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CHAPTER 2

REVIEW OF BATHTUB-SHAPED FAILURE RATE DISTRIBUTIONS

2.1 Introduction

Attempts in modeling or summarizing survival data are mainly based on three types of distributions: lifetime distributions with constant failure rate, IFR and DFR. However, there seems to be an increased interest in non-monotone failure rate distributions, especially BFR distributions and UBFR distributions. These distributions serve as adequate models for the survival time of many industrial products. Such failure rate curves are also known as the U-shaped or J-shaped curves. Many parametric families of BFR distributions have been introduced in literature during past several years. The BFR distributions are widely used in reliability engineering and survival analysis. Monotonic ageing concepts are popular among many reliability engineers. However, in many practical applications, the effect of age is initially beneficial, but after a certain period of time, it is effecting adversely. Many products, especially electronic, electro-mechanical and mechanical items have BFR distribution, see Barlow and Proshan (1975).

Glaser (1980) and Lawless (1982) have been given many examples of BFR life distributions. Hjorth (1980) described BFR distributions by mixtures of a set of IFR distribution for competing risk model. Lai et al. (2001) discussed the BFR distributions. Xie et al. (2002) studied modified Weibull extension models with BFR function useful in reliability related decision making and cost analysis. Xie et al. (2003) investigated some models extending the traditional two-parameter Weibull distribution. Navarro and Hernandez (2004) studied the shape of reliability functions by using the s-equilibrium distribution of a renewal process and also studied how to obtain distribution with BFR using mixture of two positive truncated Normal distributions. Kundu (2004) proposed two parameter exponentiated Exponential distribution and discussed several properties and different estimation procedures. Wondmagegnehu et al. (2005) studied the failure rate of the mixture of an Exponential distribution and a Weibull distribution. Block et al. (2008) discussed the continuous mixture of whole families of distribution having a BFR functions. Sarhan and Kundu (2009) derived the generalized linear failure rate distribution and its properties.

Extension of Weibull distributions to make it compatible with BFR data are introduced by Mudholkar and Srivastava (1993), Xie and Lai (1995), and Xie et al. (2002). Chen (2000) also introduced a two parameter BFR model for survival data analysis. Wang (2000) studied an additive model based on the Burr XII distribution for lifetime data with BFR. Wang et al. (2014) derived Weibull extension with BFR function based on type-II censored samples.

Recently, Lemonte (2013) proposed a new exponential type distribution with constant failure rate, IFR, DFR, UBFR and BFR function which can be used in modeling survival data in reliability problems and fatigue life studies. Zhang et al. (2013) investigated the parameter estimation of 3-parameter Weibull related model with decreasing, increasing, bathtub and upside-down bathtub shaped failure rates. Parsa et al. (2014) investigated the difference between the change points of failure rate and mean residual life functions of some generalized Gamma type distribution due to the capability of these distribution in modeling various BFR functions. Wang et al. (2015) discussed new finite interval lifetime distribution model for fitting BFR curve. Shehla and Ali khan (2016) studied reliability analysis using an exponential power model with BFR function. Zeng et al. (2016) derived two lifetime distributions, one with 4 parameters and the other with 5 parameters, for the modeling of BFR data.

Shafiq and Viertl (2017) proposed generalized estimators for the parameters and failure rates of BFR distributions used to model fuzzy lifetime data. Gauss et al. (2018) introduced new Lindley Weibull distribution which accommodates unimodal and bathtub shaped failure rates. Dey et al. (2019) introduced a new distribution alpha-power transformed Lomax distribution with decreasing and UBFR distribution. Al-abbasi et al. (2019) proposed a three parameter generalized Weibull uniform distribution that extends the Weibull distribution to have BFR or DFR property. Shoaee (2019) investigated two bivariate models, viz., bivariate

Chen distribution and bivariate Chen-Geometric distribution, that has BFR or IFR functions. Ahsan et al. (2019) studied the reliability analysis of gas-turbine engine with BFR distribution. Chen and Gui (2020) discussed the estimation problem of two parameters of a lifetime distributions with a BFR functions based on adaptive progressive type-II censored data.

The aim of this chapter is to provide a review of BFR and UBFR models. Basic definitions and results are given in section 2.2. Construction of BFR or UBFR model is recalled in section 2.3.

2.2 Definitions and Results

In this section, some definitions of BFR distributions are presented.

DEFINITION 2.2.1. *(Glaser, 1980). Let F be a cdf with a failure rate function $r(t)$ which is continuous. Then F is BFR distribution if there exists a t_o such that:*

(a) *$r(t)$ is decreasing for $t < t_o$,*

(b) *$r(t)$ is increasing for $t > t_o$. i.e., $r'(t) < 0$ for $t < t_o$, $r'(t_o) = 0$ and $r'(t) > 0$ for $t > t_o$.*

Here, when $r(t)$ is increasing (decreasing), it is strictly increasing (decreasing). The bathtub curves given in this definition would probably represent some U-shaped tubs. There is no interval for which $r(t)$ is a constant.

DEFINITION 2.2.2. *(Deshpande and Suresh, 1990). A life distribution F having support on $[0, \infty)$ is said to be a BFR distribution if there exists a point t_o such* *that* $-\log \bar{F}(t)$ *is concave in* $[0, t_o)$ *and convex in* $[t_0, \infty)$.

Definition 2.2.3. *(Mitra and Basu, 1995). An absolutely continuous life distribution* F *having support* $[0, \infty)$ *is said to be a BFR distribution if there exists a,* $t_o \geq 0$ *such that* $r(t)$ *is non-increasing for* $[0, t_o)$ *and non-decreasing on* $[t_o, \infty)$.

Definition 2.2.4. *(Mi, 1995). A lifetime distribution* F *is said to be a BFR distribution if there exists* $0 \leq t_1 \leq t_2 < \infty$ *such that:*

(a) $r(t)$ *is strictly decreasing if* $0 < t < t_1$;

(b) $r(t)$ *is a constant if* $t_1 \leq t \leq t_2$; *and*

(c) strictly increasing if $t > t_2$.

The points t_1 and t_2 are the change points of $r(t)$. If $t_1 = t_2 = 0$, then $r(t)$ becomes IFR, and if $t_1 = t_2 \to \infty$, then $r(t)$ becomes DFR. In general, if $t_1 = t_2$ then the interval for which $r(t)$ is constant degenerates to a single point. The points in the interval (t_1, t_2) are not change points according to Mi (1995).

If F is not absolutely continuous, then BFR property can be explained through the conditional reliability function

$$\bar{F}(x|t) = \frac{\bar{F}(t+x)}{\bar{F}(t)}, \quad \bar{F}(t) = 1 - F(t) > 0,\, t > 0,\, x > 0. \tag{2.2.1}$$

Definition 2.2.5. *(Haupt and Scabe, 1997).* F *is said to be BFR if there exists a* t_o *such that*

- $\bar{F}(x|t)$ *is increasing in* t *for* $0 \leq t < t_o,\ 0 \leq x \leq (t_o - t)$,
- $\bar{F}(x|t)$ *is decreasing in* t *for* $t_o < t < \infty,\ x \geq 0$.

2.2.1 Some Basic Properties

Mitra and Basu (1996a) presented some basic properties of the survival functions and moments of a BFR distribution. Let T represent lifetime r.v with cdf F.

- Suppose F is BFR, then $\bar{F}(t) \leq \bar{G}(t)$ where G is exponential with mean $r(t_o)^{-1}$. Here t_o is a change point at which $r(t)$ is minimum.
- $E(T^k) \leq \frac{\Gamma(k+1)}{\{r(t_o)\}^k}, \quad k > 0.$
- A BFR life distribution F with $E(T^k) = \frac{\Gamma(k+1)}{\{r(t_o)\}^k}$ is necessarily an exponential.
- Convolution of BFR distributions is not necessarily BFR.
- The mixture of BFR distributions need not be BFR.
- Suppose we have a competing risk model: $\bar{F}(t) = \bar{F}_1(t)\bar{F}_2(t)$ where the lifetime of each component is BFR with a common turning point t_o. Then the lifetime of the system is again has a BFR distribution with t_o as one of its turning points.
- A parallel system of two independent BFR components need not be BFR.

In many distributions, survival functions and failure rate functions do not have an analytically tractable form. In such cases, Glaser's technique can be applied, Glaser (1980).

Assume that the pdf $f(t)$ of T is positive on $(0, \infty)$ and that it is twice differentiable on $(0, \infty)$. Let $\eta(t) = -\frac{f'(t)}{f(t)}$, $g(t) = \frac{1}{r(t)}$, then, we have following the results. Clearly $g'(t) = \int_t^\infty [f(y)/f(t)] [\eta(t) - \eta(y)] \, dy$.

Theorem 2.2.1. *(Glaser, 1980)*

(a) If $\eta'(t) > 0$ *for all* $t > 0$*, then* F *is IFR.*

(b) If $\eta'(t) < 0$ *for all* $t > 0$*, then* F *is DFR.*

(c) $\exists\ t_o > 0$ *such that* $\eta'(t_o) = 0$, $\eta'(t) < 0\ \ \forall\ \ t \in (0, t_o)$ *and* $\eta'(t) > 0\ \ \forall\ \ t > t_o$.

   (i) If $\exists\ y_o > 0$ *such that* $g'(y_o) = 0$*, then* F *is BFR.*

   (ii) If there does not exists $y_o > 0$ *such that* $g'(y_o) = 0$*, then* F *is IFR.*

(d) $\exists\ t_o > 0$ *such that* $\eta'(t_o) = 0$, $\eta'(t) > 0\ \ \forall\ \ t \in (0, t_o)$, *and* $\eta'(t) < 0\ \ \forall\ \ t > t_o$.

   (i) If $\exists\ y_o > 0$ *such that* $g'(y_o) = 0$*, then* F *is UBFR.*

   (ii) If there does not exists $y_o > 0$ *such that* $g'(y_o) = 0$*, then* F *is DFR.*

We can use $\eta'(t)$ when the failure rate function is very complicated or not determined.

Following two results from Glaser further ease the computations by avoiding the complications which usually arise because of the function $g(t)$.

Lemma 2.2.2. *(a) Let* $\epsilon = \lim_{t \downarrow 0} f(t)$*. If the condition (c) of Theorem 2.2.1 hold and if* $\epsilon = \infty$*, then the corresponding distribution is BFR. If the condition (d) of Theorem 2.2.1 hold and if* $\epsilon = 0$*, then the corresponding distribution is UBFR.*

(b) Let $\delta = \lim_{t \downarrow 0} g(t)\eta(t)$*. If the condition (c) of Theorem 2.2.1 hold and if* $\delta > 1$*, then the corresponding distribution is BFR. If the condition (d) of Theorem 2.2.1 hold and if* $\delta < 1$*, then the corresponding distribution is UBFR.*

The conditions on ϵ and δ in the above results are reflections of the high infant mortality rate, a characteristic of the bathtub distributions.

2.3 Construction Techniques for Bathtub Distributions

There are lots of ways available for the construction of BFR distribution. Schabe (1994a) has constructed BFR distributions from DFR distributions by truncations. Techniques for construction of a BFR model are given below.

1. **Convex function:** Define a BFR by choosing a positive convex function $r(t)$ over $(0, \infty)$ such that $\int r(t)\ dt = \infty$ (Rajarshi and Rajarshi, 1988). The distribution having a failure rate function $r(t) = \exp\{\alpha + \beta t + \gamma t^2\}$, a strictly increasing function of BFR function, is BFR.

2. **Glaser's technique:** Glaser's theorem (Theorem 2.2.1 above) can be applied to derive new bathtub distributions. i.e., we can choose a function $\eta(t)$ that satisfies the conditions of the theorem.

3. **Function of random variables:** This procedure is due to Griffith (1982). Let T be a continuous r.v and let $g(u)$ be a strictly increasing function which is differentiable on $[0, \infty)$ with $g(0) = 0$. Let g^{-1} be the inverse function of g. Then the failure rate function of $g(T)$ is given by

$$r_{g(T)}(t) = r_T(g^{-1}(t))[g^{-1}(t)]' \qquad (2.3.1)$$

Let T be an exponential r.v and let $g(u)$ be a strictly increasing differentiable function on $[0, \infty)$ with $g(0) = 0$. If g is a convex on $(0, \tau]$ and concave on $[\tau, \infty)$ for a positive τ, then the distribution of $g(T)$ is a bathtub distribution.

4. **Series system (competing risk model):** Suppose we have a series system of two independent components. Everyone knows that the system failure rate is the sum of the two component failure rates. If one of them has IFR distribution and the other has DFR distribution, the system distribution may be BFR. This type of models are obtained by Murthy et al. (1973).

5. **Mixtures:** Mixtures of distributions often give rise to bathtub distributions. For example, Glaser (1980) showed that Gamma mixture has BFR distribution. The mixture of the two increasing linear failure rate distributions provided a BFR distribution, Block et al. (2008).

6. **Sectional models:** Shooman (1968) reported BFR with piecewise linear shape in three areas. Other sectional models that give rise to bathtub distributions were given in Murthy and Jiang (1997).

7. **Polynomial of finite order:** Jaisingh et al. (1987) suggested a polynomial of finite order failure model: $r(t) = a_0 + a_1 t + \ldots + a_n t_n$. As the constants $a_i,\ i = 0, \ldots, n$ may be positive or negative, bathtub shapes for $r(t)$ can be achieved.

8. **TTT Transform:** In Kunitz (1989) and Haupt and Schabe (1997), the TTT transform was used to construct parametric bathtub life distributions.

9. **Truncation of DFR distribution:** Schabe (1994a) has constructed BFR distributions from DFR distributions by truncations.

2.4 Some BFR and UBFR Distributions

Several parametric families of BFR and UBFR life distributions have been constructed in various contexts over the past two decades. Ideally, we should classify them into groups according to some common characteristics. (1) Lifetime distributions with explicit expressions for failure rates and (2) Lifetime distributions with inefficient or unknown failure rate functions.

Quadratic model and its generalization

Bain (1978) considered a quadratic failure rate model,

$$r(x) = \alpha + \beta x + \gamma x^2, \quad \alpha \geq 0, \quad -2(\alpha\gamma)^{1/2} \leq \beta < 0, \ \gamma > 0, \, x > 0, \tag{2.4.1}$$

which has a bathtub shape. Here, $r(0) = \alpha, \ r(x) \to \infty$ as $x \to \infty$. It is easy to verify that $\hat{r}(x) = e^{r(x)}$ also has a bathtub shape if $r(x)$ has a bathtub shape.

A flexible family

Gaver and Acar (1979) have proposed a BFR model with $r(x) = \lambda + g(x) + k(x)$ where $g(x)$ is a non-negative decreasing function of x with $\lim\limits_{x\to\infty} g(x) \to 0$ whereas $k(x)$ is an increasing function of x such that $k(0) = 0, \ \lim\limits_{x\to\infty} k(x) \to \infty$ and λ is any real number such that $r(x) \to 0$. This is a popular method for making BFR functions. Several special cases of this family are presented below:

- $r(x) = \lambda + \frac{\theta}{x+\phi} + \alpha x^p, \quad x > 0, \phi > 0, \alpha > 0, \theta > 0, p > 0$; the model is an extension of Murthy et al. (1973). If both $g(x)$ and $k(x)$ are failure rate functions, then this model is simply a competing risks model involving three distributions.

- $r(x) = \theta_1 \alpha_1 x^{\alpha_1 - 1} + \theta_2 + \theta_3 \alpha_3 x^{\alpha_3 - 1},\ \alpha_3 > 2,\ 0 < \alpha_1 < 1$. $r(x) \to \infty$ as $x \to 0$ or ∞, the model is studied by Canfield and Borgman (1975).

Additive Weibull Distribution

Xie and Lai (1995) considered a competing risk model involving two Weibull distributions. For $\alpha > 0$, $\theta > 0$, $\beta > 0$, $\gamma < 1$, pdf, cdf and failure rate function are

$$f(x) = \left(\alpha\theta x^{\theta-1} + \beta\gamma x^{\gamma-1}\right) e^{-\alpha x^{\theta} - \beta x^{\gamma}},\ x > 0,$$

$$F(x) = 1 - e^{-\alpha x^{\theta} - \beta x^{\gamma}},\ x > 0$$

and

$$r(x) = \alpha\theta x^{\theta-1} + \beta\gamma x^{\gamma-1},\ x > 0. \qquad (2.4.2)$$

The function $r(x)$ has a bathtub shape when $\alpha < 1$ and $\beta > 1$. Also $r(0) = r(\infty) = \infty$. The turning point x_o is given by

$$x_o = \left[\frac{\alpha(1-\alpha)\theta^{\alpha}}{\beta(1-\beta)\gamma^{\beta}}\right]^{\frac{1}{\alpha-\beta}}.$$

Additive Burr XII Distribution

Wang (2000) considered an additive Burr XII model that combines two Burr XII distributions, one with DFR property and another with IFR property. The failure rate function of the additive Burr XII is given by

$$r(x) = \frac{k_1 c_1 (x/s_1)^{c_1 - 1}}{s_1[1 + (x/s_1)^{c_1}]} + \frac{k_2 c_2 (x/s_2)^{c_2 - 1}}{s_2[1 + (x/s_2)^{c_2}]}, \quad x > 0, \tag{2.4.3}$$

where $k_1 \geq 0$, $k_2 \geq 0$, $s_1 \geq 0$, $s_2 \geq 0$, $0 < c_1 < 1 \geq 0$, $c_2 > 2$. It was shown that $r(x)$ has bathtub shape.

Modified Weibull Distribution

Lai et al. (2003) proposed a modified Weibull (MW) distribution with cdf

$$F(x) = 1 - e^{-\beta x^{\gamma} e^{\lambda x}}, \quad \beta > 0,\, \gamma,\, \lambda \geq 0,\, x > 0,$$

where at most one of γ, λ is equal to zero. The pdf of the MW distribution is

$$f(x) = \beta(\gamma + \lambda x)\; x^{\gamma - 1} e^{\lambda x} e^{-\beta x^{\gamma} e^{\lambda x}}, \qquad \beta > 0,\, \gamma,\, \lambda \geq 0,\, x > 0.$$

The corresponding failure rate function is

$$r(x) = \beta(\gamma + \lambda x)\; x^{\gamma - 1} e^{\lambda x}, \quad \beta > 0,\, \gamma,\, \lambda \geq 0,\, x > 0. \tag{2.4.4}$$

The pdf of the MW distribution can be unimodal or decreasing. The failure rate function can be increasing or bathtub shaped.

Sectional model with two Weibull Distributions

Murthy and Jiang (1997) have considered two sectional models involving two Weibull distributions having failure rate function

$$r(x) = \begin{cases} (\alpha_1/\beta_1)(x/\beta_1)^{\alpha_1 - 1}; & 0 \le x \le x_o, \alpha_1 > 0,\ \beta_1 > 0 \\ (\alpha_2/\beta_2)\left(\frac{x-\gamma}{\beta_2}\right)^{\alpha_2 - 1}; & x_o < x < \infty, \alpha_2 > 0,\ \beta_2 > 0 \end{cases}$$

with change point $x_o = [\beta_1^{\alpha_1}(\alpha/\beta_2)^{\alpha_2}]^{1/(\alpha_1-\alpha_2)},\ \gamma = (1-\alpha)x_o$ where $\alpha = \frac{\alpha_2}{\alpha_1}$, and $r(x)$ is continuous at x_o. i.e.,

For $\alpha_1 < \alpha_2,\ r(x)$ have a bathtub shape if $\alpha_1 < 1$ and $\alpha_2 > 1$.

Exponential power Distribution

Smith and Bain (1975) studied the exponential power model having density function

$$f(x) = \lambda\alpha(\lambda x)^{\alpha-1}\exp\{-(e^{(\lambda x)^\alpha} - (\lambda x)^\alpha - 1)\},\ x > 0,\ \alpha > 0,\ \lambda > 0.$$

The survival function is

$$\bar{F}(x) = \exp\{-(e^{(\lambda x)^\alpha} - 1)\},\ x > 0,\ \alpha > 0,\ \lambda > 0. \tag{2.4.5}$$

The failure function is given by

$$r(x) = \lambda\alpha(\lambda x)^{\alpha-1}e^{(\lambda x)^{\alpha}},\ x > 0,\ \alpha > 0,\ \lambda > 0. \tag{2.4.6}$$

For $\alpha < 1,\ r(x) \to \infty$ when $x \to 0$ or $x \to \infty$, $r(x)$ has bathtub shape.

Weibull Extension Distribution

Consider the case $\lambda = 1$ in the exponential power model. Then (2.4.5) becomes

$$\bar{F}(x) = \exp\{-(e^{x^{\alpha}} - 1)\}, \alpha > 0,\ x > 0. \tag{2.4.7}$$

Chen (2000) introduced another parameter λ to the distribution specified in (2.4.7), so that the new cdf becomes

$$\bar{F}(x) = \exp\{-\lambda(e^{x^{\alpha}} - 1)\},\ \alpha > 0,\ \lambda > 0,\ x > 0$$

with failure rate function

$$r(x) = \lambda\alpha x^{\alpha-1}e^{x^{\alpha}}. \tag{2.4.8}$$

The parameter λ here does not alter the shape of the failure rate function so (2.4.8) behaves similarly to the function given in (2.4.6). In particular, $r(x)$ is increasing for $\alpha \geq 1$ and $r(x)$ is bathtub shaped for $\alpha < 1$.

Double Exponential power Distribution

Paranjpe et al. (1985) considered the following model having failure rate function

$$r(x) = \beta\alpha x^{\alpha-1} e^{(\beta x^{\alpha})} \exp\{e^{(\beta x^{\alpha})} - 1\},\ \alpha < 1,\ \beta > 0,\ x > 0. \qquad (2.4.9)$$

The above expression is obviously quite complex. Clearly, $r(x) \to \infty$ as $x \to 0$ or $x \to \infty$, so a bathtub shape is obtained.

Power-function Distribution

Mukherjee and Islam (1983) proposed a finite range distribution with a bathtub failure rate:

$$r(x) = \frac{p x^{p-1}}{\theta^p - x^p},\quad 0 \leq x < \theta,\ p < 1,\ \theta > 0 \qquad (2.4.10)$$

and $r(x) \to \infty$ when $x \to 0$ or θ, thus bathtub is obtained.

Beta failure rate Distribution

Moore and Lai (1994) proposed another finite range distribution, an extension of beta function, with BFR function,

$$r(x) = c(x+p)^{a-1}(q-x)^{b-1},\quad 0 < a < 1,\ b < -1,\ 0 \leq x < q,\ c > 0,\ p \geq 0. \qquad (2.4.11)$$

Clearly $r(0) = cp^{a-1}q^{b-1},\ r(x) \to \infty$ as $x \to q$.

Integrated beta failure rate Distribution

Lai et al. (1998) considered a lifetime distribution with failure rate function

$$r(x) = x^{a-1}(1-x)^{b-1}\{a-(a+b)x\},\ 0<x<1,\ a>0,\ b>0. \tag{2.4.12}$$

It is obvious that $r(x) \to \infty$ as $x \to 0$ or 1 and hence $r(x)$ is bathtub shaped.

J-shaped Distribution

Topp and Leone (1955) proposed J-shaped distribution. Nadarajah and Kotz (2003) discussed moments of J-shaped distribution. For $0 < v < 1$, $b > 0$, J-shaped distribution has pdf and cdf

$$f(x) = \frac{2v}{b}\left(\frac{x}{b}\right)^{v-1}\left(1-\frac{x}{b}\right)\left(2-\frac{x}{b}\right)^{v-1},\ 0<x<b,$$

and

$$F(x) = \begin{cases} \left(\frac{x}{b}\right)^{v}\left(2-\frac{x}{b}\right)^{v}; & \text{if } 0 \le x \le b < \infty \\ 0; & \text{if } x < 0 \\ 1; & \text{if } x > b. \end{cases}$$

The failure rate function $r(x)$ is

$$r(x) = \frac{2v}{b}\frac{y(1-y^2)^{v-1}}{1-(1-y^2)^{v}}, \tag{2.4.13}$$

where $y = 1-(x/b)$. $r(x) \to \infty$ as $x \to 0$ and $x \to b$ for all $v \in (0,1)$ and $r(x)$ attains a minimum at $x = x_0$, where $y_0 = 1-(x_0/b)$ is the root of the equation

$(1-y)^v = 1 - 2vy/(1+y)$.

Beta Distribution

For $p > 0$, $q > 0$, $0 < x < 1$, the standard beta distribution has pdf and cdf

$$f(x) = \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}$$

and

$$F(x) = \frac{B_x(p,q)}{B(p,q)}$$

respectively, where $B_x(.,.)$ is the incomplete beta function defined by

$$B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}dt,\ 0 < x < 1,\ p > 0,\ q > 0.$$

The failure rate function $r(x)$ can be expressed as

$$r(x) = \frac{x^{p-1}(1-x)^{q-1}}{B(p,q) - B_x(p,q)},\ 0 < x < 1,\ p > 0,\ q > 0. \tag{2.4.14}$$

Ghitany (2004) showed that $r(x)$ is bathtub-shaped if $p < 1$.

Mukherjee and Roy's Distribution

Mukherjee and Roy (1993) defined a distribution having pdf

$$f(x) = (\delta(|x-a| + |x-b|)/(b-a)) \exp\left[-\delta a^2/(b-a) - \delta x + (-1)^{k(x-a,|x-a|)}\right]$$
$$\exp\left[((b-a)\delta/4)((|x-a|+|x-b|)/(b-a)-1)^2\right],\ x > 0,\ 0 < a < b < \infty,\ \delta > 0,$$

and cdf

$$\begin{aligned} F(x) =& 1 - \exp\big[- \delta a^2/(b-a) - \delta x \\ & + (-1)^{k(x-a,|x-a|)}((b-a)\delta/4)((|x-a| + |x-b|)/(b-a) - 1)^2\big], \end{aligned}$$

where $k(x, y)$ is Kronecker's function taking the value 1 when $x = y$ and 0 whenever $x \neq y$. The failure rate function $r(x)$ can be expressed as

$$r(x) = \delta(|x-a| + |x-b|)/(b-a). \tag{2.4.15}$$

It is clear that $r(x)$ take bathtub shape for all values of δ, a and b.

Haupt and Schabe's Distribution

Haupt and Schabe (1992) developed a distribution having pdf

$$f(x) = \begin{cases} \frac{1+2\beta}{2T\sqrt{\beta^2+(1+2\beta)x/T}}; & \text{if } 0 \leq x \leq T,\ \infty < \beta < \infty,\ T > 0, \\ 0; & \text{otherwise} \end{cases}$$

and cdf

$$F(x) = \begin{cases} 1; & \text{if } x \geq T, \\ -\beta + \sqrt{\beta^2 + (1+2\beta)x/T}; & \text{if } 0 \leq x \leq T, \\ 0; & \text{otherwise.} \end{cases}$$

The failure rate function $r(x)$ can be expressed as

$$r(x) = \begin{cases} \frac{1+2\beta}{2T\sqrt{\beta^2+(1+2\beta)x/T}(1+\beta-\sqrt{\beta^2+(1+2\beta)x/T})}; & \text{if } 0 \leq x \leq T, \\ 0; & \text{otherwise} \end{cases} \tag{2.4.16}$$

$r(x)$ is bathtub-shaped if $-1/3 < \beta < 1$ and $r(x)$ attaining the minimum at $x_0 = T(1+2\beta-3\beta^2)/[4(1+2\beta)]$.

Schabe's Distribution

Schabe (1994a) considered a simple distribution having pdf

$$f(x) = \frac{2\gamma + (1-\gamma)x/\theta}{\theta(\gamma + x/\theta)^2},\ x \leq \theta,\ \theta > 0,\ -\infty < \gamma < \infty$$

and cdf

$$F(x) = \frac{(1+\gamma)x}{\theta\gamma + x}.$$

The failure rate function $r(x)$ can be expressed as

$$r(x) = \frac{1}{\theta(\gamma + x/\theta)} + \frac{1}{\theta(1 - x/\theta)}. \tag{2.4.17}$$

$r(x)$ is bathtub-shaped if $\gamma < 1$ and minimum of $r(x)$ occurring at $x_0 = (\theta/2)(1-\gamma)$. $r(x) \to \infty$ as $x \to \infty$.

Hjorth's Distribution

For $\delta > 0,\ \beta > 0,\ \theta > 0$, Hjorth (1980) proposed a distribution with pdf and cdf

$$f(x) = \frac{[(1+\beta x)\delta x + \theta]exp(-\delta x^2/2)}{(1+\beta x)^{\frac{\theta}{\beta+1}}},\ x > 0$$

and

$$F(x) = 1 - \frac{\exp(-\delta x^2/2)}{(1+\beta x)^{\frac{\theta}{\beta}}},\ x > 0$$

respectively. The failure rate function $r(x)$ can be expressed as

$$r(x) = \delta x + \frac{\theta}{1+\beta x}. \qquad (2.4.18)$$

It is easily seen that $r(x)$ is bathtub-shaped if $0 < \delta < \theta\beta$.

Gamma mixture Distribution

Gupta and Warren (2001) examined the mixture of Gamma distributions having pdf

$$f(x) = \frac{px^{\alpha_1-1}e^{-x/\beta}}{\beta^{\alpha_1}\Gamma(\alpha_1)} + \frac{(1-p)x^{\alpha_2-1}e^{-x/\beta}}{\beta^{\alpha_2}\Gamma(\alpha_2)},$$

for $x > 0,\ \alpha_1 > 0,\ \alpha_2 > 0,\ \beta > 0$ and $0 < p < 1$. Corresponding failure rate function is bathtub-shaped if either $\beta_1 = \beta_2,\ \alpha_1 > 1$ and $\alpha_2 < 1$ or $\alpha_1 > 2$.

Normal mixture Distribution

Navarro and Hernandez (2004) examined the mixture of truncated Normal distributions with pdf

$$f(x) = \frac{p}{\sqrt{2\Pi}\sigma_0\Phi(\mu_0/\sigma_0)} \exp\left[-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right] + \frac{1-p}{\sqrt{2\Pi}\sigma_1\Phi(\mu_1/\sigma_1)} \exp\left[-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right],$$

for $x > 0,\ -\infty < \mu_0 < \infty,\ -\infty < \mu_1 < \infty,\ \sigma_0 > 0,\ \sigma_1 > 0$ and $0 < p < 1$, in which the corresponding failure rate function exhibiting a bathtub shape.

Exponentiated Weibull Distribution

Mudholkar et al. (1995) introduced the exponentiated Weibull distribution with pdf

$$f(x) = a\alpha\lambda^\alpha x^{\alpha-1} e^{-(\lambda x)^\alpha}(1-e^{-(\lambda x)^\alpha})^{a-1},\ x > 0,\ a > 0,\ \alpha > 0,\ \lambda > 0$$

and cdf

$$F(x) = (1-e^{-(\lambda x)^\alpha})^a,\ x > 0,\ a > 0,\ \alpha > 0,\ \lambda > 0$$

respectively. The failure rate function $r(x)$ can be expressed as

$$r(x) = \frac{a\alpha(\lambda x)^{\alpha-1}e^{-(\lambda x)^\alpha}(1-e^{-(\lambda x)^\alpha})^{a-1}}{1-(1-e^{-(\lambda x)^\alpha})^a} \tag{2.4.19}$$

in which $r(x)$ is bathtub-shaped if $\alpha > 1$ and $a\alpha < 1$.

Stacy's Weibull Distribution

Stacy (1962) proposed a distribution with pdf

$$f(x) = c\beta^{-c\alpha}(\Gamma\alpha)^{-1}x^{c\alpha-1}e^{-(x/\beta)^c}$$

and cdf

$$F(x) = (\Gamma\alpha)^{-1}\gamma\left(\alpha, \left(\frac{x}{\beta}\right)^c\right),$$

where $x > 0,\ c > 0,\ \alpha > 0,\ \beta > 0,\ \gamma(\cdot, \cdot)$ denotes the incomplete Gamma function defined by

$$\gamma(a,\ x) = \int_0^x t^{a-1}e^{-t}dt,\ x > 0,\ a > 0.$$

The failure rate function $r(x)$ can be expressed as

$$r(x) = \frac{c\beta^{-c\alpha}x^{c\alpha-1}e^{-(x/\beta)^c}}{(\Gamma\alpha)^{-1}\gamma\left(\alpha, \left(\frac{x}{\beta}\right)^c\right)}. \tag{2.4.20}$$

$r(x)$ is bathtub-shaped if $c > 1$.

Truncated Weibull Distribution

McEwen and Parresol (1991) proposed doubly truncated Weibull distribution having pdf

$$f(x) = \frac{p(x)}{P(b) - P(a)}$$

and cdf

$$F(x) = \frac{P(x) - P(a)}{P(b) - P(a)},$$

where $0 \leq a < x < b < \infty$, $p(x)$ and $P(x)$ are the pdf and cdf, respectively, of the traditional Weibull distribution, $p(x) = \alpha\lambda(\lambda x)^{\alpha-1}e^{-(\lambda x)^{\alpha}}$, and $P(x) = 1 - e^{-(\lambda x)^{\alpha}}$, for $x > 0$, $\alpha > 0$ and $\lambda > 0$. Corresponding failure rate function has bathtub shape.

Xie et al.'s Weibull Distribution

Xie et al. (2002) proposed a modification of the Weibull distribution having pdf

$$f(x) = \lambda\beta\left(\frac{x}{\alpha}\right)^{\beta-1}\exp\left[\left(\frac{x}{\alpha}\right)^{\beta} + \lambda\alpha\left\{1 - \exp\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right]$$

and cdf

$$F(x) = 1 - \exp\left[\lambda\alpha\left\{1 - \exp\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right],$$

where $x > 0$, $\lambda > 0$, $\alpha > 0$ and $\beta > 0$. The failure rate function $r(x)$ can be expressed as

$$r(x) = \lambda\beta\left(\frac{x}{\alpha}\right)^{\beta-1}\exp\left(\frac{x}{\alpha}\right)^{\beta}. \tag{2.4.21}$$

It is easily seen that $r(x)$ is bathtub-shaped if $0 < \beta < 1$ with $r(x)$ attaining the minimum at $x_o = \alpha(1/\beta - 1)^{1/\beta}$ and $r(x)$ increases to ∞ as $x \to 0$ and ∞.

Generalized Lindley Distribution

Nadarajah et al. (2011) proposed Generalized Lindley (GL) distribution whose cdf is

$$F(x) = \left[1 - \frac{1 + \lambda + \lambda x}{1 + \lambda}e^{-\lambda x}\right]^{\alpha}$$

for $x > 0$, $\lambda > 0$, and $\alpha > 0$. The failure rate function is given by

$$r(x) = \frac{\alpha\lambda^2}{1+\lambda}(1+x)\left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}\right]^{\alpha-1} e^{-\lambda x}[1 - V^{\alpha}(x)]^{-1} \qquad (2.4.22)$$

for $x > 0$, $\lambda > 0$, $\alpha > 0$, where $V(x) = \left[1 - \frac{1+\lambda+\lambda x}{1+\lambda}e^{-\lambda x}\right]$. The shape of the failure rate function appears monotonically decreasing or to initially decrease and then increase, a bathtub shape if $\alpha < 1$, the shape appears monotonically increasing if $\alpha \geq 1$.

Burr XII Distribution

The Burr XII distribution was first introduced by Burr (1942). Burr XII distribution having reliability function

$$\bar{F}(x) = \frac{1}{(1+x^c)^k},\ k,\ c > 0,\ x > 0.$$

The failure rate function is

$$r(x) = \frac{kcx^{c-1}}{(1+x^c)} \qquad (2.4.23)$$

For $c \leq 1$, the slope is always negative, for $c > 1$ the slope is positive for $x^c < c-1$ and negative for $x^c > c-1$. Thus $r(x)$ is decreasing for $c \leq 1$ and UBFR if $c > 2$.

Birnbaum and Saunders Distribution

Birnbaum and Saunders (1969a,b) introduced a lifetime distribution

$$\begin{aligned} F(x) &= \Phi\left\{\frac{1}{\alpha}.\left[\left(\frac{x}{\beta}\right)^{1/2} - \left(\frac{x}{\beta}\right)^{-1/2}\right]\right\} \\ &= \Phi\left\{\frac{1}{\alpha}\xi\left(\frac{x}{\beta}\right)\right\},\ x > 0, \end{aligned}$$

where $\xi(x) = x^{1/2} - x^{-1/2},\ \alpha,\ \beta > 0$ and $\Phi(.)$ denotes the cdf of the standard normal. While the failure rate of Birnbaum and Saunders is zero at $x = 0$, then increases to a maximum for some x_0 and finally decreases to a finite positive value (i.e., $r(x)$ is UBFR) when $\beta = 1$ and $\alpha > 0.8$, the failure rate of the log-Normal also has a UBFR function but decreases to zero.

Inverse Gaussian Distribution

The name 'inverse Gaussian' was first applied to a certain class of distributions by Tweedie (1947). The density function of the inverse Gaussian is

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\left[-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right],\ \lambda > 0,\ x \geq 0.$$

The corresponding distribution function is

$$F(x) = \Phi\left\{\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} - 1\right)\right\} + e^{2\lambda/\mu}\Phi\left\{-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} + 1\right)\right\}$$

and the expression for $r(x)$ is quite complicated. By Glaser's theorem, $r(x)$ is UBFR.

Inverse Weibull Distribution

The cdf of two-parameter inverse Weibull distribution is given by

$$F(x) = \exp\left\{-\left(\frac{\alpha}{x}\right)^{\beta}\right\},\ \alpha, \beta > 0,\ x > 0.$$

The failure rate function is

$$r(x) = \beta\alpha^{\beta}x^{-(\beta+1)}\exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\left\{1 - \exp\left[-\left(\frac{\alpha}{x}\right)^{\beta}\right]\right\}^{-1}. \tag{2.4.24}$$

It has been shown that $\lim_{x\to 0} r(x) = \lim_{x\to\infty} r(x) = 0$ and $r(x) \in$ UBFR.

Log-Logistic Distribution

The pdf, cdf and failure rate of Log-Logistic distribution are given by

$$f(x) = \frac{k\rho(x\rho)^{k-1}}{[1 + (\rho x)^k]^2},\ x > 0,\ \rho > 0,\ \ k > 0$$
$$F(x) = 1 - \frac{1}{1 + (\rho x)^k}$$

and

$$r(x) = \frac{k\rho(\rho x)^{k-1}}{1 + (\rho x)^k}. \tag{2.4.25}$$

It can be shown easily that $r(x)$ is DFR when $k \leq 1$; $r(x)$ is UBFR when $k > 1$.

Log-Normal Distribution

The distribution function and failure rate function of log-Normal distribution are

$$F(x) = \Phi\left(\frac{\log x - \mu}{\sigma}\right)$$

and

$$r(x) = \frac{(1/2\pi x\sigma)\exp\{-(\log ax)^2/2\sigma^2\}}{1 - \Phi\{\log ax/\sigma\}}, \tag{2.4.26}$$

where $a = e^{-\alpha}$. $r(0) = 0$ and $r(x) \to 0$ when $x \to \infty$.

2.5 Optimal burn-in time

Burn-in plays an important role in reliability engineering, Jensen and Petersen (1982). In this section, we discuss the concepts of burn-in.

2.5.1 Concepts of burn-in

Due to the high failure rate (most importantly, silicon and integrated circuits) in the early stages of module life, burn-in is widely accepted as a method of screening failures before sending or delivering these components to the field operations. That is, before delivery to the customers, the components are tested in approximate electrical or thermal conditions that approximate the working conditions in field operation. The components that fail in the burn-in process are removed or repaired, and only those that survive the burn-in process are considered to be of high quality. These are how users send them to the field or to the field function.

With adequate burn-in, a high initial failure rate results in high maintenance costs. A general background on burn-in can be found in Kuo and Kuo (1983), and Kuo (1984).

Estimating change points is particularly relevant for BFR in the context of maintenance policies, naturally, do not consider replacing a component of such a life distribution before the 'threshold' unknown age t_o is reached. Kulasekera and Saxena (1991) have constructed kernel density estimators and empirical cdf to estimate f, F, r and its change point.

Mitra and Basu (1995) considered the problem of estimating change points in different monotonic aging models. Suresh (1992) also obtained two estimates of the change point; one by using the definition of BFR distribution, another by a characterization of BFR distribution in terms of TTT transform. Nguyen et al. (1984) considered the estimation of the turning point of a two-step piecewise linear failure rate function. Pham and Nguyen (1993) considered estimating the turning point of a truncated BFR function. Gupta et al. (1999) considered estimation of the turning point of the failure rate function in the case of the log-Logistic model. Details about optimal burn-in time can be seen in Myung and Young (2002). The expressions of long run average cost function per unit time for obtaining optimal burn-in time and optimal age for various distributions is an open problem.

This chapter gave a comprehensive review of known BFR and UBFR distributions. Some tools and methods that will be used for data analysis are also reviewed.