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CHAPTER 3

NEW BATHTUB SHAPED FAILURE RATE DISTRIBUTIONS

3.1 Introduction

[1] In many applied sciences such as medicine, engineering, bio statistics, survival analysis etc modeling and analysis of lifetime data are crucial, Deshpande and Suresh (1990). In analyzing lifetime data one often uses the Exponential, Gamma, Weibull and Generalized Lindley distributions. It is well known that Exponential distribution has constant hazard function, Generalized Lindley distribution has a BFR function whereas Weibull and Gamma distributions have constant or monotone (increasing/decreasing) failure rate functions. In this chapter we present two new simple distributions which have BFR function. The proposed distributions

[1] Some contents of this chapter are based on Chacko and Deepthi (2018 & 2019).

are capable of modeling the real problems.

In this chapter, Generalized X-Exponential distribution and Weibull-Lindley Distribution are discussed in section 3.2 and 3.3. Summary are given in section 3.4.

3.2 Generalized X-Exponential distribution

Here we consider a new distribution named as Generalized X-Exponential distribution, having BFR function, which generalizes the distribution having df $F(x) = 1 - (1 + \lambda x^2)e^{-\lambda x}$, $x > 0$, $\lambda > 0$, Chacko (2016). The failure rate function of X-Exponential distribution appears monotonically decreasing and bathtub shape. The generalization considered is the distribution of a series system having distribution function $F(x) = 1 - (1 + \lambda x^2)e^{-\lambda(x^2+x)}$, $x > 0$, $\lambda > 0$, for its components.

In section 3.2.1, the distribution function of the Generalized X-Exponential distribution (GXE) is given. In section 3.2.2 discussed the statistical properties of the distribution. In section 3.2.3 discussed the distribution of maximum and minimum to address the reliability problems of parallel system and series system, respectively. The maximum likelihood estimation of the parameters is explained in section 3.2.4. In section 3.2.5 discussed the asymptotic confidence bounds of MLEs of the distribution. In section 3.2.6 a simulation study is given. Two real data sets are analyzed in section 3.2.7 and the results are compared with some existing distributions.

3.2.1 Generalized X-Exponential Distribution

Let X be a life time r.v having cdf with parameter α and λ,

$$F(x) = \left(1 - (1 + \lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}, \quad x > 0,\ \alpha > 0,\ \lambda > 0. \tag{3.2.1}$$

Clearly $F(0) = 0$, $F(\infty) = 1$, F is non-decreasing and right continuous. More over F is absolutely continuous. Then the pdf of the r.v X is given by

$$f(x) = \alpha e^{-\lambda(x^2+x)}(\lambda(1 + \lambda x^2)(2x + 1) - 2\lambda x)\left(1 - (1 + \lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1}, \quad x > 0,\ \alpha, \lambda > 0. \tag{3.2.2}$$

Here α and λ are the shape and scale parameters. It is clear that F is a positively skewed distribution. The distribution with pdf of the form (3.2.2) is named as GXE distribution with parameters α and λ and denoted by GXE(α, λ). Failure rate function of GXE distribution is

$$r(x) = \frac{\alpha e^{-\lambda(x^2+x)}(\lambda(1 + \lambda x^2)(2x + 1) - 2\lambda x)\left(1 - (1 + \lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1}}{1 - (1 - (1 + \lambda x^2)e^{-\lambda(x^2+x)})^{\alpha}}, \quad x > 0,\ \alpha, \lambda > 0. \tag{3.2.3}$$

Considering the behavior near the change point $x_0, x_0 > 0$ and if $\frac{d}{dx}h(x_0) = 0$.

(i) If $0 < \alpha < 1/2$, and $0 < \lambda < 1$, then $\frac{d}{dx}h(x) < 0$ when $0 < x < x_0$ and $\frac{d}{dx}h(x) > 0$ when $x > x_0$, $\frac{d^2}{dx^2}h(x) > 0$ for $x = x_0$.

(ii) If $0 < \alpha < 1/2$, and $\lambda > 1$, then $\frac{d}{dx}h(x) < 0$ when $0 < x < x_0$, $\frac{d}{dx}h(x) > 0$

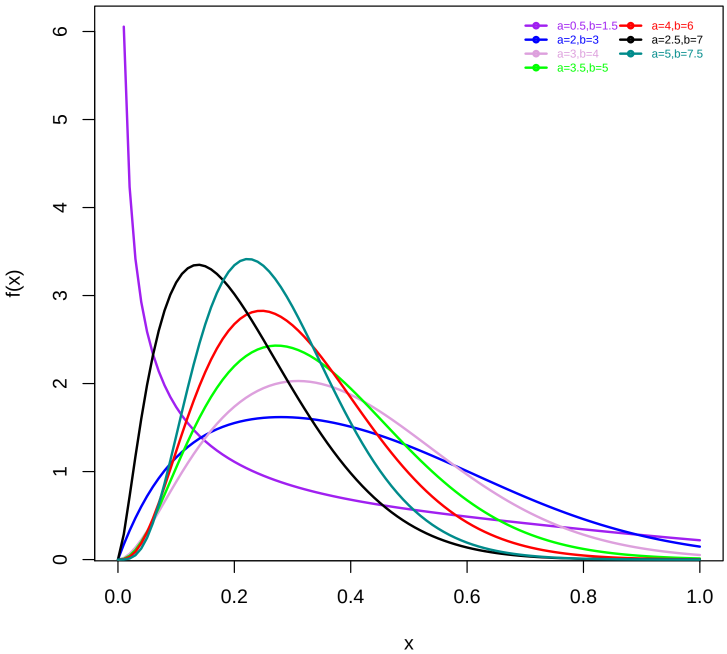

Figure 3.1: PDF of GXE distribution for values of parameters $\alpha = 0.5, 2, 3, 3.5, 4, 2.5, 5$ and $\lambda = 1.5, 3, 4, 5, 6, 7, 7.5$ with color shapes purple, blue, plum, green, red, black and dark cyan, respectively.

when $x > x_0$, $\frac{d^2}{dx^2}h(x) > 0$ for $x = x_0$.

(iii) If $1/2 < \alpha < 1$, and $0 < \lambda < 1$, then $\frac{d}{dx}h(x) < 0$ when $0 < x < x_0$ and $\frac{d}{dx}h(x) > 0$ when $x > x_0$, $\frac{d^2}{dx^2}h(x) > 0$ for $x = x_0$.

(iv) If $1/2 < \alpha < 1$, and $\lambda > 1$, then $\frac{d}{dx}h(x) < 0$ when $0 < x < x_0$ and $\frac{d}{dx}h(x) > 0$ when $x > x_0$, $\frac{d^2}{dx^2}h(x) > 0$ for $x = x_0$.

(v) If $\alpha > 1$, and $\lambda > 1$, then $\frac{d}{dx}h(x) > 0$ for $x > 0$.

The shape of (3.2.3) appears monotonically decreasing or to initially decrease and then increase, a bathtub shape if $\alpha < 1$. GXE(α, λ) allows for monotonically decreasing, monotonically increasing and bathtub shapes for its failure rate function. As α decreases from 1 to 0, the graph shift above whereas if λ increases from 1 to ∞ the shape of the graph concentrate near to 0, see Figures. 3.1, 3.2, 3.3.

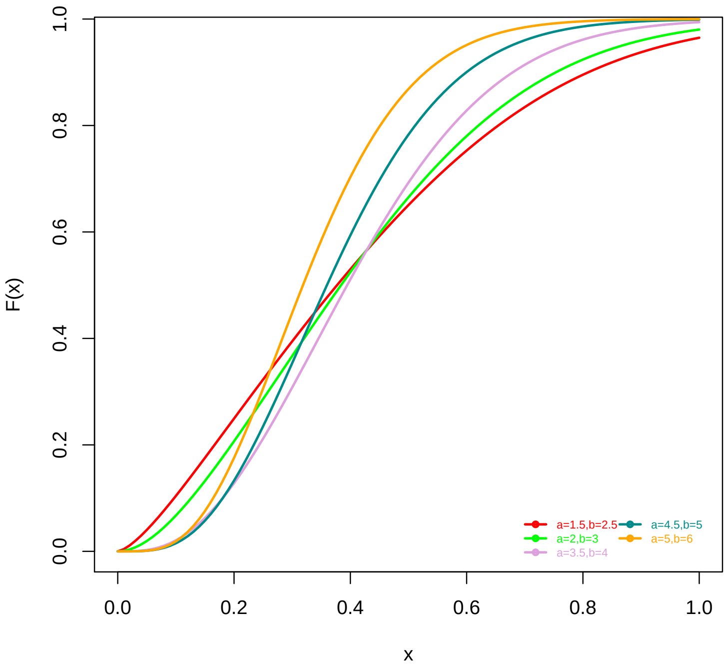

Figure 3.2: CDF of GXE distribution for values of parameters $\alpha = 1.5, 2, 3.5, 4.5, 5$ and $\lambda = 2.5, 3, 4, 5, 6$ with color shapes red, green, plum, dark cyan and orange respectively.

3.2.2 Moments

In order to calculate moments of X, we require the following lemma.

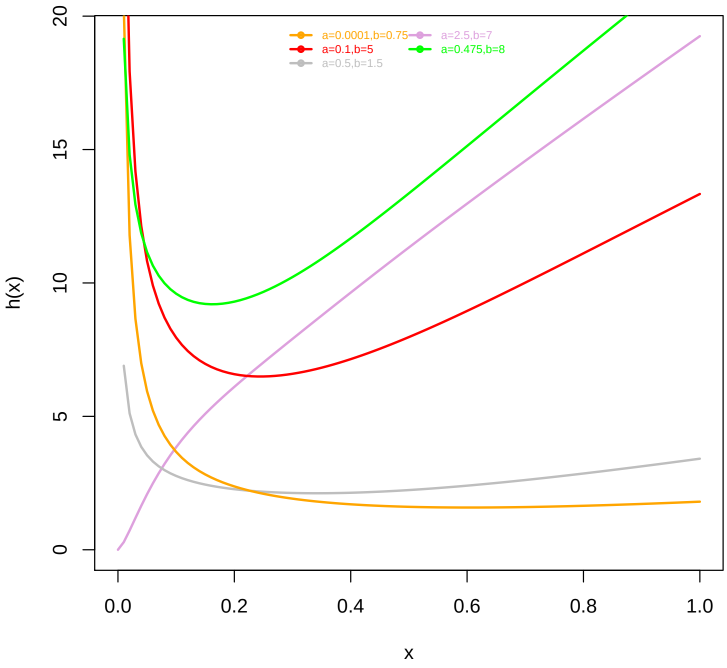

Figure 3.3: Failure rate function GXE distribution for values of parameters $\alpha = 0.0001, 0.1, 0.5, 2.5, 0.475$ and $\lambda = 0.75, 5, 1.5, 7, 8$ with color shapes orange, red, grey, plum and green, respectively.

Lemma 3.2.1. *For* $\alpha, \lambda > 0,\ x > 0,$

$$K(\alpha, \lambda, c) = \int_0^\infty x^c \left(1 - (1 + \lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1} e^{-\lambda(x^2+x)}\ dx.$$

Then,

$$K(\alpha, \lambda, c) = \sum_{i=0}^{\alpha-1} \sum_{j=0}^{i} \binom{\alpha-1}{i} \binom{i}{j} (-1)^i \lambda^j \int_0^\infty x^{2j+c} e^{-\lambda(x^2+x)}\ dx.$$

Proof. Using Binomial expansion, $(1-z)^{\alpha-1} = \sum_{i=0}^{\alpha-1} \binom{\alpha-1}{i} (-1)^i z^i$, we have,

$$\begin{aligned}
K(\alpha,\lambda,c) &= \int_0^\infty x^c \left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1} e^{-\lambda(x^2+x)}\, dx \\
&= \sum_{i=0}^{\alpha-1} \binom{\alpha-1}{i} (-1)^i \int_0^\infty x^c \left[(1+\lambda x^2)e^{-\lambda(x^2+x)}\right]^i e^{-\lambda(x^2+x)}\, dx \\
&= \sum_{i=0}^{\alpha-1} \binom{\alpha-1}{i} (-1)^i \int_0^\infty x^c \sum_{j=0}^{i} \binom{i}{j} (\lambda x^2)^j e^{-(i+1)\lambda(x^2+x)}\, dx \\
&= \sum_{i=0}^{\alpha-1} \sum_{j=0}^{i} \binom{\alpha-1}{i} \binom{i}{j} (-1)^i \lambda^j \int_0^\infty x^{2j+c} e^{-\lambda(x^2+x)}\, dx.
\end{aligned}$$

The result of the lemma follows by the definition of the Gamma function. The first raw moment is

$$E(X) = \alpha\lambda K(\alpha,\lambda,1) + 2\alpha\lambda^2 K(\alpha,\lambda,4) + \alpha\lambda^2 K(\alpha,\lambda,3).$$

The n^{th} raw moment is

$$E(X^n) = \alpha\lambda K(\alpha,\lambda,n) + 2\alpha\lambda^2 K(\alpha,\lambda,n+3) + \alpha\lambda^2 K(\alpha,\lambda,n+2).$$

□

Moment Generating Function

Moment generating function can be obtained from following formula

$$M_X(t) = \int_0^\infty e^{tx} \alpha e^{-\lambda(x^2+x)} \left(\lambda + \lambda x^2 + 2\lambda^2 x^3\right) \left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1}\, dx$$

$$= \int_0^\infty \alpha \left(\lambda + \lambda x^2 + 2\lambda^2 x^3\right) \left(1 - (1+\lambda x^2) e^{-\lambda(x^2+x)}\right)^{\alpha-1} e^{-\lambda(x^2+x)+tx} \, dx.$$

Characteristic Function

Characteristic function can be obtained from following formula

$$\begin{aligned} \phi_X(t) &= \int_0^\infty e^{itx} \alpha e^{-\lambda(x^2+x)} \left(\lambda + \lambda x^2 + 2\lambda^2 x^3\right) \left(1 - (1+\lambda x^2) e^{-\lambda(x^2+x)}\right)^{\alpha-1} \, dx \\ &= \int_0^\infty \alpha \left(\lambda + \lambda x^2 + 2\lambda^2 x^3\right) \left(1 - (1+\lambda x^2) e^{-\lambda(x^2+x)}\right)^{\alpha-1} e^{-\lambda(x^2+x)+tx} \, dx. \end{aligned}$$

Mean Deviation About Mean

The scatter in a population is measured by using Mean deviation about the mean μ is defined by

$$\begin{aligned} \mathrm{MD(mean)} &= 2\mu F(\mu) - 2\mu + 2\int_\mu^\infty x f(x) \, dx \\ &= 2\mu F(\mu) - 2\mu + 2\left(\alpha\lambda L(\alpha,\lambda,1,\mu) + 2\alpha\lambda^2 L(\alpha,\lambda,4,\mu)\right) \\ &+ 2\alpha\lambda^2 L(\alpha,\lambda,3,\mu) \end{aligned}$$

where

$$\begin{aligned} L(\alpha,\lambda,c,\mu) &= \int_\mu^\infty x^c \left(1 - (1+\lambda x^2) e^{-\lambda(x^2+x)}\right)^{\alpha-1} e^{-\lambda(x^2+x)} \, dx \\ &= \sum_{i=0}^{\alpha-1} \sum_{j=0}^{i} \binom{\alpha-1}{i} \binom{i}{j} (-1)^i \lambda^j \left(\int_\mu^\infty x^{2j+c+1} e^{-(j+1)\lambda(x^2+x)} dx\right). \end{aligned}$$

Similarly, Mean deviation about the Median M is defined by

$$
\begin{aligned}
\text{MD(Median)} &= -M + 2\int_{M}^{\infty} x f(x)\, dx \\
&= -M + 2\left(\alpha\lambda L(\alpha,\lambda,1,M) + 2\alpha\lambda^2 L(\alpha,\lambda,4,M)\right) \\
&\quad + 2\alpha\lambda^2 L(\alpha,\lambda,3,M).
\end{aligned}
$$

3.2.3 Distribution of Maximum and Minimum

Series, Parallel, Series-Parallel and Parallel-Series systems are general system structure of many engineering systems. The theory of order statistics provides a useful tool for analysing life time data of such systems. Let $X_1, X_2, \ldots, X_n$ be a random sample from GXE distribution with cdf and pdf as in (3.2.1) and (3.2.2), respectively. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics obtained from this sample. The pdf of $X_{(r)}$ is given by,

$$
\begin{aligned}
f_{(r:n)}(x) = &\frac{1}{B(r, n-r+1)}\left[\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{r-1} \\
&\left[1-\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{n-r}\alpha e^{-\lambda(x^2+x)}(\lambda(1+\lambda x^2)(2x+1)-2\lambda x) \\
&\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1}, \quad x>0,\ \alpha,\lambda>0. \qquad (3.2.4)
\end{aligned}
$$

The cdf of $X_{(r)}$ is given by

$$
\begin{aligned}
F_{r:n}(x) &= \sum_{j=r}^{n}\binom{n}{j}F^j(x)[1-F(x)]^{n-j} \\
&= \sum_{j=r}^{n}\binom{n}{j}\left[\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{j}
\end{aligned}
$$

$$\left[1-\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{n-j}, \quad x>0,\ \alpha,\lambda>0. \tag{3.2.5}$$

The pdf of the smallest and largest order statistics $X_{(1)}$ and $X_{(n)}$ are respectively given by:

$$f_1(x)=\frac{1}{B(1,n)}\left[1-\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{n-1}\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1}$$
$$\alpha e^{-\lambda(x^2+x)}(\lambda(1+\lambda x^2)(2x+1)-2\lambda x),\ x>0,\ \alpha,\lambda>0, \tag{3.2.6}$$

and

$$f_n(x)=\frac{1}{B(n,1)}\left[\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{n-1}$$
$$\alpha e^{-\lambda(x^2+x)}(\lambda(1+\lambda x^2)(2x+1)-2\lambda x)\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha-1},$$
$$x>0,\ \alpha,\lambda>0. \tag{3.2.7}$$

The cdf of the smallest and largest order statistics $X_{(1)}$ and $X_{(n)}$ are respectively given by

$$F_1(x)=1-\left[1-\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{n}, \quad x>0,\ \alpha,\lambda>0$$

and

$$F_n(x)=\left[\left(1-(1+\lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^{n}, \quad x>0,\ \alpha,\lambda>0.$$

Reliability of a series system having n components with GXE(α, λ) is

$$R(x) = \left[\left(1 - (1 + \lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^n.$$

Reliability of a parallel system having n components having GXE(α, λ) is

$$R(x) = 1 - \left[\left(1 - (1 + \lambda x^2)e^{-\lambda(x^2+x)}\right)^{\alpha}\right]^n.$$

Both the reliability functions can be used in various reliability calculations.

3.2.4 Parameter Estimation

In this section, estimation of the unknown parameters of the GXE by using the method of moments and method of maximum likelihood is explained.

Let $X_1, X_2, \ldots, X_n$ are random sample taken from GXE. Let $m_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $m_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. Equating sample moments to population moments we get moment estimators for parameters.

$$m_1 = \alpha\lambda K(\alpha, \lambda, 1) + 2\alpha\lambda^2 K(\alpha, \lambda, 4) + \alpha\lambda^2 K(\alpha, \lambda, 3)$$
$$m_2 = \alpha\lambda K(\alpha, \lambda, 2) + 2\alpha\lambda^2 K(\alpha, \lambda, 5) + \alpha\lambda^2 K(\alpha, \lambda, 4)$$

where $K(\alpha, \lambda, 1) = \sum_{i=0}^{\alpha-1}\sum_{j=0}^{i}\binom{\alpha-1}{i}\binom{i}{j}(-1)^i\lambda^j\int_0^{\infty} x^{2j+1}e^{-\lambda(x^2+x)}\,dx$. The solution of these equations are moment estimators.

To find MLE, consider likelihood function as,

$$
\begin{aligned}
L(x;\alpha,\lambda) &= \prod_{i=1}^{n} f(x_i)\\
&= \alpha^n e^{-\lambda\sum_{i=1}^{n}(x_i^2+x_i)}\prod_{i=1}^{n}(\lambda(1+\lambda x_i^2)(2x_i+1)-2\lambda x_i)\\
&\prod_{i=1}^{n}\left(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\right)^{\alpha-1}.
\end{aligned}
$$

The log-likelihood function is,

$$
\begin{aligned}
l = \log L(x;\alpha,\lambda) = n\log\alpha &+ \sum_{i=1}^{n}\log\left(\lambda(1+\lambda x_i^2)(2x_i+1)-2\lambda x_i\right)\\
&-\lambda\sum_{i=1}^{n}(x_i^2+x_i)+(\alpha-1)\sum_{i=1}^{n}\log\left(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\right).
\end{aligned}
$$

The first partial derivatives of the log-likelihood function with respect to the two-parameters are

$$
\begin{aligned}
\frac{\partial l}{\partial\alpha} &= \frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\right)\\
\frac{\partial l}{\partial\alpha} &= 0\\
\Longrightarrow\ \hat{\alpha} &= -\frac{1}{n}\log\left(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\right) \qquad (3.2.8)\\
\text{and}\ \frac{\partial l}{\partial\lambda} &= -\sum_{i=1}^{n}(x_i^2+x_i)+\sum_{i=1}^{n}\frac{((2x_i+1)(1+2\lambda x_i^2)-2x_i)}{(\lambda(1+\lambda x_i^2)(2x_i+1)-2\lambda x_i)}\\
&+(\alpha-1)\sum_{i=1}^{n}\frac{((1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}(x_i^2+x_i)-e^{-\lambda(x_i^2+x_i)}x_i^2)}{\left(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\right)}\\
\frac{\partial l}{\partial\lambda} &= 0
\end{aligned}
$$

$$\Longrightarrow \sum_{i=1}^{n}(x_i^2 + x_i) = \sum_{i=1}^{n} \frac{((2x_i+1)(1+2\lambda x_i^2) - 2x_i)}{(\lambda(1+\lambda x_i^2)(2x_i+1) - 2\lambda x_i)}$$
$$+ (\alpha - 1)\sum_{i=1}^{n} \frac{((1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}(x_i^2+x_i) - e^{-\lambda(x_i^2+x_i)}x_i^2)}{\left(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\right)}. \tag{3.2.9}$$

Solving this system of equations (3.2.8) and (3.2.9) in α and λ gives the MLE of α and λ. Estimates can be obtained by using 'nlm' package in R software with arbitrarily initial values.

3.2.5 Asymptotic Confidence bounds

In this section, since the MLEs of the unknown parameters $\alpha > 0$ and $\lambda > 0$ cannot be obtained in closed forms, we derive the asymptotic confidence intervals of these parameters when $\alpha > 0$ and $\lambda > 0$, by using variance covariance matrix I^{-1}, where I^{-1} is the inverse of the observed information matrix which is defined as follows

$$I = \begin{pmatrix} E(-\frac{\partial^2 l}{\partial \alpha^2}) & E(-\frac{\partial^2 l}{\partial \alpha \lambda}) \\ E(-\frac{\partial^2 l}{\partial \lambda \alpha}) & E(-\frac{\partial^2 l}{\partial \lambda^2}) \end{pmatrix}$$
$$= \begin{pmatrix} \mathrm{Var}(\hat{\alpha}) & \mathrm{Cov}(\hat{\alpha}, \hat{\lambda}) \\ \mathrm{Cov}(\hat{\lambda}, \hat{\alpha}) & \mathrm{Var}(\hat{\lambda}) \end{pmatrix}.$$

The second partial derivatives are

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{n}{\alpha^2},$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\lambda^2} &= \sum_{i=0}^{n} \frac{\Big(\lambda(1+\lambda x_i^2)(2x_i+1) - 2\lambda x_i\Big)[2x_i^2(2x_i+1)]}{\Big(\lambda(1+\lambda x_i^2)(2x_i+1) - 2\lambda x_i\Big)^2} \\
&- \sum_{i=0}^{n} \frac{((2x_i+1)(1+2\lambda x_i^2) - 2x_i)^2}{\Big(\lambda(1+\lambda x_i^2)(2x_i+1) - 2\lambda x_i\Big)^2} \\
&+ \sum_{i=1}^{n} \frac{\Big(x_i^2 e^{-\lambda(x_i^2+x_i)} - (1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}(x_i^2+x_i)\Big) + e^{-\lambda(x_i^2+x_i)}(x_i^2+x_i)}{\big(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\big)} \\
&- \sum_{i=1}^{n} \frac{\Big((1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}(x_i^2+x_i) - x_i^2 e^{-\lambda(x_i^2+x_i)}\Big)^2}{\big(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\big)^2}\end{aligned}$$

and $$\frac{\partial^2 l}{\partial\alpha\lambda} = \sum_{i=1}^{n} \frac{(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}(x_i^2+x_i) - x_i^2 e^{-\lambda(x_i^2+x_i)}}{\big(1-(1+\lambda x_i^2)e^{-\lambda(x_i^2+x_i)}\big)}.$$

We can derive the $(1-\delta)100\%$ confidence intervals of the parameters α and λ by using variance matrix as in the form

$$\hat{\alpha} \pm Z_{\frac{\delta}{2}}\sqrt{\mathrm{Var}(\hat{\alpha})},\ \ \hat{\lambda} \pm Z_{\frac{\delta}{2}}\sqrt{\mathrm{Var}(\hat{\lambda})}$$

where $Z_{\frac{\delta}{2}}$ is the upper $(\frac{\delta}{2})^{\text{th}}$ percentile of the SN distribution.

3.2.6 Simulation

To understand the performance of the MLEs given by (3.2.8) and (3.2.9) with respect to sample size n, a simulation study for assessment is considered:

(i) Generate five thousand samples from (3.2.2). Using Newton Raphson method, values of the GXE random variable are generated using

$$(1+\lambda x^2)e^{-\lambda(x^2+x)} = 1 - u^{\frac{1}{\alpha}}$$

where $u \sim U(0,1)$.

(ii) Compute the MLEs for the five thousand samples, say (α_i, λ_i) for $i = 1, 2, \ldots, 5000$.

(iii) Compute the biases of the estimator and mean squared errors using $\text{bias}_h(n) = \frac{1}{5000}\sum_{i=1}^{5000}(\hat{h}_i - h)$ and $\text{MSE}_h(n) = \frac{1}{5000}\sum_{i=1}^{5000}(\hat{h}_i - h)^2$ for $h = (\alpha, \lambda)$.

We repeated these steps for $n = 10, 20, \ldots, 100$ with different values of parameters, for computing $\text{bias}_h(n)$ and $\text{MSE}_h(n)$ for $n = 10, 20, \ldots, 100$.

3.2.7 Data Analysis

In this section, we present the analysis of a real data for using the GXE(α, λ) model and compare it with Generalized Lindley (GL) distribution using AIC, BIC and K-S statistic. We considered the survival data for psychiatric inpatients (Klein and Moesch Berger (1997)) to estimate the parameter values. The data are given in Table 3.2. Table 3.3 provides the parameter estimates, standard errors obtained by inverting the observed information matrix and log-likelihood values. Table 3.4 provides values of AIC, BIC, and p-values based on the K-S statistic. The corresponding probability plots and histogram are shown in Figure 3.4 and 3.5. We can see that the GXE distribution provides the smallest AIC

Table 3.1: Simulation study on different choices of parameter values

	$\alpha = 2.5$		$\lambda = 0.75$		$\alpha = 0.5$		$\lambda = 0.001$	
n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
10	0.37515	1.40735	0.03116	0.00971	0.00256	6.537×10^{-5}	-7.531×10^{-6}	5.6723×10^{-10}
20	0.00777	0.00121	0.00967	0.00187	-0.001132	2.5621×10^{-5}	6.3116×10^{-6}	7.9673×10^{-10}
30	-0.00347	0.000361	-0.00133	5.2976×10^{-5}	0.00992	0.001967	3.9965×10^{-6}	3.1944×10^{-10}
40	0.002158	0.0001863	-0.001515	9.1865×10^{-5}	0.001568	9.8350×10^{-5}	-1.1745×10^{-6}	5.5179×10^{-11}
50	-0.008035	0.003228	0.001646	0.0001354	0.000202	2.0495×10^{-6}	5.5596×10^{-6}	1.5455×10^{-9}
60	0.00111	7.4300×10^{-5}	-7.695×10^{-6}	3.5528×10^{-9}	-0.000341	6.9802×10^{-6}	-2.9708×10^{-6}	5.2953×10^{-10}
70	0.002488	0.0004332	-0.0001722	2.0761×10^{-6}	-0.000331	7.6762×10^{-6}	-3.4171×10^{-6}	8.1738×10^{-10}
80	0.001709	0.0002337	5.6173×10^{-5}	2.5243×10^{-7}	2.4323×10^{-5}	4.7331×10^{-8}	7.1859×10^{-7}	4.1309×10^{-11}
90	0.001912	0.0003292	0.0001403	1.7714×10^{-6}	0.000429	1.6546×10^{-5}	-4.3202×10^{-7}	1.6798×10^{-11}
100	0.001929	0.0003720	0.000216	4.6694×10^{-6}	0.001706	0.0002910	2.5672×10^{-6}	6.5907×10^{-10}
	$\alpha = 1$		$\lambda = 0.5$		$\alpha = 1.5$		$\lambda = 1$	
10	0.00465	0.000216	-0.00458	0.00021	-0.05683	0.03229	-0.04672	0.02183
20	-0.00292	0.000171	0.02258	0.0102	-0.020347	0.008281	-0.00697	0.000973
30	-9.052×10^{-5}	2.4583×10^{-7}	-0.00184	0.000102	0.001420	6.0464×10^{-5}	0.005746	0.000991
40	-0.000876	3.0695×10^{-5}	-0.000901	3.2446×10^{-5}	-0.003607	0.0005203	0.006072	0.001475
50	0.00128	8.1932×10^{-5}	-0.000622	1.9383×10^{-5}	0.00479	0.001149	0.002649	0.000351
60	0.000594	2.117×10^{-5}	0.00118	8.3813×10^{-5}	0.001702	0.000174	-0.000463	1.288×10^{-5}
70	-0.00155	0.000168	-0.000892	5.5728×10^{-5}	0.003831	0.001027	0.000398	1.1106×10^{-5}
80	-0.000983	7.7277×10^{-5}	-0.000934	6.9806×10^{-5}	0.00285	0.000649	0.000584	2.7314×10^{-5}
90	0.001544	0.000214	-5.4667×10^{-5}	2.6896×10^{-7}	0.000841	6.359×10^{-5}	0.000968	8.4369×10^{-5}
100	0.001396	0.000195	0.000373	1.3918×10^{-5}	0.000537	2.8865×10^{-5}	0.000275	7.5845×10^{-6}
	$\alpha = 2.5$		$\lambda = 1.25$		$\alpha = 3$		$\lambda = 1.75$	
10	0.008654	0.000749	0.01351	0.001825	0.03068	0.009415	0.02618	0.006855
20	-0.005014	0.000503	-0.00401	0.000321	0.017103	0.00585	0.009213	0.001698
30	0.00994	0.002963	0.002236	0.000149	-0.00845	0.002143	0.00851	0.002172
40	-0.004481	0.000803	-0.000823	2.7113×10^{-5}	-0.00231	0.000213	0.003504	0.000491
50	0.00666	0.002220	-0.000140	9.8505×10^{-7}	0.005896	0.001738	0.000748	2.801×10^{-5}
60	0.000206	2.5387×10^{-6}	0.001005	6.0642×10^{-5}	0.000311	5.7859×10^{-6}	0.002391	0.000343
70	-4.04×10^{-5}	1.1425×10^{-7}	0.000722	3.6513×10^{-5}	0.001375	0.0001324	-0.000786	4.3222×10^{-5}
80	0.000882	6.2230×10^{-5}	0.001149	0.000106	0.001103	9.7265×10^{-5}	0.000192	2.9403×10^{-6}
90	0.000732	4.8281×10^{-5}	-0.000923	7.6744×10^{-5}	-0.000874	6.8733×10^{-5}	-0.000445	1.7813×10^{-5}
100	-0.000957	9.1617×10^{-5}	0.000349	1.2245×10^{-5}	8.322×10^{-5}	6.9256×10^{-7}	-0.000947	8.9755×10^{-5}

Table 3.2: Survival data for psychiatric inpatients

1	1	2	22	30	28	32	11	14	36	31	33	33
37	35	25	31	22	26	24	35	34	30	35	40	39

value and BIC value, the largest p-value based on the K-S statistic. Hence, the GXE distribution provides the better fit. The variance covariance matrix I^{-1} of

Table 3.3: MLEs of parameters, SE and Log-likelihood

Model	ML estimates	Standard error	Log L
GXE	$\hat{\alpha} = 0.6967$	0.03313	-99.360
	$\hat{\lambda} = 0.00184$	7.852×10^{5}	
GL	$\hat{\alpha} = 1.0691$	0.05637	-107.657
	$\hat{\lambda} = 0.07547$	0.00274	

Table 3.4: AIC, BIC, K-S Statistic and p-value of the model

Model	AIC	BIC	K-S Statistic	p value
GXE	202.721	205.237	0.2113	0.1962
GL	219.315	221.831	0.3011	0.0179

the MLEs of GXE distribution for the data set 1 is computed as

$$= \begin{pmatrix} 2.8532 \times 10^{-2} & 3.9810 \times 10^{-5} \\ 3.9810 \times 10^{-5} & 1.6031 \times 10^{-7} \end{pmatrix}.$$

Thus, the variances of the MLE of α and λ are $\mathrm{Var}(\hat{\alpha}) = 2.853 \times 10^{-2}$ and $\mathrm{Var}(\hat{\lambda}) = 1.603 \times 10^{-7}$, respectively. Therefore, 95% confidence intervals for α and λ are [0.6417, 0.7507] and [0.0017, 0.00197], respectively. The data set 2 consist of the lifetimes of 50 devices (Aarset (1987)) and it is provided in Table 3.5. The parameter estimates, standard errors of the estimators and the various

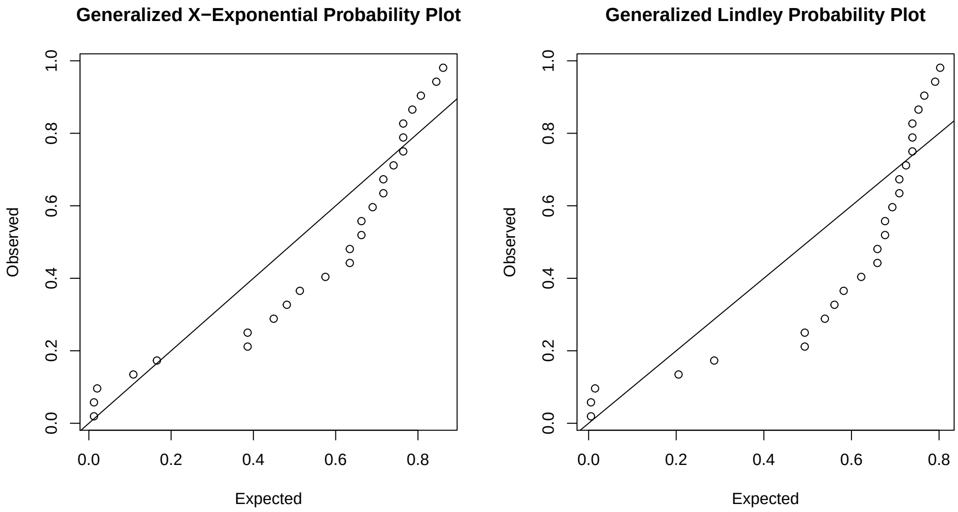

Figure 3.4: Probability plots of data set 1.

Table 3.5: Lifetimes of 50 devices

0.1	0.2	1	1	1	1	1	2	3	6	7	11	12	18	18
18	18	18	21	32	36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83	84	84	84	85	85
85	85	85	86	86										

measures are given in Table 3.6. The corresponding histogram and probability plots are shown in Figure 3.6 and 3.7.

We can see again that the GXE distribution gives the smallest AIC value, the smallest BIC value, and largest p-value based on the K-S statistic, see Table 3.7. Hence, the GXE distribution again provides the better fit. The variance covariance matrix I^{-1} of the MLEs under the GXE distribution for the data set

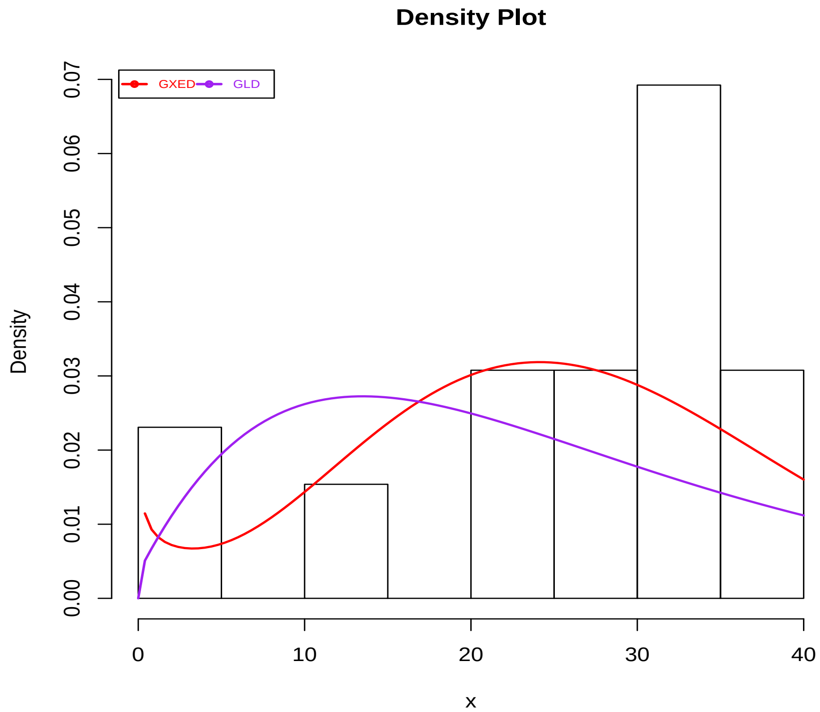

Figure 3.5: Histogram with fitted Pdfs for the data set 1.

Table 3.6: MLEs of parameters, SE and Log-likelihood

Model	ML estimates	Standard error	Log L
GXE	$\hat{\alpha} = 0.3181$ $\hat{\lambda} = 0.000302$	0.00715 1.0452×10^5	-231.609
GLD	$\hat{\alpha} = 0.4547$ $\hat{\lambda} = 0.0278$	0.01123 0.000691	-238.9909

2 is computed as

$$= \begin{pmatrix} 2.5596 \times 10^{-3} & 1.7080 \times 10^{-6} \\ 1.7080 \times 10^{-6} & 5.4625 \times 10^{-9} \end{pmatrix}.$$

Table 3.7: AIC, BIC, K-S Statistic and p-value of the model

Model	AIC	BIC	K-S Statistic	p value
GXE	467.218	471.042	0.1555	0.1783
GLD	481.982	485.806	0.1936	0.0472

Thus, the variances of the MLEs of α and λ are $\mathrm{Var}(\hat{\alpha}) = 2.56 \times 10^{-3}$ and $\mathrm{Var}(\hat{\lambda}) = 5.463 \times 10^{-9}$, respectively. Therefore, 95% confidence intervals for α and λ are [0.3064, 0.3299] and [0.00029, 0.00032], respectively.

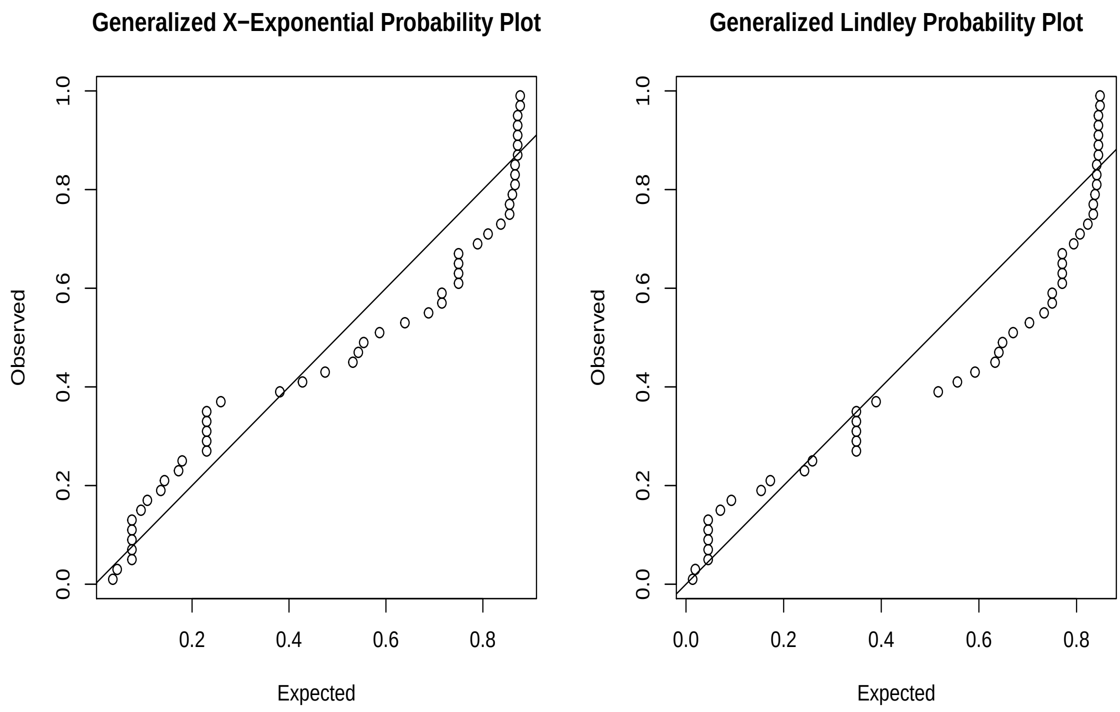

Figure 3.6: Probability plots of data set 2.

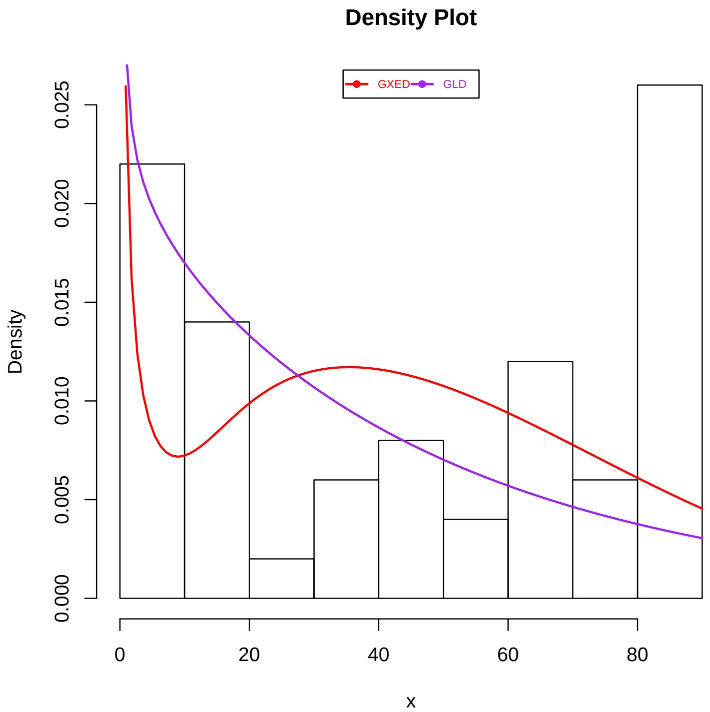

Figure 3.7: Histogram with fitted Pdfs for the data set 2.

3.3 Weibull-Lindley Distribution

Let X be a random variable with the cdf,

$$F(x;\alpha,\beta) = 1 - e^{-\alpha\left((1+x)e^{(x)^{\beta}}-1\right)}, \quad x > 0,\ \alpha,\beta > 0. \tag{3.3.1}$$

The r.v X is said to have Weibull-Lindly (WL) distribution if its distribution function is in the form (3.3.1). It will be denoted by WL(α,β). Then, the pdf corresponding to (3.3.1) is given by

$$f(x;\alpha,\beta)=\alpha\left(\beta x^{\beta-1}(1+x)e^{x^{\beta}}+e^{x^{\beta}}\right)e^{-\alpha\left((1+x)e^{x^{\beta}}-1\right)},$$
$$x>0,\ \ \alpha>0,\ \beta>0. \quad (3.3.2)$$

Here β is shape parameter. The pdf of WL distribution can be rewritten as

$$f(x;\alpha,\beta)=\alpha\beta x^{\beta-1}(1+x)e^{x^{\beta}}e^{-\alpha\left((1+x)e^{x^{\beta}}-1\right)}+\alpha e^{x^{\beta}}e^{-\alpha\left((1+x)e^{x^{\beta}}-1\right)},$$
$$x>0,\ \ \alpha>0,\ \beta>0. \quad (3.3.3)$$

By using the power series expansion for the exponential function, we obtain

$$e^{-\alpha\left((1+x)e^{x^{\beta}}-1\right)}=\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\left((1+x)e^{x^{\beta}}-1\right)^i. \quad (3.3.4)$$

Substituting (3.3.4) in (3.3.3), we get

$$f(x;\alpha,\beta)=\alpha\beta x^{\beta-1}(1+x)e^{x^{\beta}}\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\left((1+x)e^{x^{\beta}}-1\right)^i$$
$$+\alpha e^{x^{\beta}}\sum_{i=0}^{\infty}\frac{(-1)^i\alpha^i}{i!}\left((1+x)e^{x^{\beta}}-1\right)^i,\quad x>0,\ \ \alpha>0,\ \beta>0. \quad (3.3.5)$$

Using the generalized binomial theorem, we have

$$\left((1+x)e^{x^{\beta}}-1\right)^i=\sum_{j=o}^{i}\frac{i!}{j!(i-j)!}\left((1+x)e^{x^{\beta}}\right)^{i-j}(-1)^j. \quad (3.3.6)$$

Inserting (3.3.6) in (3.3.5), we get

$$f(x;\alpha,\beta) = \alpha\beta x^{\beta-1}(1+x)e^{x^\beta}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^i\alpha^i}{j!(i-j)!}\left((1+x)e^{x^\beta}\right)^{i-j}$$
$$+\,\alpha e^{x^\beta}\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(-1)^i\alpha^i}{j!(i-j)!}\left((1+x)e^{x^\beta}\right)^{i-j}, \quad x>0,\ \alpha>0,\ \beta>0. \tag{3.3.7}$$

The pdf can be further simplified as

$$f(x;\alpha,\beta) = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{i+1-j}\frac{(-1)^{i+j}\alpha^{i+1}\beta(i-j+1)}{j!k!(i-j-k+1)!}e^{x^\beta(i-j+1)}x^{\beta+k-1}$$
$$+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{i-j}\frac{(-1)^{i+j}\alpha^{i+1}}{j!m!(i-j-m)!}e^{x^\beta(i-j+1)}x^m, \quad x>0,\ \alpha>0,\ \beta>0. \tag{3.3.8}$$

Figure 3.8 provide the pdfs of WL(α,β) for different parameter values. From the below figures it is immediate that the pdfs are unimodal. The survival function $S(x)$, failure rate function $r(x)$, reversed failure rate function $h(x)$ and cumulative failure rate function $H(x)$ of X are

$$S(x;\alpha,\beta) = e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}, \quad x>0,\ \alpha>0,\ \beta>0. \tag{3.3.9}$$

$$r(x;\alpha,\beta) = \alpha\left(\beta x^{\beta-1}(1+x)e^{x^\beta}+e^{x^\beta}\right), \quad x>0,\ \alpha>0,\ \beta>0. \tag{3.3.10}$$

$$h(x;\alpha,\beta) = \frac{\alpha\left(\beta x^{\beta-1}(1+x)e^{x^\beta}+e^{x^\beta}\right)e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}}{1-e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}}, \quad x>0,\ \alpha>0,\ \beta>0. \tag{3.3.11}$$

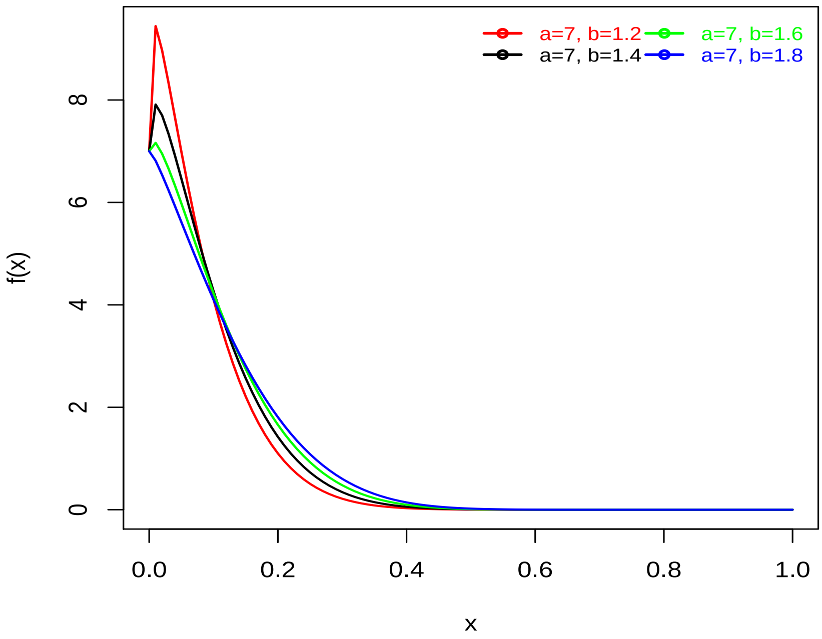

Figure 3.8: PDF of the WL(α, β).

and

$$H(x; \alpha, \beta) = \int_0^x r(t; \alpha, \beta)\, dt = \alpha(1+x)e^{x^\beta}, \quad x > 0,\ \alpha > 0,\ \beta > 0. \qquad (3.3.12)$$

The failure rate function of the WL(α, β) exhibit increasing, decreasing and bathtub shapes. We can see from that

$$\lim_{x \to 0} r(x) = \begin{cases} \infty, & \beta < 1 \\ 2\alpha, & \beta = 1 \\ \alpha, & \beta > 1. \end{cases}$$

Figure 3.9 provide the failure rate functions of WL(α, β) for different parameter values. From the below figures it is immediate that the failure rate function can be increasing, decreasing or bathtub shaped. It is clear that the pdf and the

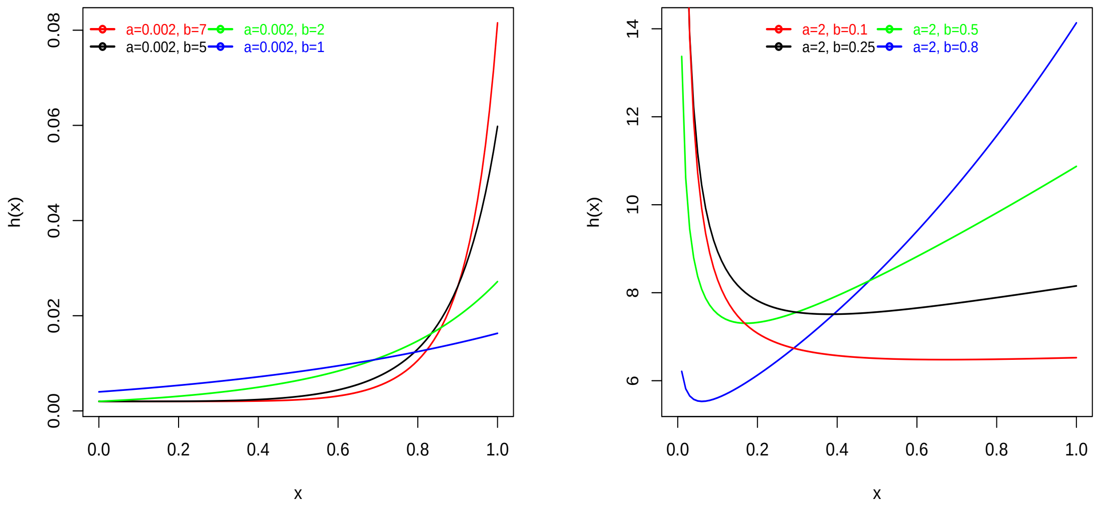

Figure 3.9: Failure rate function of the WL(α, β).

failure rate function have many different shapes, which allows this distribution to fit different types of lifetime data. For fixed α, the failure rate function is non-decreasing function if $\beta > 1$ (left) and non-increasing and bathtub function if $\beta < 1$ (right).

3.3.1 Statistical Properties

In this section, we study the statistical properties for the WL(α, β), specially Quantile function, Median, Mode, Moments etc.

Quantile and Median

We obtain the $100p^{\text{th}}$ percentile,

$$(1+x)e^{x^{\beta}} = -\frac{1}{\alpha}\log(1-p)+1. \tag{3.3.13}$$

Setting in (3.3.13), we get the median of WL(α,β) from

$$(1+x)e^{x^{\beta}} = \frac{1}{\alpha}\log\left(\frac{1}{1-0.5}\right)+1.$$

x_p is the solution of above monotone increasing function. Software can be used to obtain the Quantiles/Percentiles.

Mode

Mode can be obtained as solution of

$$[h'(x;\alpha,\beta)-(h(x;\alpha,\beta))^2].S(x;\alpha,\beta)=0. \tag{3.3.14}$$

It is not possible to get an analytic solution in x for (3.3.14). It can be obtained numerically by using methods such as fixed-point or bisection method.

Moments

We obtain the r^{th} moment of WL(α, β) in the form

$$
\begin{aligned}
\mu'_r &= \int_0^\infty x^r f(x;\alpha,\beta)\ dx \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+j}\alpha^{i+1}}{j!k!(i-j-k+1)!} \int_0^\infty x^{r+\beta+k-1} e^{x^\beta(i-j+1)}\ dx \\
&+ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i+j}\alpha^{i+1}}{j!m!(i-j-m)!} \int_0^\infty x^{r+m} e^{x^\beta(i-j+1)}\ dx.
\end{aligned}
$$

By using the definition of Gamma function, we get

$$
\begin{aligned}
\mu'_r &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{i+1-j} \frac{(-1)^{i+j+\frac{\beta+r+k}{\beta}}\alpha^{i+1}}{j!k!(i-j-k+1)!} \frac{\Gamma(\frac{\beta+r+k}{\beta})}{\beta(i-j+1)^{\frac{\beta+r+k}{\beta}}} \\
&+ \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{i-j} \frac{(-1)^{i+j+\frac{m+r+1}{\beta}}\alpha^{i+1}}{j!m!(i-j-m)!} \frac{\Gamma(\frac{m+r+1}{\beta})}{\beta(i-j+1)^{\frac{m+r+1}{\beta}}}.
\end{aligned} \tag{3.3.15}
$$

If (3.3.15) is a convergent series for any $r \geq 0$, therefore all the moments exist and for integer values of α and β, it can be represented as a finite series representation. Therefore putting $r = 1$, we obtain the mean as

$$
\begin{aligned}
E(X) &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{i+1-j} \frac{(-1)^{i+j+\frac{\beta+k+1}{\beta}}\alpha^{i+1}}{j!k!(i-j-k+1)!} \frac{\Gamma(\frac{\beta+k+1}{\beta})}{\beta(i-j+1)^{\frac{\beta+k+1}{\beta}}} \\
&+ \sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{i-j} \frac{(-1)^{i+j+\frac{m+2}{\beta}}\alpha^{i+1}}{j!m!(i-j-m)!} \frac{\Gamma(\frac{m+2}{\beta})}{\beta(i-j+1)^{\frac{m+2}{\beta}}}
\end{aligned}
$$

and putting $r=2$, we obtain the second moment as

$$E(X^2)=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{i+1-j}\frac{(-1)^{i+j+\frac{\beta+k+2}{\beta}}\alpha^{i+1}}{j!k!(i-j-k+1)!}\frac{\Gamma(\frac{\beta+k+2}{\beta})}{\beta(i-j+1)^{\frac{\beta+k+2}{\beta}}}$$
$$+\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{i-j}\frac{(-1)^{i+j+\frac{m+3}{\beta}}\alpha^{i+1}}{j!m!(i-j-m)!}\frac{\Gamma(\frac{m+3}{\beta})}{\beta(i-j+1)^{\frac{m+3}{\beta}}}.$$

It can be used to obtain the higher central moments and variance.

Moment Generating Function and Characteristic Function

The moment generating function, $M_X(t)$, of WL(α,β) is obtained as

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu'$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{i+1-j}\frac{t^r(-1)^{i+j+\frac{\beta+r+k}{\beta}}\alpha^{i+1}}{r!j!k!(i-j-k+1)!}\frac{\Gamma(\frac{\beta+r+k}{\beta})}{\beta(i-j+1)^{\frac{\beta+r+k}{\beta}}}$$
$$+\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{i-j}\frac{t^r(-1)^{i+j+\frac{m+r+1}{\beta}}\alpha^{i+1}}{r!j!m!(i-j-m)!}\frac{\Gamma(\frac{m+r+1}{\beta})}{\beta(i-j+1)^{\frac{m+r+1}{\beta}}}.$$

The characteristic function, $\phi_X(t)$, of WL(α,β) is obtained as

$$\phi_X(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{i+1-j}\frac{(it)^r(-1)^{i+j+\frac{\beta+r+k}{\beta}}\alpha^{i+1}}{r!j!k!(i-j-k+1)!}\frac{\Gamma(\frac{\beta+r+k}{\beta})}{\beta(i-j+1)^{\frac{\beta+r+k}{\beta}}}$$
$$+\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{m=0}^{i-j}\frac{(it)^r(-1)^{i+j+\frac{m+r+1}{\beta}}\alpha^{i+1}}{r!j!m!(i-j-m)!}\frac{\Gamma(\frac{m+r+1}{\beta})}{\beta(i-j+1)^{\frac{m+r+1}{\beta}}}.$$

3.3.2 Distribution of Maximum and Minimum

Let $X_1, X_2, \ldots, X_n$ be a random sample from WL(α, β) with cdf and pdf as in (3.3.1) and (3.3.2), respectively. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics obtained from this sample. The pdf of $X_{(r)}$ is given by,

$$f_{(r:n)}(x) = \frac{1}{B(r, n-r+1)} \left[1 - e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}\right]^{r-1} \left[e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}\right]^{n-r}$$
$$\alpha\left(\beta x^{\beta-1}(1+x)e^{x^\beta} + e^{x^\beta}\right)e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}, \quad x > 0,\ \alpha > 0,\ \beta > 0. \quad (3.3.16)$$

The cdf of $X_{(r)}$ is given by,

$$F_{r:n}(x) = \sum_{j=r}^{n} \binom{n}{j} \left[1 - e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}\right]^{j} \left[e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}\right]^{n-j},$$
$$x > 0,\ \alpha > 0,\ \beta > 0. \quad (3.3.17)$$

The cdf of $X_{(1)}$ is

$$F_{X_{(1)}}(x;\alpha,\beta) = P(X_{(1)} \le x) = 1 - \left[1 - e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}\right]^{n}, \quad x > 0,\ \alpha > 0,\ \beta > 0.$$

The cdf of $X_{(n)}$ is

$$F_{X_{(n)}}(x;\alpha,\beta) = P(X_{(n)} \le x) = \left[e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}\right]^{n}, \quad x > 0,\ \alpha > 0,\ \beta > 0.$$

Reliability of a series system having n components with WL(α, β) is

$$R(x) = \left[1 - e^{-\alpha\left((1+x)e^{x^\beta}-1\right)}\right]^{n}.$$

Reliability of a parallel system having n components with WL(α, β) is

$$R(x) = 1 - \left[e^{-\alpha\left((1+x)e^{x^\beta}-1\right)} \right]^n.$$

3.3.3 Parameter Estimation

In this section, point estimation of the unknown parameters of the WL(α, β) are conducted using MLE. First partial derivatives of the log-likelihood function with respect to the two-parameters are

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}(1+x_i)e^{x_i^\beta} \tag{3.3.18}$$

and

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{n} x_i^\beta \log x_i - \alpha \sum_{i=1}^{n}\left((1+x_i)e^{x_i^\beta} x_i^\beta \log x_i\right) + \sum_{i=1}^{n} \frac{(1+x_i)\left[x_i^{\beta-1} + \beta x_i^{\beta-1}\log x_i\right]}{\beta x_i^{\beta-1}(1+x_i)+1}. \tag{3.3.19}$$

Setting the left side of the above two equations to zero, we get the likelihood equations as a system of two non-linear equations in α and β. Solving this system in α and β gives the MLE of α and β. It is very easy to obtain estimates using R software by numerical methods.

3.3.4 Asymptotic Confidence bounds

In this section, we derive the asymptotic confidence intervals of the parameters α and β, since the MLEs of the unknown parameters α and β cannot be obtained in closed forms. Let the variance covariance matrix be denoted by I^{-1}, where I^{-1} is the inverse of the observed information matrix which defined as follows

$$I^{-1} = \begin{pmatrix} E(-\frac{\partial^2 l}{\partial \alpha^2}) & E(-\frac{\partial^2 l}{\partial \alpha \beta}) \\ E(-\frac{\partial^2 l}{\partial \beta \alpha}) & E(-\frac{\partial^2 l}{\partial \beta^2}) \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{Var}(\hat{\alpha}) & \mathrm{Cov}(\hat{\alpha}, \hat{\beta}) \\ \mathrm{Cov}(\hat{\beta}, \hat{\alpha}) & \mathrm{Var}(\hat{\beta}) \end{pmatrix}$$

where the second partial derivatives of log-likelihood function are

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{n}{\alpha}, \quad \frac{\partial^2 l}{\partial \alpha \beta} = \sum_{i=1}^{n} (1+x_i) e^{x_i^\beta} x_i^\beta \log x_i$$

and

$$\begin{aligned} \frac{\partial^2 l}{\partial \beta^2} &= \sum_{i=1}^{n} x_i^\beta (\log x_i)^2 - \alpha \sum_{i=1}^{n} (1+x_i) \left[e^{x_i^\beta} (x_i^\beta \log x_i)^2 + e^{x_i^\beta} x_i^\beta (\log x_i)^2 \right] \\ &+ \sum_{i=1}^{n} \frac{\left(\beta x_i^{\beta-1}(1+x_i)+1\right)\left(\beta x_i^{\beta-1}(1+x_i)(\log x_i)^2 + 2(1+x_i)x_i^{\beta-1}\right)}{\left(\beta x_i^{\beta-1}(1+x_i)+1\right)^2} \\ &- \sum_{i=1}^{n} \frac{\left((1+x_i)\left[x_i^{\beta-1} + \beta x_i^{\beta-1} \log x_i\right]\right)^2}{\left(\beta x_i^{\beta-1}(1+x_i)+1\right)^2}. \end{aligned}$$

We can derive the $(1-\xi)100\%$ confidence intervals of the parameters α and β as

$$\hat{\alpha} \pm Z_{\frac{\xi}{2}} \sqrt{\mathrm{Var}(\hat{\alpha})}, \ \hat{\beta} \pm Z_{\frac{\xi}{2}} \sqrt{\mathrm{Var}(\hat{\beta})}$$

where $Z_{\frac{\xi}{2}}$ is the upper $(\frac{\xi}{2})^{\text{th}}$ percentile of the standard Normal distribution.

3.3.5 Three parameter Weibull-Lindley Distribution

A random variable X is said to have three parameter Weibull-Lindley (3WL) distribution if its cdf is of the form,

$$F(x;\alpha,\beta,\lambda) = 1 - e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^\beta}-1\right)}, \quad x>0,\ \alpha>0, \beta>0, \lambda>0. \tag{3.3.20}$$

The pdf corresponding to Eq.(3.3.20) is given by

$$f(x;\alpha,\beta,\lambda) = \alpha\left(\beta\lambda(\lambda x)^{\beta-1}(1+\lambda x)e^{(\lambda x)^\beta} + \lambda e^{(\lambda x)^\beta}\right)e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^\beta}-1\right)}, \quad x>0,\ \alpha,\ \beta,\ \lambda>0. \tag{3.3.21}$$

Here β is shape parameter and λ is scale parameter. The distribution of this form with parameters α, β, and λ will be denoted by 3WL(α,β,λ). The survival function $S(x;\alpha,\beta,\lambda)$, failure rate function $r(x;\alpha,\beta,\lambda)$, reversed failure rate function $h(x;\alpha,\beta,\lambda)$ and cumulative failure rate function $H(x;\alpha,\beta,\lambda)$ of X are

$$S(x;\alpha,\beta,\lambda) = 1 - F(x;\alpha,\beta,\lambda) = e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^\beta}-1\right)}, \quad x>0,\ \alpha,\ \beta,\ \lambda>0, \tag{3.3.22}$$

$$r(x;\alpha,\beta,\lambda) = \alpha\left(\beta\lambda(\lambda x)^{\beta-1}(1+\lambda x)e^{(\lambda x)^\beta} + \lambda e^{(\lambda x)^\beta}\right), \quad x>0,\ \alpha,\ \beta,\ \lambda>0, \tag{3.3.23}$$

$$h(x;\alpha,\beta,\lambda) = \frac{\alpha\left(\beta\lambda(\lambda x)^{\beta-1}(1+\lambda x)e^{(\lambda x)^\beta} + \lambda e^{(\lambda x)^\beta}\right)e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^\beta}-1\right)}}{1 - e^{-\alpha\left((1+\lambda x)e^{(\lambda x)^\beta}-1\right)}}, \quad x>0,\ \alpha,\ \beta,\ \lambda>0 \tag{3.3.24}$$

and

$$H(x;\alpha,\beta,\lambda) = \int_0^x r(t;\alpha,\beta,\lambda)dt = \alpha(1+\lambda x)e^{(\lambda x)^\beta}, \;\; x>0, \; \alpha, \; \beta, \; \lambda > 0 \quad (3.3.25)$$

respectively. Figure 3.10 and Figure 3.11 provide the pdfs and the failure rate functions of 3WL(α,β,λ) for different parameter values. From the below figures it is immediate that the pdfs can be unimodal and the failure rate function can be increasing, decreasing or bathtub shaped. It is clear that the pdf and the

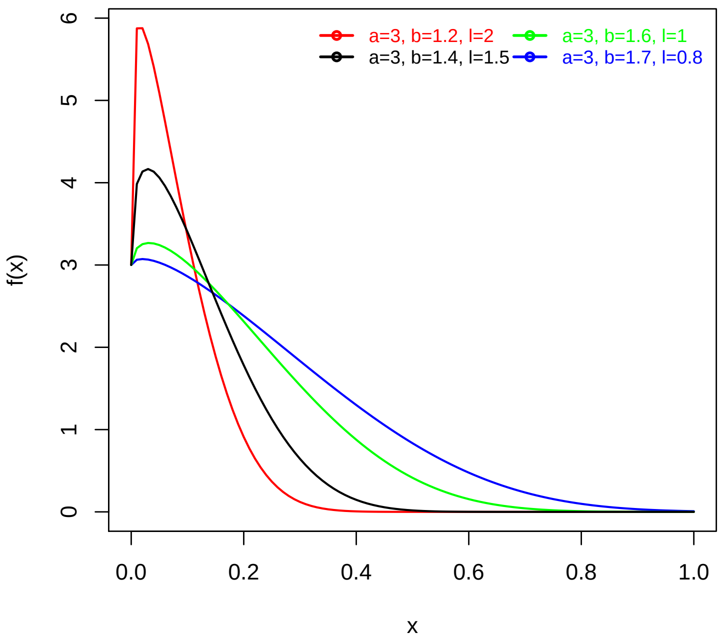

Figure 3.10: PDF of the 3WL(α,β,λ).

failure rate function have many different shapes, which allows this distribution to fit different types of lifetime data. For fixed α, F is IFR if $\beta > 1$ and $\lambda > 1$, (left)

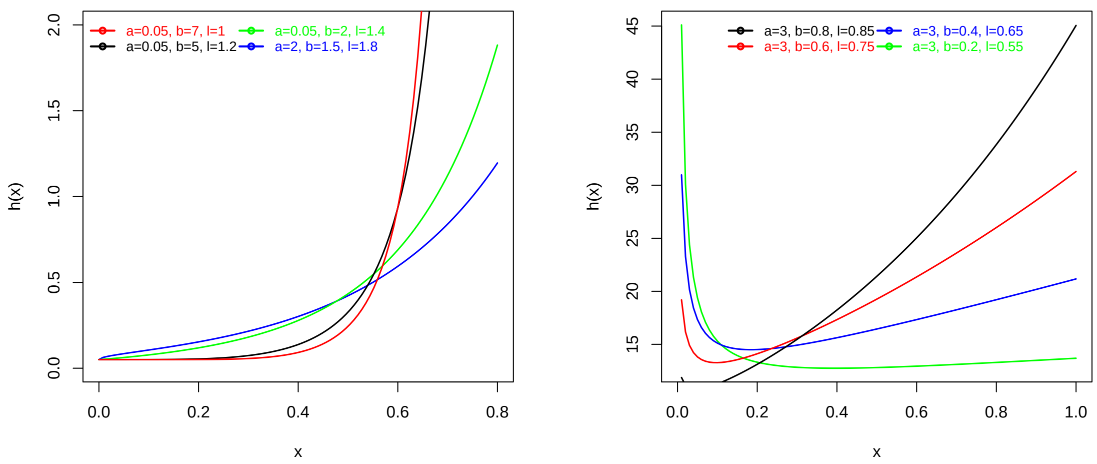

Figure 3.11: Failure rate function of the 3WLD(α, β, λ).

and DFR and BFR if $\beta < 1$ and $\lambda < 1$ (right).

Parameter Estimation

In this section, point estimation of the unknown parameters of the 3WL(α, β, λ) are done by method of maximum likelihood. The first partial derivatives of the log-likelihood function with respect to the three-parameters are

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=0}^{n} \big(1 + \lambda x_i\big) e^{(\lambda x_i)^\beta} + n, \tag{3.3.26}$$

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{n} (\lambda x_i)^\beta \log(\lambda x_i) - \alpha \sum_{i=1}^{n} \Bigg((1 + \lambda x_i) e^{(\lambda x_i)^\beta} (\lambda x_i)^\beta \log(\lambda x_i) \Bigg) + \sum_{i=1}^{n} \frac{(1 + \lambda x_i)\big((\lambda x_i)^{\beta - 1} + \beta (\lambda x_i)^{\beta - 1} \log(\lambda x_i)\big)}{\beta (\lambda x_i)^{\beta - 1} (1 + \lambda x_i) + 1} \tag{3.3.27}$$

and

$$\frac{\partial L}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n} x_i^{\beta} \beta \lambda^{\beta-1} - \alpha \sum_{i=1}^{n} \left((1+\lambda x_i) e^{(\lambda x_i)^{\beta}} x_i \beta (\lambda x_i)^{\beta} + x_i e^{(\lambda x_i)^{\beta}} \right) + \sum_{i=1}^{n} \frac{\beta x_i^{\beta} \left((\beta-1)\lambda^{(\beta-2)} + x_i \beta \lambda^{\beta-1} \right)}{\left(\beta (\lambda x_i)^{\beta-1} (1+\lambda x_i) + 1 \right)}. \quad (3.3.28)$$

Setting the left side of the above three equations to zero, we get the likelihood equations as a system of three non-linear equations in α, β and λ. Solving this system in α, β and λ gives the MLEs of α, β and λ. It is very easy to obtain estimates using R software by numerical methods.

3.3.6 Application

In this section, we present the analysis of a real data set using the WL(α, β) and 3WL(α, β, λ) model and compare it with the other bathtub models such as Generalized Lindley distribution (GL), Nadarajah et al. (2011) and Exponentiated Weibull distribution (EW), Pal et al. (2006), using K-S statistic. We considered two sets of data, which are strengths of 1.5 cm glass fibers data, Smith and Naylor (1987) and infection for AIDS data, Klein and Moesch Berger (1997).

Data Set 1: The data are the strengths of 1.5 cm glass fibers, Smith and Naylor (1987), measured at the National Physical Laboratory, England. The data set 1 is given in Table 3.8. Table 3.9 gives MLEs of parameters of the WL(α, β), GL, EW and 3WL(α, β, λ) and goodness of fit statistics.

3WL(α, β, λ) gives the smallest K-S value and largest p-value. The second

Table 3.8: Strengths of 1.5 cm glass fibres

0.55	0.93	1.25	1.36	1.49	1.52	1.58	1.61	1.64	1.68
1.73	1.81	2	0.74	1.04	1.27	1.39	1.49	1.53	1.59
1.61	1.66	1.68	1.76	1.82	2.01	0.77	1.11	1.28	1.42
1.5	1.54	1.6	1.62	1.66	1.69	1.76	1.84	2.24	0.81
1.13	1.29	1.48	1.5	1.55	1.61	1.62	1.66	1.7	1.77
1.84	0.84	1.24	1.3	1.48	1.51	1.55	1.61	1.63	1.67
1.7	1.78	1.89							

Table 3.9: MLEs of parameters, Log-likelihood

Model	**MLEs of Parameters**	**log L**	**K-S**	p**-value**
WL	$\hat{\alpha}$=0.0285 $\hat{\beta}$=1.893	-16.639	0.1368	0.189
GL	$\hat{\alpha}$=26.172 $\hat{\lambda}$=2.9901	-30.6199	0.2264	0.00314
EW	$\hat{\alpha}$=7.285 $\hat{\beta}$=0.67122 $\hat{\lambda}$=0.582	-14.676	0.146	0.135
3WL	$\hat{\alpha}$=0.000212 $\hat{\beta}$=0.8378 $\hat{\lambda}$=5.3257	-14.4228	0.1256	0.273

smallest K-S value and largest p-value are obtained for the WL distribution. The second largest log-likelihood value is given by the EW distribution. Fitted pdfs and probability plots of the three best fitting distributions for data set 1 are given in Figure 3.12 and 3.13.

Data Set 2: Consider times to infection for AIDS for two hundred and ninety five

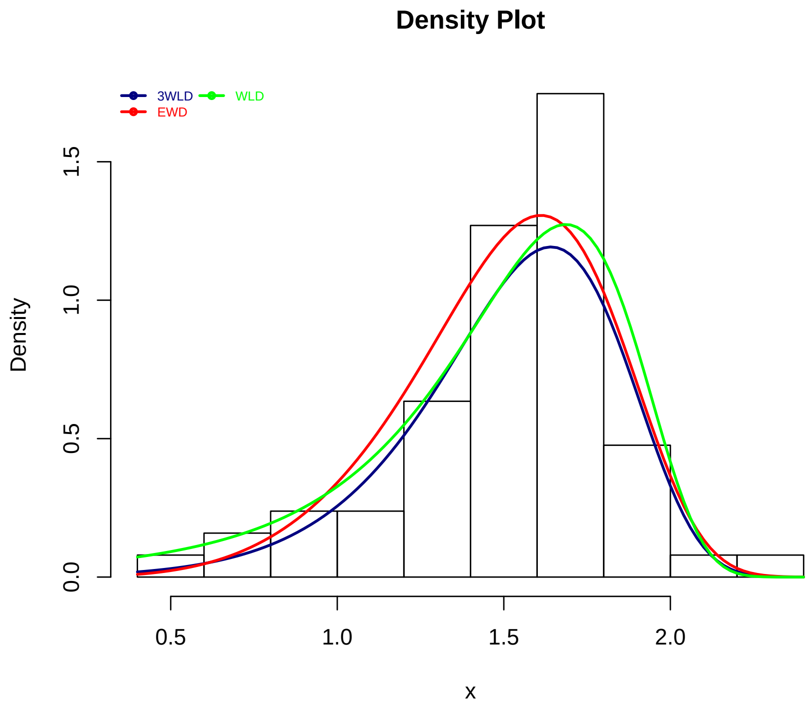

Figure 3.12: Fitted Pdfs of the three best fitting distributions for data set 1.

patients, Klein and Moesch Berger (1997). Table 3.10 gives MLEs of parameters of the WL(α, β), GL, EW and 3WL(α, β, λ) and goodness of fit statistics.

Here, the smallest K-S value and largest p-value are obtained for 3WL(α, β, λ). EW gives the largest log-likelihood value and second largest p-value. The third largest log-likelihood value and p-value based are obtained for WL(α, β). It is observed that 3WL(α, β, λ) fits as the best in the first data set whereas EW fits as the best in the second data in terms of likelihood and in terms of KS Statistic. Therefore, it is not guaranteed the 3WL(α, β, λ) will behave always better than WL(α, β) or EW but at least it can be said in certain circumstances 3WL(α, β, λ) might work better than WL(α, β) or EW. Fitted pdfs and probability plots of the

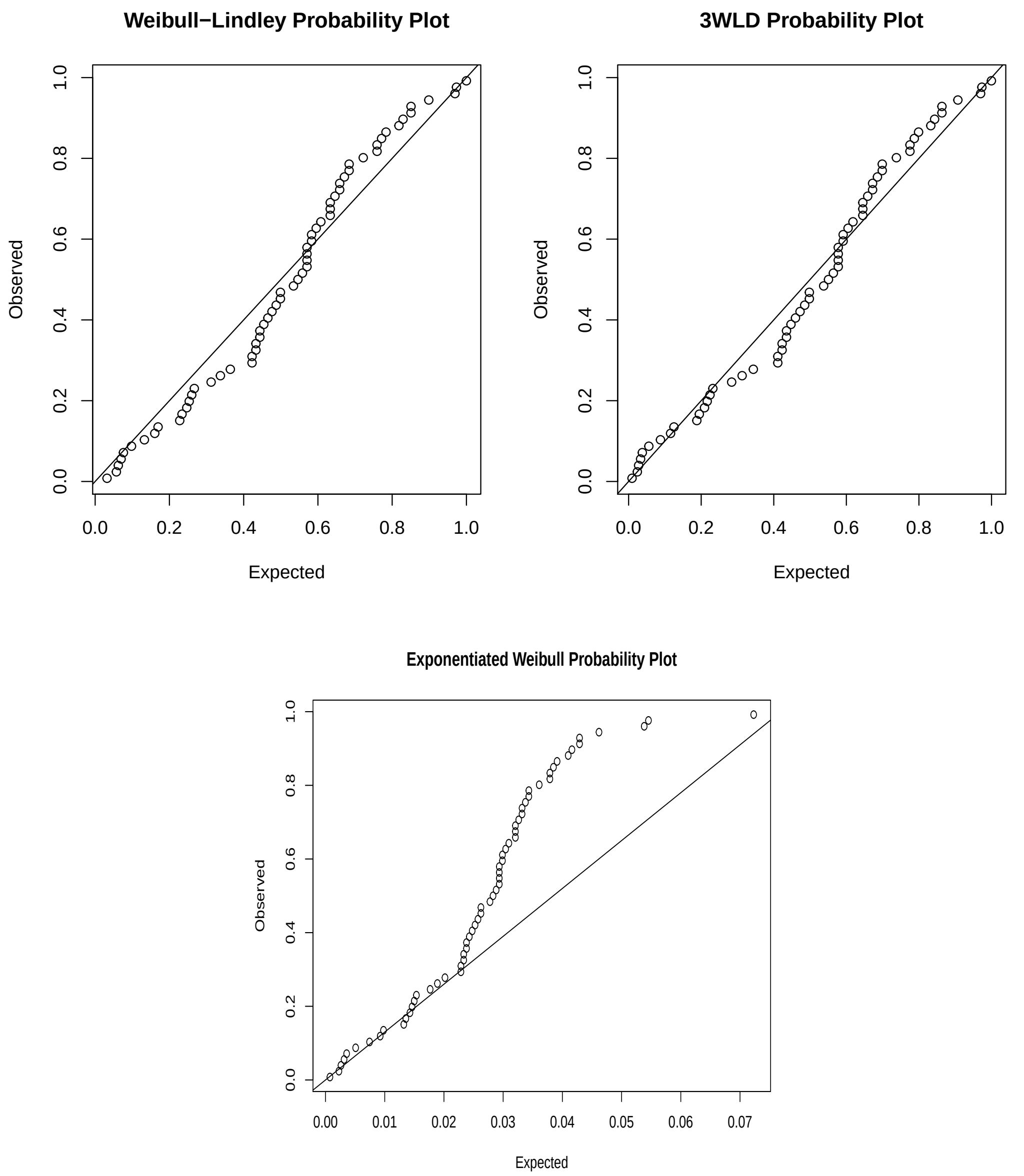

Figure 3.13: Probability plots of the three best fitting distributions for data set 1.

three best fitting distributions for data set 2 are given in Figure 3.14 and 3.15.

Table 3.10: MLEs of parameters, Log-likelihood

Model	MLEs of Parameters	log L	K-S	p-value
WL	$\hat{\alpha}$=0.0355 $\hat{\beta}$=0.5712	-457.302	0.0776	0.0893
GL	$\hat{\alpha}$=2.414 $\hat{\lambda}$=0.8929	-453.523	0.717	2.22×10^{-16}
EW	$\hat{\alpha}$=1.9566 $\hat{\beta}$=0.9598 $\hat{\lambda}$=0.3213	-450.131	0.064	0.2426
3WL	$\hat{\alpha}=8.752\times10^{-04}$ $\hat{\beta}$=0.2994 $\hat{\lambda}$=15.0999	-451.875	0.0619	0.2755

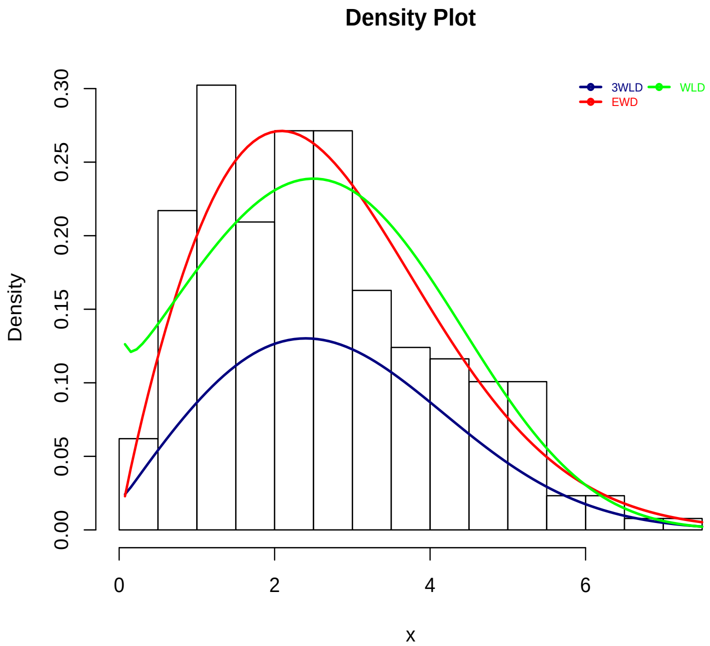

Figure 3.14: Fitted Pdfs of the three best fitting distributions for data set 2.

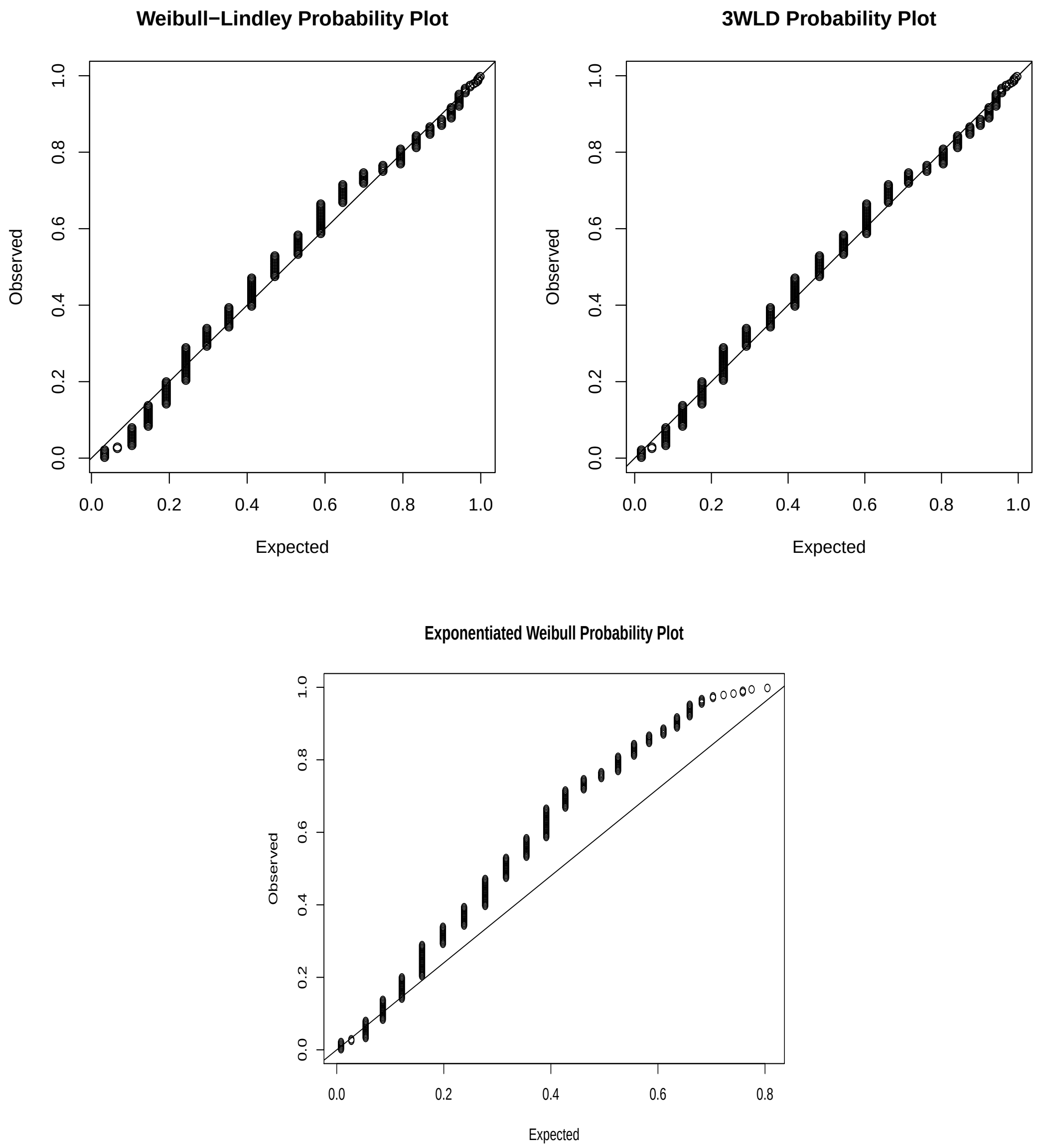

Figure 3.15: Probability plots of the three best fitting distributions for data set 2.

3.4 Summary

GXE is generalizes the X-Exponential distribution. Several properties of the distribution, hazard rate function, moments, moment generating function etc are derived. Also we presented the maximum likelihood estimation of this distribution. A simulation study is performed for validate MLE. Two real data sets are analyzed. The first data set provided smallest AIC and BIC value, the largest p-value for GXE distribution than GLD distribution. And also the second data set also provided smallest AIC and BIC value, the largest p-value for GXE distribution than GLD distribution. It shows that the proposed distribution is a better alternative among BFR models.

We proposed Weibull-Lindley distribution which exhibits bathtub shaped failure rate function, with high initial failure rate, which decreases rapidly and then slowly increases. Three parameter Weibull-Lindley distribution (3WLD) is introduced for avoid scale problem. We have studied maximum likelihood estimators and the parameters estimation is carried out in the presence of real data. We present two real life data sets, where in one data set it is observed that 3WLD has a better fit compare to EW or WLD but in the other the EW has a better fit than 3WLD or WLD.