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CHAPTER 1

INTRODUCTION

1.1 Introduction

Reliability theory deals with the interdisciplinary use of probability, statistics and stochastic modeling, combined with engineering insights into the system design and the scientific understanding of the failure mechanisms. The study of reliability characteristics and performances requires a comprehensive understanding of many different concepts of the system. System reliability describes the probability of completing the mission with in a pre-specified time interval. An engineer need to find the probability of successful functioning of many engineering systems such as Airplanes, linear accelerators, power generation system etc. The computation of reliability or expected system performance is a problem for engineers and manufacturers. Traditional reliability theory is built on a statistical framework in which the system and its components can be in one of the two states such as functioning state or failed state. As a result, the system structure function is a binary function of binary variables. Reliability calculation is an important task for increasingly sophisticated technological systems.

The reliability of a unit (a system or a component) is also defined as the probability that a unit can perform satisfactorily for a specified period of time without failure. Proper modeling of lifetime with appropriate statistical distributions makes reliability computation easy. Moreover reliability and maintenance activities can be planned with the help of distribution of lifetime of the system. Identification of failure rate model is crucial in reliability analysis to select appropriate distribution for the given data. Many of the distributions available in literature is not sufficient for explain distributional properties and reliability analysis for the given data. So searching for more appropriate distributions for the given data is an open challenge among researchers. Most of the systems are subjected to certain type of stresses, so reliability computation of stress-strength models using different distributions is an important research problem. Moreover identification of failure rate model for the given data or transformed data makes the selection of distribution for the given data easy. While considering lifetime data, study on burn-in process is unavoidable to understand the length of time to burn-in.

1.2 Binary state system

In the binary state system, the components in the system are assumed to be in one of the two states, functioning or failed, see Barlow and Proschan (1975). For a system with n components, let x_i indicate the state of the i^{th} component, $i = 1, 2, \ldots, n$. That is,

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is functioning} \\ 0 & \text{if component } i \text{ is failed.} \end{cases}$$

for $i = 1, 2, \ldots, n$. Similarly, let Φ be a binary random variable indicating the state of the system

$$\Phi = \begin{cases} 1 & \text{if the system is functioning} \\ 0 & \text{if the system is failed.} \end{cases}$$

We assume that the state of the system is a function of the states of the components, so that we may write $\Phi = \Phi(\underline{x})$, structure function or structure, where $\underline{x} = (x_1, x_2, \ldots, x_n)$. For a series system,

$$\Phi(\underline{x}) = \prod_{i=1}^{n} x_i = \min(x_1, x_2, \ldots, x_n)$$

$$= \begin{cases} 1, & \text{if each component is functioning} \\ 0, & \text{if atleast one of the components is failed.} \end{cases}$$

For a parallel system,

$$\Phi(\underline{x}) = \coprod_{i=1}^{n} x_i = \max(x_1, x_2, \ldots, x_n)$$

$$= \begin{cases} 1, & \text{if atleast one component is functioning} \\ 0, & \text{if all components are failed} \end{cases}$$

where $\coprod_{i=1}^{n} = 1 - \prod_{i=1}^{n}(1 - x_i)$.

A system of components is coherent if its structure function $\Phi(\underline{x})$ is increasing in each component and each component is relevant.

The reliability of a system is given by $P(\Phi(\underline{x}) = 1) = h = E\Phi(\underline{x})$. Under the assumption of independence of components, we may represent system reliability as a function of component reliabilities, p_i, $i = 1, 2, \ldots, n$. That is $h = h(\underline{p})$, $\underline{p} = (p_1, p_2, \ldots, p_n)$. Accordingly the reliability function of series structure is $h(\underline{p}) = \prod_{i=1}^{n} p_i$ and the reliability function of parallel structure is $h(\underline{p}) = 1 - \prod_{i=1}^{n}(1 - p_i)$.

In Barlow and Proschan (1975), we find that "reliability is the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered". Generally the period of time intended is $[0, t]$. In many problems, we consider the life lengths of the components of the system, for the reliability analysis. By lifetime we mean the maximum period for which the unit can work satisfactorily whereas age of the unit is the time which it requires for the completion of a particular mission without failure. In general, life lengths are random variables and therefore lead us to a study of life distributions. The reliability of a fresh unit corresponding to a mission of duration t is, by definition, $\bar{F}(t) = P(T > t) = 1 - F(t)$, where F is the cumulative distribution function (cdf) of lifetime of the unit.

There is a reason to believe that in many applications, the lifetime of the system will be reduced beyond the specified age. That is, survival decreases as the system ages. If units have this behavior, the corresponding life distributions are called positive ageing distributions. To understand which real life distributions are appropriate in reliability data analysis, we need to consider a notion of ageing. Ageing can be conveniently defined in terms of the failure rate function.

1.2.1 Notion of ageing

The notion of ageing plays a significant role in reliability theory. Several classes of life distributions based on the notion of ageing have been studied and explored during the past several years. Most common ageing properties like increasing failure rate (IFR), decreasing failure rate (DFR), increasing failure rate average (IFRA), decreasing failure rate average (DFRA), new better than used (NBU), new worse than used (NWU), new better than used in expectation (NBUE), new worse than used in expectation (NWUE), increasing mean residual life (IMRL), decreasing mean residual life (DMRL) etc. are discussed by Barlow and Proschan (1975) and Deshpande et al. (1986). The discrete and continuous versions of these classes have become very common in literature. Concepts of ageing describes how a component or system improves or deteriorates with age. The most popular lifetime distributions such as Exponential, Weibull, Gamma, Rayleigh, Pareto and Gompertz have monotonic failure rate functions, see Lawless (1982). If we observe a constant failure rate pattern for a data, then Exponential distribution serves as very useful model for reliability analysis. The Poisson process has many direct and indirect applications in reliability, especially in formulating shock models. ‘No ageing’ means that the age of a component does not influence the distribution of the remaining life of the component or system. Positive ageing means adverse effect of age exist on the random residual life of the component or system whereas negative ageing means beneficial effect of age exist on the random residual life of the component or system. Positive ageing describes the situation where the residual lifetime tends to decrease, in some probabilistic sense, with increasing age of the component. This situation is common in reliability engineering, as increased wear and tear may worsen over time. Negative ageing, on the other hand, has the opposite effect during the rest of life. Negative ageing is also known as beneficial ageing. In other words, the residual lifetime is monotonic with respect to age. However, in many practical applications, the effect of age is initially beneficial but after a certain period of time, adverse indicating a ‘ware-out’ phase where age is positive. Certain lifetime data, for example, human mortality, machine life cycles and data from some biological and medical studies require non-monotonic shapes like bathtub shape or upside-down bathtub shape. Initially, the failure rate (death rate) of the newborn babies is very high especially in the first six months after birth, caused by deformities, heart dysfunctions or other infant diseases. Then, the risk of death decreases rapidly until it reaches its lowest level and remains approximately constant for a long period. At some point, during the ages between 50 and 80 the death risk increases over time. This kind of non-monotonic ageing phenomenon is often modeled using life distributions that display bathtub shaped failure rate (BFR). Model the lifetime data using a distribution with BFR is important in reliability analysis. An upside-down bathtub shape is analyzed by Efron (1988) in the context of head and neck cancer data, in which the failure rate initially increased, reached a maximum and then decreased before it finally stabilized because of a therapy.

Let X be a continuous non-negative random variable (r.v) representing the lifetime of a unit which is in operation. This unit may be a living organism, a mechanical component, a system of components etc. Now we describe the failure rate function or hazard function, see Barlow and Proschan (1975). Let $F(x)$ be the cdf of X, then the survival function of a fresh unit is $\bar{F}(x) = 1 - F(x) = P(X > x)$. Also let $F(x|t) = P(X > t + x|X > t)$ be the survival probability or reliability of a unit which has attained the age t. It can be seen that $\bar{F}(x|t) = \bar{F}(x+t)/\bar{F}(t)$. Note that this represents the survival function of a unit of age t, *i.e.*, the conditional probability that a unit of age t will survive for an extra x units of time. When $t = 0$, $\bar{F}_0(x) = \bar{F}(x)$ is the survival function of a new unit.

When the derivative $F'(t) = f(t)$ exists, where $f(t)$ be the probability density function (pdf), failure rate (hazard rate) of a component is defined as $r(t) = f(t)/\bar{F}(t)$, $\bar{F}(t) > 0$. This can also be written as $r(t) = \lim\limits_{\Delta \to 0} \frac{Pr(t \le X < t+\Delta | t \le X)}{\Delta}$. Thus for small Δ, $r(t)\Delta$ is approximately the probability of a failure occurring in $(t, t+\Delta]$ given no failure has occurred in $(0, t]$.

It follows that, if $r(t)$ exists, then $-\log\ \bar{F}(t) = \int_0^t r(x)\ dx$ represents the cumulative failure rate (cumulative hazard rate) which may be denoted by $H(t)$. Hence $\bar{F}(t) = \exp\left\{-\int_0^t r(x)\ dx\right\} = \exp\{-H(t)\}$.

Now we consider a unit which does not age stochastically, that is, probability distribution of the residual lifetime at age t of the unit does not depend on t. Hence $\bar{F}_t(x) = \bar{F}_0(x) \ \forall \ t, \ x > 0$. This is equivalent to $\bar{F}(t+x) = \bar{F}(t)\bar{F}(x) \ \forall \ t, \ x > 0$. It is well known that among the continuous survival functions only the exponential survival function $\bar{F}(x) = e^{-\lambda x}, \ x > 0, \ \lambda > 0$ satisfies the above equation, and this property is known as lack of memory property or no-ageing property of Exponential distribution in reliability theory.

We recall the definition of failure rate behaviors IFR, DFR, IFRA, DFRA, NBU, NWU, NBUE, NWUE, IMRL and DMRL (Barlow and Proschan (1975)).

Definition 1.2.1. *When the density exists, IFR (DFR) is equivalent to* $r(t) = f(t)/\bar{F}(t)$ *is increasing (decreasing) in* $t \geq 0$. *When* F *is not absolutely continuous,* F *is said to be IFR (DFR) distribution if* $\bar{F}(x|t)$ *is decreasing (increasing) in* $t, \ 0 \leq t < \infty$ *for each* $x > 0$. F *is IFR (DFR) iff* $-\log \bar{F}(t)$ *is convex (concave).*

Definition 1.2.2. F *is said to be IFRA (DFRA) if and only if* $\int_0^t r(x) \ dx/t$ *increasing (decreasing) in* $t \geq 0$ *equivalently* $-(1/t) \log \bar{F}(t)$ *is increasing (decreasing) in* $t \geq 0$ *(This is equivalent to* $-\log \bar{F}(t)$ *being a star-shaped function) equivalently* $\bar{F}(\alpha t) \geq \bar{F}^{\alpha}(t), \ 0 < \alpha < 1, \ t \geq 0$.

Definition 1.2.3. F *is said to be NBU (NWU) if* $\bar{F}(x|t) \leq (\geq) \bar{F}(x)$ *equivalently* $\bar{F}(x+t) \leq (\geq) \ \bar{F}(x)\bar{F}(t)$ *for* $x, \ t \geq 0$ *equivalently* $\log \bar{F}(x+t) \leq (\geq) \ \log \bar{F}(x) + \log \bar{F}(t)$ *equivalently* $\int_0^t r(u) \ du \leq (\geq) \int_x^{x+t} r(u) \ du$.

Definition 1.2.4. F *is said to be NBUE (NWUE) if* $\int_0^\infty \bar{F}(x|t) \ dx \leq (\geq) \ \mu$ *for* $t \geq 0$.

Definition 1.2.5. *F is said to be IMRL (DMRL) if $\mu(t) = \int_0^\infty \bar{F}(x|t)\,dx$ is increasing (decreasing) in t, i.e., $\mu(s) \geq \mu(t)$ for $0 \leq s \leq t$.*

We assume that the failure rate function $r(t)$ is a real-valued differentiable function $r(t) : R^+ \to R^+$. As usual, by increasing we mean non-decreasing and by decreasing, we mean non-increasing. $r(t)$ is said to be

1. strictly increasing if $r'(t) > 0$ for all t;
2. strictly decreasing if $r'(t) < 0$ for all t;
3. bathtub shaped if $r'(t) < 0$ for $t \in (0, t_o)$, $r'(t_o) = 0$, $r'(t) > 0$ for $t > t_o$;
4. upside-down bathtub shaped if $r'(t) > 0$ for $t \in (0, t_o)$, $r'(t_o) = 0$, $r'(t) < 0$ for $t > t_o$;
5. modified bathtub shaped if $r(t)$ is first increasing and then bathtub shaped;
6. roller-coaster shaped if there exist n consecutive change points $0 < t_1 < t_2 < \ldots < t_n < \infty$ such that in each interval $[t_{j-1}, t_j]$, $1 \leq j \leq n+1$, where $t_o = 0, t_{n+1} = \infty$, $r(t)$ is strictly monotone and it has opposite monotonicity in any two adjacent such intervals.

A class of life distributions that has received considerable attention is the class of BFR life distributions, see Rajarshi and Rajarshi (1988) for a systematic review. We say that F is BFR model, if failure rate decreases first, then remains constant for a period, and eventually increases over time. In other words, the failure rate function has bathtub shape. This corresponds to the three distinct phases of a unit: early life, useful life and wear out as shown in Figure 1.1. In the initial region that begins at time zero, product is characterized by a high but rapidly decreasing failure rate. This region is known as the early failure period (also referred to as infant mortality period). Next, the failure rate levels off and remain roughly constant for the majority of the life of the product. This long period of a constant failure rate is known as the useful period. Many systems spend most of their lifetimes operating in this flat portion of the bathtub curve. Finally, if unit remain in use long enough, the failure rate increase as materials wear out and degradation failures occur at an ever increasing rate. This is called the wear-out period (Rajarshi and Rajarshi (1988)).

Another important family of life distributions is known as the upside-down bathtub-shaped failure rate (UBFR) class. Chang (2000) proposed a UBFR model. We say that F is its upside-down bathtub shaped failure rate, its failure rate increases first, then remains constant for a period, and eventually decreases over time.

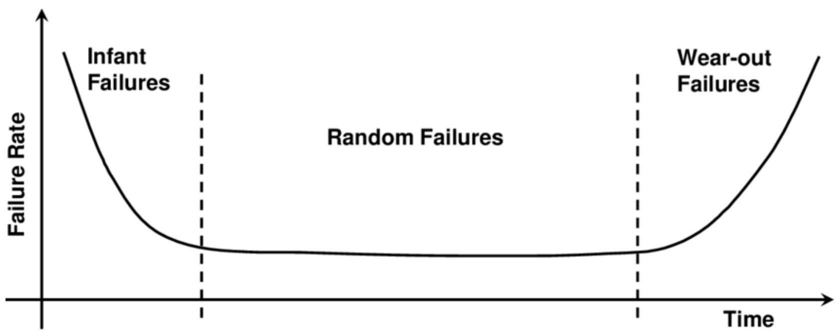

Figure 1.1: Bathtub failure rate curve

1.3 Failure Rates of Mixtures of Distributions

Mixture of distributions arise naturally in a number of reliability situations. For example, suppose a manufacturer produces $p100$ percent of a certain product in production line 1 and $(1-p)100$ percent in production line 2, $0 \leq p \leq 1$. Suppose, the life length of a unit produced in production line 1 has distribution F_1, where as the life length of a unit produced in production line 2 has distribution $F_2(\neq F_1)$. After production, units from both production lines will be allowed to campaign together, in such a way that outgoing lots consist of a random mixture of the output of the two production lines. Then, a unit selected at random from a lot would have life distribution $F = pF_1 + (1-p)F_2$, a mixture of the two underlying distributions. More generally, the distribution being mixed may be uncountably infinite in number. See Barlow and Proschan (1975) for more details.

Mixtures are important in burn-in procedures. The pdf of a mixture of two sub-populations with density functions f_1 and f_2 is $f(t) = pf_1(t) + (1-p)f_2(t),\ t \geq 0,\ 0 \leq p \leq 1$. Survival function of a mixture is also a mixture of the two survival functions, i.e., $\bar{F}(t) = p\bar{F}_1(t) + (1-p)\bar{F}_2(t)$. The mixture failure rate is $r(t) = \frac{pf_1(t)+(1-p)f_2(t)}{p\bar{F}_1(t)+(1-p)\bar{F}_2(t)}$ where $f_i(t),\ \bar{F}_i(t)$ are the pdf and survival function of the distribution having failure rate $r_i(t), i = 1, 2$, see Lai and Xie (2006) for more details.

Below Lai and Xie (2006) has given some examples in mixture failure rates.

Example 1.3.1. Consider two IFR Weibull distributions with pdfs $f_1(t) = 2t\exp\{-t^2\}$, $t > 0$ and $f_2(t) = 3t^2\exp\{-t^3\}$, $t > 0$. If $p = 0.5$, $r(t)$ is IFR.

The next example shows that a mixture of two IFR distributions results in a DFR distribution.

Example 1.3.2. Let $r_1(t) = 1 - \exp\{-5t\}$, $t > 0$, $r_2(t) = 6 - \exp\{-5t\}$, $t > 0$. We note that $r_1(t)$ strictly increases to 1 and $r_2(t)$ strictly increases to 6. However, if $p = 0.5$, $r(t)$ is DFR and strictly decreases to 1.

Example 1.3.3. Take $f_1(t) = \exp\{-t\}$, $t > 0$, pdf of exponential distribution, $f_2(t) = 16t\exp\{-4t\}$, $t > 0$, pdf of Gamma distribution with IFR property. Let $p = 0.5$. In this case, $r(t)$ is UBFR.

Example 1.3.4. Let $f_1(t) = 4\exp\{-4t\}$, $t > 0$, pdf of exponential distribution, $f_2(t) = t\exp\{-t\}$, $t > 0$, pdf of Gamma distribution with IFR property. Let $p = 0.5$. Then $r(t)$ is BFR.

Example 1.3.5. Consider two Weibull distributions, $f_1(t) = 2t\exp\{-t^2\}$, $t > 0$ and $f_2(t) = 4t^3\exp\{-t^4\}$, $t > 0$. Let $p = 0.5$; both $r_1(t)$ and $r_2(t)$ increases to ∞. The mixture failure rate $r(t)$ is BFR, see Jiang and Murthy (1998).

Example 1.3.6. Consider the mixture of two Gamma probability densities: $f(t) = pf_1(t) + (1-p)f_2(t)$ where $f_i(t) = \frac{\lambda^{\alpha_i} t^{\alpha_i}}{\Gamma(\alpha_i)} e^{-\lambda t}$, $t > 0$, α_i, $\lambda > 0$, $i = 1, 2$. Assuming $\alpha_1 < \alpha_2$, Glaser (1980) was able to determine the shape of the failure rate of the distribution in all cases except for one case: $\alpha_1 > 1$, $\alpha_2 - \alpha_1 > 0$ with $\alpha_1 - 1 < (\alpha_2 - \alpha_1 - 1)^2/4$. For this case, he conjectured that the mixture density is IFR.

Jiang and Murthy (1998) categorized the possible shapes of failure rate function for a mixture of two Weibull distributions. The mixture failure rate of two strictly IFR Weibull distributions with the same shape parameter can be either BFR or IFR. The asymptotic behavior of mixtures of exponentials has been studied by Clarotti and Spizzichino (1990). Al-Hussaini and Sultan (2001) has given a comprehensive review on reliability and failure rates of mixture models. Finkelstein and Esaulova (2001) considered several types of continuous mixtures of IFR distributions.

1.3.1 Mean Residual Life

Let $\bar{F}$ be the survival function of an item with a finite first moment μ and X be the r.v that corresponds to $\bar{F}$ assuming $F(0) = 0$. The residual life r.v at age t is same as the remaining lifetime after the time of inspection. The mean residual life (MRL) (also known as the mean remaining life) is defined as $\mu(t) = E(X-t|X > t)$ which can be given as

$$\mu(t) = E(X - t|X > t) = \left[\frac{1}{\bar{F}(t)} \int_t^{\infty} \bar{F}(x)\ dx\right]. \tag{1.3.1}$$

Clearly, $\mu(0) = \mu = E(X)$. If F has a density f, then we can write

$$\mu(t) - t = \left(\int_t^{\infty} x\ f(x)\ dx\right) / \bar{F}(t). \tag{1.3.2}$$

Park (1985) found that, the time at which a bathtub failure rate is a minimum does not maximize the mean residual life. The mean residual life function $\mu(t)$ in the constant failure rate region of a bathtub shaped failure curve is not constant but decreasing.

1.3.2 Decreasing Percentile Residual Life Function

The α-percentile residual life function (α-percentile RLF) was first defined by Haines and Singpurwalla (1974). Joe and Proschan (1984) showed that this function may be expressed as

$$q_{\alpha,F}(t) = F^{-1}(1 - (1-\alpha)\bar{F}(t)). \tag{1.3.3}$$

A distribution is DFRL-α, if and only if for some α, $0 < \alpha < 1$, $q_{\alpha,F}(t)$ decreases in t.

Launer (1993) has shown that a BFR distribution is DPRL-α for all $\alpha_o < \alpha < 1$ for some $\alpha_o > 0$, provided there exists a t_o with $r(t_o) \geq r(0)$.

1.4 Stress-strength Reliability

The stress-strength reliability model has attracted a great deal of attention in the fields of reliability engineering, medicine and psychology. In manufacturing process, the information about the mechanical reliability of design through stress-strength model prior to production can significantly decrease the cost of production. The concept of stress and strength in engineering devices have been become the deciding factors of failure of the devices. It has been customary to define safety factors for longer lives of systems in terms of the inherent strength that they have and the external stress being experienced by the systems. If x_0 is the fixed strength and y_0 is the fixed stress that a system is experiencing, then the ratio $\frac{x_0}{y_0}$ is called safety factor and the difference $x_0 - y_0$ is called safety margin. Thus in the deterministic stress-strength situation the system survives only if the safety factor is greater than 1 or equivalently safety margin is positive, see Pratapa (2012).

In the traditional approach to design a system, the safety factor or the safety margin is constructed to resolve uncertainties in the values of stress and strength. Uncertainties in the stress and strength of a system therefore tend to cause the system life to be viewed as random variables. However, the probabilistic analysis demands the use of random variables for the concepts of stress and strength for the evaluation of survival probabilities of such systems. This analysis is particularly useful in situations in which no fixed bound can be put on the stress. For example, with earthquakes, floods and other natural phenomena, stress can lead to failures of systems with unusually small strengths. Similarly when economics is the primary criterion rather than safety, it is best to compare survival performance by understanding the increase in the likelihood of failure when stress and strength are close to each other.

1.5 Total Time on Test Transform

Total time of test (TTT) transform is widely accepted as a statistical tool with applications in various fields such as reliability analysis, econometrics, cryptocurrency modeling, tail ordering, order of delivery, etc. An important part of the literature on TTT transformation deals with reliability issues, including the nature of aging features, model identification, testing of assumptions, age replacement policies, adjusting life distributions, and defining new types of life distributions. TTT transform of a lifetime distribution F is defined as

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} (1 - F(x))\, dx \quad t \in [0, 1]$$

where $F^{-1}(t) = \inf\{x : F(t) \geq t\}$. The scaled TTT transform is defined as $\phi(t) = \frac{H_F^{-1}(t)}{H_F^{-1}(1)}$. A detailed description on TTT can be seen in Barlow and Campo (1975).

1.6 Burn-in

Burn-in is a widely used engineering screening technique to eliminate vulnerable units. Systems can be electronic systems such as circuit boards having different types of chips and printed circuits. An air conditioner having a condenser, fan and circuits is an example for a typical mechanical system. A population of components may include both strong components with long lifetimes and weak components with very short lifetimes. To ensure that only strong components are given to the customer, a manufacturer can subject all components to tests in normal or harsh use conditions so that the weak components will fail during the test, leaving only the strong components. This type of test can be performed on systems to determine weak or strong components or to detect defects during assembly. These tests are usually called burn-in tests in reliability.

Let the lifetime T of a component have a continuous bathtub shaped failure rate $r(t)$. This component is required to accomplish a mission which lasts for time τ . The reliability of completing the mission is thus $\bar{F}(\tau)$. If we burn-in the component for a time b and if the component survives the burn-in, then the conditional reliability of accomplishing the mission is given by

$$\frac{\bar{F}(b+\tau)}{\bar{F}(b)} = \exp\left\{-\int_b^{b+\tau} r(t)\ dt\right\}.$$

1.7 Goodness of Fit Tests

There are different methods that can be used for testing whether a given random sample $x_1, x_2, \ldots, x_n$, of n observations, are coming from a population with specific distribution or for comparing the underlying distribution with other distributions for fitting a given data set. Some of the test for the confirmation of distributions are given below.

1.7.1 Kolmogorov-Smirnov Test

Kolmogorov (1933) proposed the Kolmogorov-Smirnov test (K-S test) for testing whether a given random sample $x_1, x_2, \ldots, x_n$ belongs to a population with a specific distribution or not. The K-S test calculates the distance between the empirical distribution function of the given sample and the estimated cdf of the distribution. The null and alternative hypotheses are H_0 : sample follow the specific distribution versus $H_1 : H_0$ is false.

Let $F(x_i)$ denote the value of the cumulative distribution function of the candidate distribution at x_i and $\hat{F}(x_i)$ denote the value of the empirical distribution function at x_i. The value of the K-S test statistic is defined by

$$\text{K-S test statistic} = \max\left\{|F(x_i) - \hat{F}(x_i)|, |F(x_i) - \hat{F}(x_{i-1})|\right\},$$

where $\hat{F}(x_i) = \frac{\{x_j : x_j \leq x_i\}}{n}$.

The computed K-S statistic is then compared with the tabulated K-S value at a pre-specified significance level to decide whether a distribution is appropriate or not. Moreover, if there are more than one distributions to be compared, the distribution with smaller K-S value will be more appropriate to fit the given sample.

1.7.2 Anderson-Darling Test

The Anderson-Darling (A-D) test is used to test if a sample of data is coming from a population with a specific distribution. It is a modification of the K-S test and gives more weight to the tails than does the K-S test. The A-D test makes use of the specific distribution in calculating critical values. This has the advantage of allowing a more sensitive test and the disadvantage that critical values must be calculated for each distribution, see Stephens (1974). The A-D test is defined as: H_0 : The data follow a specified distribution. H_1 : The data do not follow the specified distribution. The test statistic is

$$A^2 = -N - S$$

where

$$S = \sum_{i=1}^{N} \frac{(2i-1)}{N} [\log F(Y_i) + \log 1 - F(Y_{N+1-i})].$$

F is the cdf of the specified distribution. Note that the Y_i are the ordered data.

1.7.3 Akaike's information criterion

Akaike's information criterion (AIC) compares the quality of a set of statistical models to each other. AIC will take each model and rank it from the best to the worst. The "best" model may be inappropriate or overly compatible. AIC is usually calculated with software. The basic formula is defined as:

$$\mathrm{AIC} = -2l + 2K,$$

where:

- K is the number of model parameters.
- l denotes the log-likelihood function. Log-likelihood is a measure of model fit. The higher the number, the better the fit.

1.7.4 Bayesian information criterion

Bayesian information criterion (BIC) is a criterion for model selection among a finite set of models. It is based, in part, on the likelihood function, and it is closely related to AIC. Mathematically BIC can be defined as

$$\mathrm{BIC} = -2l + K \log n,$$

- K is the number of model parameters.
- l denotes the log-likelihood function. Log-likelihood is a measure of model fit.
- n is the sample size.

The theory of AIC requires that the log-likelihood has been maximized. When comparing models fitted by maximum likelihood to the same data, the smaller the BIC, the better the fit. Standard Normal (SN) distribution is uses for obtaining asymptotic distribution of estimators.

1.8 Objectives of the Study

The objectives of the study are listed below.

1. To study on bathtub shaped failure rate distributions and its applications for modeling life time data.
2. To propose new bathtub shaped failure rate models.
3. To compare existing bathtub shaped failure rate distributions.
4. To study on the stress-strength reliability models and its estimation process.

5. To enhance the application of bathtub shaped failure rate distribution in system engineering and other scientific area.

6. To develop the theory and application of TTT transformation in identification of bathtub shaped failure rate model.

7. To explore the applications of bathtub shaped failure rate models in Burn in process.

1.9 An Outline of the Present Work

The thesis is arranged into eight chapters. Two new lifetime distributions for modeling bathtub shaped failure rate distributions and one lifetime distribution for modeling upside down bathtub shaped failure rate distribution are proposed. Stress-strength reliability estimation in the context of multi-component reliability data has been done using Three-Parameter Generalized Lindley (TPGL) distribution and Power Lindley (PL) distribution. Identification procedure of failure rate distribution of increasing convex (concave) transformation of lifetime data is given.

The chapters of thesis are organized as below.

In *chapter 1*, basic concepts and definitions used in this thesis are given.

In *chapter 2*, extensive reviews of some of these bathtub (or upside down bathtub) shaped failure rate distributions have been presented. This review includes the existing bathtub life distributions that have been proposed in the last several years. In order to attain the results of proposed research work, a review study has been conducted on increasing, decreasing, bathtub shaped, upside down bathtub shaped and constant failure rate distributions. The importance of bathtub shaped failure rate distribution and its practical relevance are studied.

In *chapter 3*, two new bathtub shaped failure rate distributions, Generalized X-Exponential Distribution and Weibull-Lindley distribution are proposed and studied in detail. The new distributions provided a better fit than other well known distributions. Some of the mathematical properties, moments, moment generating function, characteristics function and order statistics, etc., are studied. The estimation of parameters by maximum likelihood is discussed. The proposed distributions are applied to several real data sets and compared with some other bathtub shaped life distributions.

In *chapter 4*, a new upside down bathtub shaped failure rate distribution, based on DUS transformation using Lomax distribution as baseline, is proposed. A very few study on upside down bathtub shaped failure rate models are available in literature. The shapes of its probability density and failure rate functions are investigated. Some of the properties including moments, moment generating function, characteristic function, quantiles, entropy of DUS Lomax distribution are studied. Distributions of minimum and maximum are obtained. Estimation of parameters of the distribution is performed via maximum likelihood method. Reliability of stress-strength models is derived. Using a simulation study the performance of the maximum likelihood estimators (MLE) with respect to biases and mean squared errors are studied. The proposed distribution is applied to three real data sets and compared with other lifetime distributions.

In manufacturing, if we have any information about the mechanical reliability of design through stress-strength model prior to production, a manufacturer can significantly decrease the cost of production. The inherent strength and external stress being experienced by the systems are customary to define safety factors for their long lives. In *chapter 5*, stress-strength reliability in two different cases using three parameter Generalized Lindley distribution and Power Lindley distribution are discussed. The procedure of estimating reliability of single component and multi-component stress-strength models are considered. Performance of the MLEs are presented by the way of a simulation study. Two applications are provided to show how the distribution work in practice using real data sets.

The total time on test transforms is a widely accepted statistical tool, which has applications in different fields such as reliability analysis, econometrics, stochastic modeling, tail ordering, ordering of distributions, etc. TTT transform technique is discussed in *chapter 6* for the problem of identification of failure rate behavior of increasing convex (concave) function of random variable based on distributional properties of the baseline lifetime variable. In this chapter, various properties of TTT transform of increasing convex (concave) function of random variable are studied. Some results about the ageing patterns are investigated.

Burn-in is a technology that used to improve the quality of components and systems which delivered to a customer using the item under normal or accelerated environmental conditions prior to export. If the burn-in procedure is effective, the items delivered to the user are better than those delivered without burn-in. In *chapter 7*, expression of long run average cost function per unit time for obtaining optimal burn-in time and optimal age using Weibull Lindley and and Generalized

X-Exponential distributions are given.

In *Chapter 8*, the conclusion of the thesis is given and presented possible future work. The references are appended at the end of the thesis.