the values of Kolmogorov-Smirnov (K-S) statistic, p-value, log-likelihood value, Akaike information criterion (AIC) and Bayesian information criterion (BIC).

The Kolmogorov-Smirnov test is defined by:

$$D = \max_{1 \le i \le N}\left(F(Y_i) - \frac{i-1}{N}, \frac{i}{N} - F(Y_i) \right)$$

where F is the cdf of the distribution and N is the number of classes being ordered.

The AIC and BIC are defined by

$$AIC = 2k - 2\ln\left(\hat{L}\right)$$

and

$$BIC = \ln(n)k - 2\ln\left(\hat{L}\right)$$

where $\hat{L}$ denote the maximum value of the likelihood function and k is the number of parameters and n is the sample size. By comparing the values of K-S statistic, p-values, likelihood value, AIC and BIC, Table 6 shows that EW(α, λ) distribution has smallest K-S statistic value and largest p-value among them. The AIC and BIC values of EW(α, λ) distribution indicates that the amount of information lost by the model is less than that of Exponential and Weibull distributions. This points out that the proposed model provided a better fit as well as more precise estimates.

Table 6: *Parameter Estimates, log-likelihood, p-value, AIC and BIC values of model fitted*

MODEL	PARAMETER ESTIMATES	LOG LIKELIHOOD	K-S STATISTIC	P-VALUE	AIC	BIC
MEW	a=1.396574502 b=0.009055284	-397.1651	0.1459	0.09327	798.3302	802.8836
WEIBULL	a=1.393170797 b=0.009045585	-421.9635	0.14645	0.09114	847.927	852.4804
EXPONENTIAL	b= 0.01001842	-422.1097	0.2116	0.003168	846.2193	848.496

15. Conclusions

In this paper, the mixture of Exponential and Weibull distributions are considered. The failure rate or hazard rate function is given. Moments, generating functions, Conditional moments, Quantile function, Mean deviation, distributions of order statistics, and Bonferroni and Lorenz Curves are obtained. Reliability in stress-streangth model is computed. Method of estimation of parameters using maximum likelihood estimation method is described and a simulation study is conducted to validate the maximum likelihood estimation procedure. A real data analysis is given, which shows the mixture distribution, EW(α, λ) distribution is a better alternative in certain situations of reliability and survival analysis.

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