

DECLARATION

I hereby declare that the matter embodied in this thesis entitled '**Modelling and Analysis of Reliability Data using Bathtub shaped Failure rate Distributions**', submitted to the St. Thomas' College (Autonomous), Thrissur in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy in Statistics**, is the result of my studies and this has been composed by me under the guidance and supervision of Dr. Chacko V.M., Assistant Professor & Head of the Department, Department of Statistics, St. Thomas' College (Autonomous), Thrissur during 2017-2020.

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

Thrissur,

14th December 2020

Deepthi K.S.

I dedicate this work

to my parents who paved the way for me during their lifetime and

*to my beloved husband Mr. Praveen R., and my child for their unconditional love
and support.*

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