Deepthi K.S. “Modelling and analysis of reliability data using bathtub shaped failure rate distributions.” Thesis. Research & Post Graduate Department of Statistics, St. Thomas’ College (Autonomous), University of Calicut, 2020.

CHAPTER 4

DUS TRANSFORMATION OF LOMAX DISTRIBUTION: AN UPSIDE-DOWN BATHTUB SHAPED FAILURE RATE MODEL

4.1 Introduction

[1] A little research works have been discussed on the upside-down bathtub (UBT) shaped failure rate distributions. Efron (1988) discussed head and neck cancer data having UBT shape for failure rate because of a therapy. Inverse Lindley (IL) distribution is used to model UBT data, Sharma et al. (2014 & 2015). In reliability literature, the stress-strength model describes the life of a component that has a random strength X and is subjected to random stress Y. The system fails if and

[1] Some contents of this chapter are based on Deepthi and Chacko (2020).

only if the stress is greater than strength. The estimation of a stress-strength model when X and Y are having a specified distribution, has been discussed by many researchers, Al-Mutairi et al. (2013).

Use of heavy tailed distribution required for many of the lifetime data analysis. Pareto distribution is one of the heavy tailed distributions which usually models nonnegative data. It was introduced by Pareto (1897) as a model for the distribution of incomes. Several different forms of Pareto distribution have been studied by many authors including Lomax (1954), Davis and Feldstein (1979), Grimshaw (1993) and Nadarajah and Gupta (2008). One of the popular hierarchy of Pareto distribution is Pareto Type II which has been named as Lomax distribution. Lomax distribution has been applied in a variety of fields such as engineering, reliability and life testing. In statistical literature, there are several methods to propose new distribution by the use of some baseline distribution. Dinesh et al. (2015) proposed a method, DUS transformation, to get new distribution by the use of Exponential baseline distribution and studied its properties with application to survival data analysis. If $f(x)$ and $F(x)$ be the pdf and cdf of some baseline distribution, then the pdf $g(x)$ of the corresponding DUS Transformation distribution is given by

$$g(x) = \frac{1}{e-1} f(x) e^{F(x)}. \tag{4.1.1}$$

The cdf and failure rate function corresponding to the pdf $g(x)$ is given by

$$G(x) = \frac{1}{e-1}[e^{F(x)} - 1] \tag{4.1.2}$$

and

$$h(x) = \frac{1}{e - e^{F(x)}} f(x) e^{F(x)} \tag{4.1.3}$$

respectively. It is a transformation, not a generalization, hence it produces a parsimonious distribution in terms of computation and interpretation as it never contains any new parameter other than the parameter(s) involved in the baseline distribution.

The aim of this chapter to derive DUS Transformation of Lomax distribution which possesses the upside-down bathtub-shaped failure rate function. The proposed distribution is thus capable of modeling the real problems.

The following sections are organized as follows. The pdf, distribution function, failure rate function and its characteristics are given in section 4.2. In section 4.3, shapes of the pdf and failure rate function are given. Moments, moment generating function, characteristic function, quantile function, entropy, skewness and kurtosis are discussed in section 4.4. In section 4.5, distribution of maximum and minimum order statistics are discussed. The maximum likelihood estimation is discussed in section 4.6. In section 4.7, stress-strength reliability and its MLE are derived. In section 4.8, a simulation study is given. Three real data sets are analyzed in section 4.9. Conclusions are given in section 4.10.

4.2 DUS Transformation of Lomax Distribution

In this section, we consider DUS transformation of Lomax distribution with two parameters.

Consider Lomax distribution with pdf,

$$f(x) = \alpha\beta(1+\beta x)^{-(\alpha+1)}, \quad x > 0,\ \alpha > 0,\ \beta > 0 \tag{4.2.1}$$

and the corresponding cdf is given by,

$$F(x) = 1 - (1+\beta x)^{-\alpha}, \quad x > 0,\ \alpha > 0,\ \beta > 0. \tag{4.2.2}$$

Using (4.2.1) in (4.1.1), the pdf of DUS transformation of Lomax distribution is obtained by,

$$g(x) = \frac{1}{e-1}\alpha\beta(1+\beta x)^{-(\alpha+1)}e^{1-(1+\beta x)^{-\alpha}}, \quad x > 0, \alpha > 0,\ \beta > 0. \tag{4.2.3}$$

The distribution having pdf (4.2.3) is named as DUS-Lomax distribution and is denoted by DUS-Lomax(α, β). Here α and β are the shape and scale parameters respectively. The cdf and failure rate function of DUS-Lomax(α, β) are, respectively, given by

$$G(x) = \frac{1}{e-1}\left[e^{1-(1+\beta x)^{-\alpha}} - 1\right], \quad x > 0, \alpha > 0,\ \beta > 0 \tag{4.2.4}$$

and

$$r(x) = \alpha\beta(1+\beta x)^{-(\alpha+1)}\left[e^{(1+\beta x)^{-\alpha}} - 1\right]^{-1}, \quad x > 0,\ \alpha > 0,\ \beta > 0. \tag{4.2.5}$$

4.3 Shapes

Here, we discuss the shapes of the pdf and failure rate function of DUS-Lomax(α, β) distribution.

4.3.1 Shape of Probability Density Function

We can see from (4.2.3) that

$$\lim_{x \to 0} g(x) = \begin{cases} \frac{\alpha\beta}{(e-1)}, & \alpha < 1 \\ \frac{\beta}{(e-1)}, & \alpha = 1 \\ \frac{\alpha\beta}{(e-1)}, & \alpha > 1 \end{cases}$$

and

$$\frac{1}{(1+\beta x)} \text{tends to zero, as } x \to \infty.$$

So $\lim_{x\to\infty} g(x) = 0$. The first derivatives of $g(x)$ is

$$g'(x) = \frac{e}{e-1} \alpha\beta e^{-(1+\beta x)^{-\alpha}} \left[-(\alpha+1)\alpha(1+\beta x)^{-\alpha-2} + \alpha\beta(1+\beta x)^{-2\alpha-2} \right].$$

So the mode of DUS-Lomax(α, β) is $\frac{1}{\beta}\left[\left(1+\frac{1}{\alpha}\right)^{-\frac{1}{\alpha}} - 1\right]$. Clearly, $g(x)$ is unimodal. Figure 4.1 shows the pdf of DUS-Lomax(α, β) for various choices of the parameters.

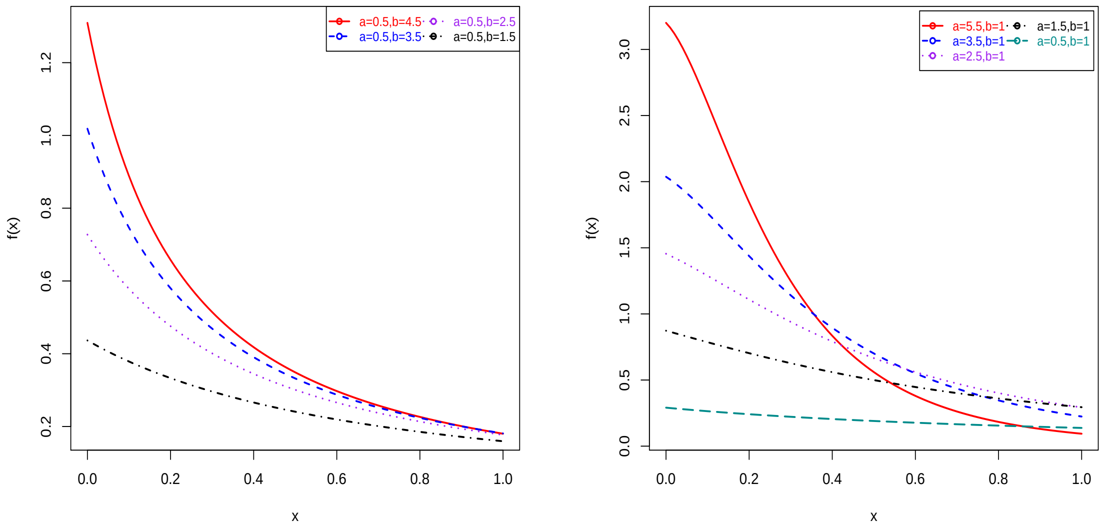

Figure 4.1: PDF of DUS-Lomax(α, β) for values of parameters $\alpha = 0.5$ and $\beta = 4.5, 3.5, 2.5, 1.5$ with color shapes red, blue, purple, black (left) and $\alpha = 5.5, 3.5, 2.5, 1.5, 0.5$ and $\beta = 1$ with color shapes red, blue, purple, black, dark cyan, respectively (right).

4.3.2 Shape of Failure Rate Function

For discussing the shape property of failure rate function, we apply Glaser's technique, see Glaser (1980). Let $\eta(x) = \frac{-g'(x)}{g(x)}$ where $g(x)$ is the density function and $g'(x)$ is the first derivative of $g(x)$ with respect to x. Then

$$\eta(x) = \frac{(\alpha+1)\beta(1+\beta x)^{-\alpha-2} - \alpha\beta(1+\beta x)^{2\alpha-2}}{(1+\beta x)^{\alpha+1}}$$

and its first derivative is

$$\eta'(x) = -(\alpha+1)\beta^2(1+\beta x)^{-\alpha-2}\Bigg[(1+\beta x)^{\alpha} - \alpha\Bigg].$$

If $\alpha > 1$, $\beta > 0$, $\eta'(x) > 0$ for $x \in (0, x_0)$, $\eta'(x_0) = 0$, $\eta'(x) < 0$ for $x \in (x_0, \infty)$ where $x_0 = \frac{\alpha^{1/\alpha} - 1}{\beta}$, the shape of failure rate function, $r(x)$, appears UBFR shapes if $\alpha > 1$. If $\alpha \leq 1$, $\beta > 0$, $\eta'(x) < 0$, the shape of failure rate function appears monotonically decreasing. The failure rate function of the DUS-Lomax(α, β) distribution exhibit monotonically decreasing and UBFR shapes, see Figure 4.2.

From (4.2.5),

$$\lim_{x \to 0} r(x) = \begin{cases} \alpha\beta(e-1), & \alpha < 1 \\ \beta(e-1), & \alpha = 1 \\ \alpha\beta(e-1), & \alpha > 1 \end{cases}$$

and

$$\frac{1}{(1+\beta x)} \text{tends to zero, } \; e^{\frac{1}{(1+\beta x)^{\alpha}}} \text{ tends to 1, as } x \to \infty.$$

So $\lim_{x \to \infty} r(x) = 0$.

4.4 Statistical Properties

In this section, we study the statistical properties for the two parameter DUS-Lomax(α, β) distribution, moments, moment generating function, characteristic function, quantile function, skewness, kurtosis etc.

4.4.1 Moments

If X be a random variable having the pdf in (4.2.3), then the r^{th} raw moment is

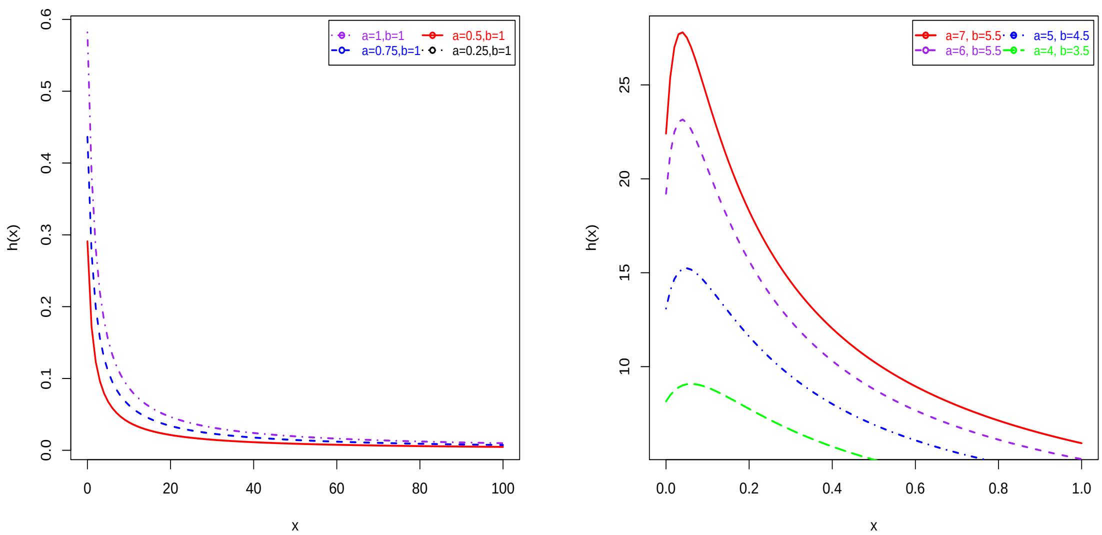

Figure 4.2: Failure rate function of the DUS-Lomax(α, β) for different parameter values $\alpha = 1, 0.75, 0.5, 0.25$ and $\beta = 1$ with color shapes purple, blue, red, black (left) and $\alpha = 7, 6, 5, 4$ and $\beta = 5.5, 5.5, 4.5, 3.5$ with color shapes red, purple, blue, green respectively (right).

$$\begin{aligned}
\mu_r' &= \frac{e}{e-1}\alpha \sum_{j=0}^{\infty} (-1)^j \binom{\alpha+j}{j} \beta^{j+1} \int_0^{\infty} x^{j+r} e^{-(1+\beta x)^{-\alpha}} dx \\
&= \frac{e}{e-1} \sum_{j=0}^{\infty} (-1)^j \binom{\alpha+j}{j} \beta^{j+1} \frac{1}{\beta^{j+r+1}} \sum_{k=0}^{j+r} (-1)^k \binom{j+r}{r} \int_0^1 u^{-\frac{j+r-k+1}{\alpha}-1} e^{-u} du \\
&= \frac{e}{e-1} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{j+r} (-1)^{j+k+n} \binom{\alpha+j}{j} \binom{j+r}{r} \frac{1}{n!\beta^r} \frac{\alpha}{\alpha n - r + k - j - 1}.
\end{aligned}$$

The mean μ and variance σ^2 of DUS-Lomax(α, β) distribution are, respectively,

$$\mu = \frac{e}{e-1} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{j+1} (-1)^{j+k+n} \binom{\alpha+j}{j} \frac{1}{n!\beta} \frac{(j+1)\alpha}{\alpha n + k - j - 2}$$

and

$$\sigma^2 = \frac{e}{e-1}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{j+2}(-1)^{j+k+n}\binom{\alpha+j}{j}\binom{j+2}{2}\frac{1}{n!\beta^2}\frac{\alpha}{\alpha n+k-j-3}$$
$$-\left(\frac{e}{e-1}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{j+1}(-1)^{j+k+n}\binom{\alpha+j}{j}\frac{1}{n!\beta}\frac{(j+1)\alpha}{\alpha n+k-j-2}\right)^2.$$

The skewness and kurtosis can be obtained using

$$\text{Skewness} = \frac{(\mu_3'-3\mu\mu_2'+2\mu^3)^2}{(\mu_2'-\mu^2)^3}\text{and Kurtosis} = \frac{(\mu_4'-4\mu\mu_3'+6\mu^2\mu_2'-3\mu^4)}{(\mu_2'-\mu^2)^2}.$$

4.4.2 Moment Generating Function

The mgf of DUS-Lomax(α,β) distribution is

$$M_X(t) = \frac{e}{e-1}\alpha\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+k}{k}\beta^{k+1}\int_0^{\infty}x^k e^{tx}e^{-(1+\beta x)^{-\alpha}}dx$$
$$= \frac{e}{e-1}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+k}{k}\frac{1}{\beta^m}\sum_{j=0}^{k+m}(-1)^j\binom{k+m}{j}\sum_{m=0}^{\infty}\frac{t^m}{m!}$$
$$\int_0^1 u^{-\frac{k+m-j+1}{\alpha}-1}e^{-u}du$$
$$= \frac{e}{e-1}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{k+m}\sum_{m=0}^{\infty}\frac{t^m}{m!}(-1)^{k+j+n}\binom{\alpha+k}{k}\binom{k+m}{j}$$
$$\frac{1}{n!\beta^m}\frac{\alpha}{\alpha n+j-k-m-1}.$$

4.4.3 Characetristic Function

The characteristic function of DUS-Lomax(α, β) distribution is

$$\phi_X(t) = \frac{e}{e-1}\alpha\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+k}{k}\beta^{k+1}\int_0^{\infty} x^k e^{itx}e^{-(1+\beta x)^{-\alpha}}dx, \ i=\sqrt{-1}$$
$$= \frac{e}{e-1}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha+k}{k}\frac{1}{\beta^m}\sum_{j=0}^{k+m}(-1)^j\binom{k+m}{j}\sum_{m=0}^{\infty}\frac{(it)^m}{m!}$$
$$\int_0^1 u^{-\frac{k+m-j+1}{\alpha}-1}e^{-u}du$$
$$= \frac{e}{e-1}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{k+m}\sum_{m=0}^{\infty}\frac{(it)^m}{m!}(-1)^{k+j+n}\binom{\alpha+k}{k}\binom{k+m}{j}$$
$$\frac{1}{n!\beta^m}\frac{\alpha}{\alpha n+j-k-m-1}.$$

4.4.4 Quantile Function

For any $p \in (0,1)$, the p^{th} quantile $Q(p)$ of DUS-Lomax(α, β) is

$$Q(p) = \frac{1}{\beta}\Big[\Big(1-\log(1+p(e-1))\Big)^{-\frac{1}{\alpha}}-1\Big]. \quad (4.4.1)$$

Setting $p = 0.5$ in (4.4.1), we get the median of DUS-Lomax(α, β) as follows

$$\text{Median} = \frac{1}{\beta}\Big[\Big(1-\log(1+0.5(e-1))\Big)^{-\frac{1}{\alpha}}-1\Big]. \quad (4.4.2)$$

Setting $p = \frac{1}{4}$ in (4.4.1), we get the 1^{st} quartile of DUS-Lomax(α, β) as follows

$$Q_1 = \frac{1}{\beta}\Big[\Big(1-\log(1+\frac{1}{4}(e-1))\Big)^{-\frac{1}{\alpha}}-1\Big].$$

Setting $p = \frac{3}{4}$ in (4.4.1), we get the 3rd quartile of DUS-Lomax(α, β) as follows

$$Q_3 = \frac{1}{\beta}\Big[\Big(1 - \log(1 + \frac{3}{4}(e-1))\Big)^{-\frac{1}{\alpha}} - 1\Big].$$

A random sample X with DUS-Lomax(α, β) distribution can be simulated using

$$X = \frac{1}{\beta}\Big[\Big(1 - \log(1 + u(e-1))\Big)^{-\frac{1}{\alpha}} - 1\Big], \text{where}\, u \sim U(0,1). \tag{4.4.3}$$

4.4.5 Entropy

Suppose X is the DUS-Lomax(α, β), first we consider

$$\begin{aligned}\int f^{\gamma}(x)dx &= \left(\frac{e}{e-1}\right)^{\gamma} \alpha^{\gamma}\beta^{\gamma+i}\sum_{i=0}^{\infty}(-1)^i\binom{\gamma\alpha+\gamma+i-1}{i}\int_0^{\infty} x^i e^{-\gamma(1+\beta x)^{-\alpha}}dx \\ &= \left(\frac{e}{e-1}\right)^{\gamma}\alpha^{\gamma-1}\beta^{\gamma-1}\sum_{i=0}^{\infty}\sum_{k=0}^{i}\sum_{m=0}^{\infty}(-1)^{i+k+m} \\ &\quad \binom{\gamma\alpha+\gamma+i-1}{i}\binom{i}{k}\frac{\gamma^m}{m!}\frac{\alpha}{\alpha m+k+i-1}\end{aligned}$$

where $\gamma > 0$ and $\gamma \neq 1$. Then the Renyi entropy is

$$\begin{aligned}\tau_R(\gamma) &= \frac{1}{1-\gamma}\log\left\{\int f^{\gamma}(x)dx\right\} \\ &= \frac{1}{1-\gamma}\log\Bigg\{\left(\frac{e}{e-1}\right)^{\gamma}\alpha^{\gamma-1}\beta^{\gamma-1}\sum_{i=0}^{\infty}\sum_{k=0}^{i}\sum_{m=0}^{\infty}(-1)^{i+k+m} \\ &\quad \binom{\gamma\alpha+\gamma+i-1}{i}\binom{i}{k}\frac{\gamma^m}{m!}\frac{\alpha}{\alpha m+k+i-1}\Bigg\}.\end{aligned}$$

4.5 Distribution of Maximum and Minimum

In order to conduct reliability analysis in series structure, parallel structure, series-parallel structure, parallel-series structure and complex structures, the theory of order statistics is used as tool for analyzing life time data. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from DUS-Lomax(α, β) with cdf and pdf as in (4.2.4) and (4.2.3), respectively and let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the corresponding order statistics. The pdf of the r^{th} order statistic is

$$\begin{aligned}
f_{X_{(r)}}(x;\alpha,\beta) &= \frac{n!}{(r-1)!(n-r)!} f(x;\alpha,\beta) F^{r-1}(x;\alpha,\beta)\bar{F}^{n-r}(x;\alpha,\beta)\\
&= \frac{n!}{(r-1)!(n-r)!} \frac{\alpha\beta(1+\beta x)^{-(\alpha+1)}}{e-1} e^{1-(1+\beta x)^{-\alpha}}\\
&\left[\frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}}-1\right)\right]^{r-1}\left[1-\frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}}-1\right)\right]^{n-r},\\
&\qquad x>0,\ \alpha>0,\ \beta>0.
\end{aligned}$$

The cdf of r^{th} order statistic is

$$\begin{aligned}
F_{(r)}(x;\alpha,\beta) &= \sum_{j=r}^{n}\binom{n}{j} F^{j}(x;\alpha,\beta)[1-F(x;\alpha,\beta)]^{n-j}\\
&= \sum_{j=r}^{n}\binom{n}{j}\left[\frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}}-1\right)\right]^{j}\\
&\left[1-\frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}}-1\right)\right]^{n-j}, x>0,\ \alpha>0,\ \beta>0.
\end{aligned}$$

The pdf of the 1st order statistics $X_{(1)}$, is

$$f_{X_{(1)}}(x;\alpha,\beta) = \frac{n\alpha\beta(1+\beta x)^{-(\alpha+1)}}{e-1} e^{1-(1+\beta x)^{-\alpha}} \left[1 - \frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}} - 1\right)\right]^{n-1}, \ x > 0, \ \alpha > 0, \ \beta > 0.$$

The pdf of the n^{th} order statistics $X_{(n)}$, is

$$f_{X_{(n)}}(x;\alpha,\beta) = \frac{n\alpha\beta(1+\beta x)^{-(\alpha+1)}}{e-1} e^{1-(1+\beta x)^{-\alpha}} \left[\frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}} - 1\right)\right]^{n-1}, \ x > 0, \ \alpha > 0, \ \beta > 0.$$

The cdf of $X_{(1)}$ is

$$F_{X_{(1)}}(x;\alpha,\beta) = P(X_{(1)} \le x) = 1 - \left[1 - \tfrac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}} - 1\right)\right]^{n}, x > 0, \ \alpha > 0, \ \beta > 0.$$

The cdf of $X_{(n)}$ is

$$F_{X_{(n)}}(x;\alpha,\beta) = P(X_{(n)} \le x) = \left[\tfrac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}} - 1\right)\right]^{n}, \ x > 0, \ \alpha > 0, \ \beta > 0.$$

Reliability, $R(x;\alpha,\beta)$, of series and parallel system having n components with DUS-Lomax (α,β), respectively, are

$$\left[1 - \frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}} - 1\right)\right]^{n} \text{and } 1 - \left[\frac{1}{e-1}\left(e^{1-(1+\beta x)^{-\alpha}} - 1\right)\right]^{n}.$$

4.6 Parameter Estimation

In this section, we discuss method of moments and method of maximum likelihood for the estimation of parameters. Asymptotic bounds of the unknown parameter are also discussed.

Let $X_1, X_2, \ldots, X_n$ be an observed random sample from DUS-Lomax(α, β) with unknown parameters α and β. Let $m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $m_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2$ be the first two sample moments. Equating sample moments with population moments, we get the moment estimators of the parameters,

$$m_1 = \frac{e}{e-1}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{j+1}(-1)^{j+k+n}\binom{\alpha+j}{j}\frac{1}{n!\beta}\frac{(j+1)\alpha}{\alpha n - r + k - j - 2}$$

and

$$m_2 = \frac{e}{e-1}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{j+2}(-1)^{j+k+n}\binom{\alpha+j}{j}\binom{j+2}{2}\frac{1}{n!\beta^2}\frac{\alpha}{\alpha n - r + k - j - 3}.$$

We derive MLE of the parameters of the DUS-Lomax(α, β) distribution as below.

The likelihood function is

$$l(x; \alpha, \beta) = \prod_{i=1}^{n} f(x_i; \alpha, \beta) = \frac{e^n \alpha^n \beta^n}{(e-1)^n} e^{-\sum_{i=1}^{n}(1+\beta x_i)^{-\alpha}} \prod_{i=1}^{n}(1+\beta x_i)^{-(\alpha+1)},$$

so that the log-likelihood function becomes

$$\log l = K + n\log\alpha + n\log\beta - (\alpha+1)\sum_{i=1}^{n}\log(1+\beta x_i) - \sum_{i=1}^{n}(1+\beta x_i)^{-\alpha}, \quad (4.6.1)$$

where $K = n\log(\frac{e}{e-1})$. Then the partial derivatives of $\log L$ with respect to unknown parameters α and β are

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \log(1+\beta x_i) + \sum_{i=1}^{n} \log(1+\beta x_i)(1+\beta x_i)^{-\alpha} \tag{4.6.2}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - (\alpha+1)\sum_{i=1}^{n} \frac{x_i}{(1+\beta x_i)} + \alpha \sum_{i=1}^{n} x_i(1+\beta x_i)^{-(\alpha+1)}. \tag{4.6.3}$$

Setting the left side of the above two equations to zero, we get the likelihood equations as a system of two non-linear equations in α and β.

$$\frac{n}{\alpha} - \sum_{i=1}^{n} \log(1+\beta x_i) + \sum_{i=1}^{n} \log(1+\beta x_i)(1+\beta x_i)^{-\alpha} = 0 \tag{4.6.4}$$

$$\frac{n}{\beta} - (\alpha+1)\sum_{i=1}^{n} \frac{x_i}{(1+\beta x_i)} + \alpha \sum_{i=1}^{n} x_i(1+\beta x_i)^{-(\alpha+1)} = 0. \tag{4.6.5}$$

Solving these systems, (4.6.4) and (4.6.5), in α and β gives the MLE of α and β. These equations cannot be solved analytically and statistical software can be used to solve them numerically, by taking initial value arbitrarily.

4.6.1 Asymptotic distribution and Confidence bounds

In this section, we derived the asymptotic distribution and confidence intervals of the parameters $\alpha > 0$ and $\beta > 0$, when the MLEs of the unknown parameters α and β cannot be obtained in closed forms, using variance covariance matrix I^{-1}, where I^{-1} is the inverse of the observed information matrix which is defined as follows

$$I^{-1} = \begin{bmatrix} E(-\frac{\partial^2 \log L}{\partial \alpha^2}) & E(-\frac{\partial^2 \log L}{\partial \alpha \partial \beta}) \\ E(-\frac{\partial^2 \log L}{\partial \beta \partial \alpha}) & E(-\frac{\partial^2 \log L}{\partial \beta^2}) \end{bmatrix}^{-1}.$$

For a large sample, the asymptotic distribution of $\hat{h}$, $\hat{h} = (\alpha, \beta)$ is defined by $\sqrt{n}(\hat{h} - h) \rightarrow N(0, I^{-1})$. The second partial derivatives are as follows

$$\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \sum_{i=1}^{n} \left[\log(1+\beta x_i)\right]^2 (1+\beta x_i)^{-\alpha} \tag{4.6.6}$$

$$\frac{\partial^2 \log L}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} x_i (1+\beta x_i)^{-(\alpha+1)} - \alpha \sum_{i=1}^{n} x_i \log(1+\beta x_i)(1+\beta x_i)^{-(\alpha+1)} - \sum_{i=1}^{n} \frac{x_i}{1+\beta x_i} \tag{4.6.7}$$

$$\frac{\partial^2 \log L}{\partial \beta^2} = -\frac{n}{\beta^2} + (\alpha+1) \sum_{i=1}^{n} \frac{x_i^2}{(1+\beta x_i)^2} - \alpha(\alpha+1) \sum_{i=1}^{n} x_i^2 (1+\beta x_i)^{-(\alpha+2)}. \tag{4.6.8}$$

The approximate $100\%(1-\eta)$ confidence intervals of the parameters α and β, by using variance-covariance matrix, are $\hat{\alpha} \pm Z_{\frac{\eta}{2}} \sqrt{var(\hat{\alpha})}$ and $\hat{\beta} \pm Z_{\frac{\eta}{2}} \sqrt{var(\hat{\beta})}$ where $Z_{\frac{\eta}{2}}$ is the upper $100(\frac{\eta}{2})^{\text{th}}$ percentile of the standard Normal distribution.

4.7 Stress-Strength Reliability Estimation

Consider two independent random variables X and Y, where Y represents the 'stress' and X represents the 'strength'. The reliability of the stress-strength model is $R = P(Y < X)$, which is used in engineering statistics, quality control and other fields.

Suppose X and Y have DUS-Lomax(α, β) distribution with parameters (α, β_1) and (α, β_2) respectively. The reliability of the system is

$$\begin{aligned}
R = P[Y < X] &= \int_0^{\infty} f(x) F_Y(x) dx \\
&= \left(\frac{e}{e-1}\right)^2 \alpha\beta_1 \int_0^{\infty} (1+\beta_1 x)^{-(\alpha+1)} e^{-(1+\beta_1 x)^{-\alpha}} \left(e^{-(1+\beta_2 x)^{-\alpha}} - 1\right) dx \\
&= \left(\frac{e}{e-1}\right)^2 \alpha\beta_1^{n+1} \sum_{n=0}^{\infty} (-1)^n \binom{\alpha+n}{n} \int_0^{\infty} x^n e^{-(1+\beta_1 x)^{-\alpha}} \left(e^{-(1+\beta_2 x)^{-\alpha}} - 1\right) dx \\
&= \left(\frac{e}{e-1}\right)^2 \sum_{n=0}^{\infty} (-1)^n \binom{\alpha+n}{n} \left\{ \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{n+i} \sum_{r=0}^{\infty} \frac{\beta_2^i}{\beta_1^i} \frac{(-1)^{m+i+j+r}}{m! r!} \binom{\alpha m + i - 1}{i} \right. \\
&\left. \binom{n+i}{j} \frac{\alpha}{\alpha r - n + j - i - 1} - \sum_{k=0}^{n} \sum_{l=0}^{\infty} \frac{(-1)^{k+l}}{l!} \binom{n}{k} \frac{\alpha}{\alpha l - n + k - 1} \right\}. \qquad (4.7.1)
\end{aligned}$$

The Maximum Likelihood Estimation of R

Let $(X_1, X_2, \ldots, X_n)$ and $(Y_1, Y_2, \ldots, Y_m)$ be two independent random samples from DUS-Lomax(α, β_1), and DUS-Lomax(α, β_2) respectively. The log-likelihood function of α, β_1 and β_2 for the observed samples is

$$\begin{aligned}
\log l(x, y, \alpha, \beta_1, \beta_2) &= n \log\left(\frac{e}{e-1}\right) + (n+m)\log\alpha + n\log\beta_1 \\
&- (\alpha+1)\sum_{i=1}^{n} \log(1+\beta_1 x_i) - \sum_{i=1}^{n} (1+\beta_1 x_i)^{-\alpha} + m\log\left(\frac{e}{e-1}\right) + m\log\beta_2 \\
&- (\alpha+1)\sum_{j=1}^{m} \log(1+\beta_2 y_j) - \sum_{j=1}^{m} (1+\beta_2 y_j)^{-\alpha}. \qquad (4.7.2)
\end{aligned}$$

The estimators $\hat{\alpha}, \hat{\beta_1}$ and $\hat{\beta_2}$ of the parameters of α, β_1 and β_2 respectively can then be obtained as the solution of the following non-linear equations.

$$\frac{\partial \log l}{\partial \alpha} = \frac{n+m}{\alpha} - \sum_{i=1}^{n} \log(1+\beta_1 x_i) - \sum_{j=1}^{m} \log(1+\beta_2 y_j) + \sum_{i=1}^{n} \frac{\log(1+\beta_1 x_i)}{(1+\beta_1 x_i})^{\alpha} + \sum_{j=1}^{m} \frac{\log(1+\beta_2 y_j)^{\alpha}}{(1+\beta_2 y_j)}$$

$$\frac{\partial \log l}{\partial \beta_1} = \frac{n}{\beta_1} - (\alpha_1+1) \sum_{i=1}^{n} \frac{x_i}{1+\beta_1 x_i} + \alpha_1 \sum_{i=1}^{n} x_i (1+\beta_1 x_i)^{-(\alpha_1+1)}$$

$$\frac{\partial \log l}{\partial \beta_2} = \frac{m}{\beta_2} - (\alpha_2+1) \sum_{j=1}^{m} \frac{y_j}{1+\beta_2 y_j} + \alpha_2 \sum_{j=1}^{m} y_j (1+\beta_2 y_j)^{-(\alpha_2+1)}.$$

MLE of R, denoted by $\hat{R}^{ML}$, can obtained by replacing α, β_1 and β_2 by their MLEs.

Then $\hat{R}^{ML}$ is given by

$$\hat{R}^{ML} = \left(\frac{e}{e-1}\right)^2 \sum_{n=0}^{\infty} (-1)^n \binom{\hat{\alpha}+n}{n} \left\{ \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{n+i} \sum_{r=0}^{\infty} \frac{\hat{\beta_2}^{i}}{\hat{\beta_1}^{i}} \frac{(-1)^{m+i+j+r}}{m!r!} \binom{\hat{\alpha}m+i-1}{i} \binom{n+i}{j} \frac{\hat{\alpha}}{\hat{\alpha}r-n+j-i-1} - \sum_{k=0}^{n} \sum_{l=0}^{\infty} \frac{(-1)^{k+l}}{l!} \binom{n}{k} \frac{\hat{\alpha}}{\hat{\alpha}l-n+k-1} \right\}. \quad (4.7.3)$$

The asymptotic variance of $\hat{R}^{ML}$ is given by

$$AV(\hat{R}^{ML}) = \sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial R}{\partial \theta_i} \frac{\partial R}{\partial \theta_j} I^{-1}(\underline{\theta}), \quad (4.7.4)$$

where $\underline{\theta} = (\alpha, \beta_1, \beta_2)$ and $I^{-1}(\underline{\theta})$ is the inverse of Fisher Information Matrix. Therefore, an asymptotic $100(1-\nu)\%$ confidence interval for R can obtain as $\hat{R}^{ML} \pm Z_{\frac{\nu}{2}} \sqrt{AV(\hat{R}^{ML})}$ where $Z_{\frac{\nu}{2}}$ is the upper $\frac{\nu}{2}-$ quantile of standard Normal distribution.

4.8 Simulation Study

A simulation study is performed to verify the MLEs work for different sample sizes and different parameter values for the proposed DUS-Lomax(α, β) distribution using inversion method. Eq. (4.4.3) is used to generate a random sample from the DUS-Lomax with parameter α and β. The different sample sizes considered in the simulation are $n = 10,\ 25,\ 50,\ 100,\ 250,\ 500,\ 750$ and 1000. We have used 'optim' package in R language to find the estimate. We replicated the process 5000 times and reported the average estimates and the associated mean squared errors in Table 4.1, 4.2, 4.3 and 4.4.

The simulation is conducted for four different cases using varying true parameter values. The selected true parameter values are $\alpha = 0.5$ and $\beta = 0.01$; $\alpha = 1$ and $\beta = 0.5$; $\alpha = 2$ and $\beta = 1.5$; and $\alpha = 0.5$ and $\beta = 2$ for the first, second, third and fourth cases, respectively.
As the sample size increases, the mean square error decreases for all selected parameter values as in Tables 4.1, 4.2, 4.3 and 4.4. The bias caused by the estimates are nearer to zero. Also, when the sample size increases, absolute bias decreases. Thus the estimates tends to the true parameter values with the increase in sample size.

Table 4.1: Simulation study at $\alpha = 0.5$ and $\beta = 0.01$

n	MLE	Bias	MSE
10	$\hat{\alpha} = 0.4785$	-4.299×10^{-6}	9.240×10^{-8}
	$\hat{\beta} = 0.0213$	2.252×10^{-6}	2.536×10^{-8}
25	$\hat{\alpha} = 0.5349$	6.975×10^{-6}	2.432×10^{-7}
	$\hat{\beta} = 0.0126$	5.181×10^{-7}	1.342×10^{-9}
50	$\hat{\alpha} = 0.5222$	4.440×10^{-6}	9.857×10^{-8}
	$\hat{\beta} = 0.0105$	9.865×10^{-8}	4.866×10^{-11}
100	$\hat{\alpha} = 0.5147$	2.939×10^{-6}	4.319×10^{-8}
	$\hat{\beta} = 0.0103$	6.017×10^{-8}	1.811×10^{-11}
250	$\hat{\alpha} = 0.5051$	1.010×10^{-6}	5.102×10^{-9}
	$\hat{\beta} = 0.0101$	2.743×10^{-8}	3.762×10^{-12}
500	$\hat{\alpha} = 0.5032$	6.418×10^{-7}	2.06×10^{-9}
	$\hat{\beta} = 0.0100$	6.533×10^{-9}	2.134×10^{-13}
750	$\hat{\alpha} = 0.5015$	2.986×10^{-7}	4.458×10^{-10}
	$\hat{\beta} = 0.0101$	1.305×10^{-8}	8.510×10^{-13}
1000	$\hat{\alpha} = 0.5013$	2.529×10^{-7}	3.198×10^{-10}
	$\hat{\beta} = 0.0101$	1.067×10^{-8}	5.699×10^{-13}

4.9 Data Analysis

In this section, we illustrate the use of DUS-Lomax(α, β) distribution using three real data sets. We fit DUS-Lomax(α, β) distribution to these data sets and compare with Lomax distribution, Gompertz Lomax (GoL) distribution, Kumaraswamy Lomax (KL) distribution, DUS-Exponential distribution and Inverse Lindley (IL) distribution. The first data-sets, considered here, represent the survival times of two groups of patients suffering from head and neck cancer disease. The patients in one group were treated using radiotherapy ((RT), see Table 4.5), whereas the patients belonging to other group were treated using a combined RT

Table 4.2: Simulation study at $\alpha = 1$ and $\beta = 0.5$

n	MLE	Bias	MSE
10	$\hat{\alpha} = 1.3044$	6.088×10^{-5}	1.853×10^{-5}
	$\hat{\beta} = 0.3013$	-3.974×10^{-5}	7.897×10^{-6}
25	$\hat{\alpha} = 1.4531$	9.061×10^{-5}	4.105×10^{-5}
	$\hat{\beta} = 0.5667$	1.334×10^{-5}	8.90×10^{-7}
50	$\hat{\alpha} = 1.1232$	2.464×10^{-5}	3.035×10^{-6}
	$\hat{\beta} = 0.5086$	1.724×10^{-6}	1.487×10^{-8}
100	$\hat{\alpha} = 1.0526$	1.051×10^{-5}	5.528×10^{-7}
	$\hat{\beta} = 0.5074$	1.487×10^{-6}	1.106×10^{-8}
250	$\hat{\alpha} = 1.0199$	3.975×10^{-6}	7.899×10^{-8}
	$\hat{\beta} = 0.502$	3.997×10^{-7}	7.987×10^{-10}
500	$\hat{\alpha} = 1.009$	1.798×10^{-6}	1.616×10^{-8}
	$\hat{\beta} = 0.501$	2.806×10^{-7}	3.938×10^{-10}
750	$\hat{\alpha} = 1.0060$	1.203×10^{-6}	7.237×10^{-9}
	$\hat{\beta} = 0.5012$	2.328×10^{-7}	2.711×10^{-10}
1000	$\hat{\alpha} = 1.0033$	6.501×10^{-7}	2.113×10^{-9}
	$\hat{\beta} = 0.502$	3.128×10^{-7}	4.891×10^{-10}

and chemotherapy ((CT + RT), see Table 4.7) (Efron (1988)). Another one concerns 46 observations reported on active repair times ((hours), see Table 4.9) for an airborne communication transceiver (Chhikara and Folks (1977)).

The required numerical evaluations are carried out using the R software. Table 4.6, Table 4.8 and Table 4.10 provide the MLEs of the model parameters. The model selection is carried out using the AIC and the BIC:

$$\text{AIC} = -2l + 2k,$$

$$\text{BIC} = -2l + k \log n,$$

Table 4.3: Simulation study at $\alpha = 2$ and $\beta = 1.5$

n	MLE	Bias	MSE
10	$\hat{\alpha} = 19.2540$	0.00345	0.05954
	$\hat{\beta} = 2.0681$	0.000114	6.455×10^{-5}
25	$\hat{\alpha} = 5.3742$	0.000675	0.00228
	$\hat{\beta} = 1.6073$	2.147×10^{-5}	2.305×10^{-5}
50	$\hat{\alpha} = 2.9633$	0.000193	0.000186
	$\hat{\beta} = 1.5316$	6.317×10^{-6}	1.995×10^{-7}
100	$\hat{\alpha} = 2.3108$	6.217×10^{-5}	1.933×10^{-5}
	$\hat{\beta} = 1.4932$	-1.358×10^{-6}	9.215×10^{-9}
250	$\hat{\alpha} = 2.0899$	1.798×10^{-5}	1.616×10^{-6}
	$\hat{\beta} = 1.4981$	-3.833×10^{-7}	7.345×10^{-10}
500	$\hat{\alpha} = 2.0414$	8.286×10^{-6}	3.433×10^{-7}
	$\hat{\beta} = 1.4984$	-3.283×10^{-7}	5.389×10^{-10}
750	$\hat{\alpha} = 2.0269$	5.388×10^{-6}	1.452×10^{-7}
	$\hat{\beta} = 1.5023$	4.682×10^{-7}	1.096×10^{-9}
1000	$\hat{\alpha} = 2.0191$	3.810×10^{-6}	7.260×10^{-8}
	$\hat{\beta} = 1.5004$	8.41×10^{-8}	3.536×10^{-11}

where l denotes the log-likelihood function, k is the number of parameters and n is the sample size. Moreover, perfection of competing models is also tested using the K-S test. K-S test statistic is

$$KS = \max\left\{\frac{i}{m} - z_i, z_i - \frac{i-1}{m}\right\}, i = 1, ..., n,$$

where m denotes the number of classes and $z_i = \text{cdf}(x_i)$, the $x_i^{'}$s being the ordered observations.

Table 4.4: Simulation study at $\alpha = 0.5$ and $\beta = 2$

n	MLE	Bias	MSE
10	$\hat{\alpha} = 1.0819$	0.000116	6.772×10^{-5}
	$\hat{\beta} = 2.9260$	0.000185	0.000172
25	$\hat{\alpha} = 0.5657$	1.314×10^{-5}	8.636×10^{-7}
	$\hat{\beta} = 2.2879$	5.757×10^{-5}	1.657×10^{-5}
50	$\hat{\alpha} = 0.5314$	6.276×10^{-6}	1.970×10^{-7}
	$\hat{\beta} = 2.1124$	2.248×10^{-5}	2.528×10^{-6}
100	$\hat{\alpha} = 0.5149$	2.987×10^{-6}	4.462×10^{-8}
	$\hat{\beta} = 2.0359$	7.178×10^{-6}	2.576×10^{-7}
250	$\hat{\alpha} = 0.5045$	9.096×10^{-7}	4.137×10^{-9}
	$\hat{\beta} = 2.0277$	5.548×10^{-6}	1.539×10^{-7}
500	$\hat{\alpha} = 0.5015$	3.093×10^{-7}	4.784×10^{-10}
	$\hat{\beta} = 2.0223$	4.457×10^{-6}	9.931×10^{-8}
750	$\hat{\alpha} = 0.5015$	3.091×10^{-7}	4.778×10^{-10}
	$\hat{\beta} = 2.0065$	1.296×10^{-6}	8.396×10^{-9}
1000	$\hat{\alpha} = 0.5008$	1.677×10^{-7}	1.407×10^{-10}
	$\hat{\beta} = 2.0091$	1.830×10^{-6}	1.674×10^{-8}

4.9.1 Complete Data radiotherapy (RT)

The data set is given below: The values of the AIC, BIC and K-S Statistic are

Table 4.5: Survival times of patients treated using RT:

6.53	7	10.42	14.48	16.1	22.7	34	41.55	42	45.28
49.4	53.62	63	64	83	84	91	108	112	129
133	133	139	140	140	146	149	154	157	160
160	165	146	149	154	157	160	160	165	173
176	218	225	241	248	273	277	297	405	417
420	440	523	583	594	1101	1146	1417		

listed in Table 4.6. The variance covariance matrix of the MLEs under the DUS-

Table 4.6: MLEs of the parameters, Log-likelihoods, AIC, BIC, K-S Statistics of the fitted models in Data set 1.

Model	**MLEs**	$\log \mathbf{L}$	**AIC**	**BIC**	**KS-Statistic**	**p-value**
DUS-Lomax	$\hat{\alpha} = 4.195$ $\hat{\beta} = 0.0018$	-370.85	745.7	749.82	0.139	0.210
Lomax	$\hat{\alpha} = 6.668$ $\hat{\beta} = 0.00078$	-371.61	747.219	751.34	0.145	0.175
KL	$\hat{a} = 27.93$ $\hat{b} = 112.22$ $\hat{\alpha} = 0.1851$ $\hat{\lambda} = 0.0085$	-371.01	750.03	758.27	0.154	0.126
GoL	$\hat{\theta} = 0.0042$ $\hat{\alpha} = 0.689$ $\hat{\beta} = 1.812$ $\hat{\gamma} = 1.405$	-372.381	752.76	761.004	0.158	0.111
DUS-E(θ)	$\hat{\theta} = 0.0056$	-373.82	749.647	751.71	0.201	0.0188
ILD	$\hat{\theta} = 60.094$	-385.70	773.41	775.47	22.629	2.2×10^{-16}

Lomax distribution for the Data set 1 is computed as

$$= \begin{pmatrix} 3.2382 & -0.001953 \\ -0.001953 & 1.0991 \times 10^{-6} \end{pmatrix}.$$

Thus, the variances of the MLE of α and β is $\text{Var}(\hat{\alpha}) = 3.2382$ and $\text{Var}(\hat{\beta}) = 1.0991 \times 10^{-6}$. Therefore, 95% confidence intervals for α and β are [1.235, 7.155] and [0.000107, 0.00356] respectively. Histogram and Empirical cdf of DUS-Lomax (α, β) are given in Figure 4.3.

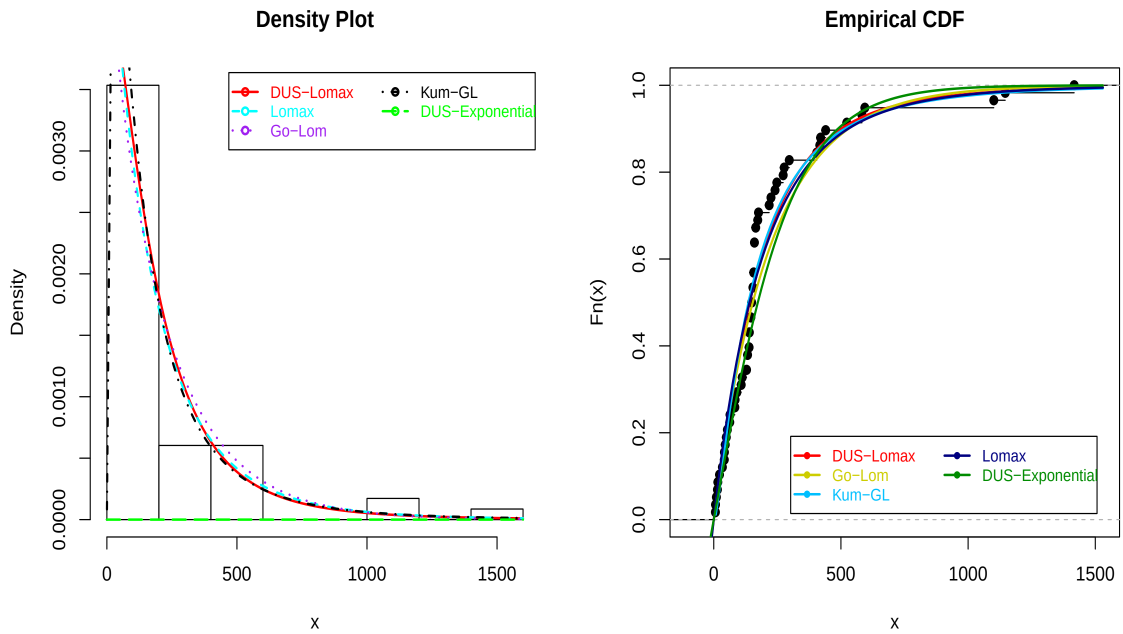

Figure 4.3: Histogram with fitted pdfs (left) and Empirical cdf with fitted cdfs (right) for the Data set 1.

4.9.2 Complete Data RT and chemotherapy (RT+CT)

The data set is given below: The values of the AIC, BIC and K-S Statistic are

Table 4.7: Survival times of patients treated using RT+CT:

12.2	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46
58.36	63.47	68.46	78.26	74.47	81.43	84	92	94
110	112	119	127	130	133	140	146	155
159	173	179	194	195	209	249	281	319
339	432	469	519	633	725	817	1776	

listed in Table 4.8. The variance covariance matrix of the MLEs under the DUS-

Table 4.8: MLEs of the parameters, Log-likelihoods, AIC, BIC, K-S Statistics of the fitted models in Data set 2.

Model	**MLEs**	$\log \mathbf{L}$	**AIC**	**BIC**	**KS-Statistic**	**p-value**
DUS-Lomax	$\hat{\alpha} = 3.165$ $\hat{\beta} = 0.0028$	-279.91	563.81	567.38	0.093	0.806
Lomax	$\hat{\alpha} = 4.40$ $\hat{\beta} = 0.0013$	-280.45	564.91	568.48	0.104	0.695
KL	$\hat{a} = 23.902$ $\hat{b} = 0.125$ $\hat{\alpha} = 8.675$ $\hat{\lambda} = 44.97$	-281.91	571.82	578.95	0.211	0.034
GoL	$\hat{\theta} = 0.0185$ $\hat{\alpha} = 0.467$ $\hat{\beta} = 0.719$ $\hat{\gamma} = 1.99$	-281.77	571.54	578.68	0.1297	0.414
DUS-E(θ)	$\hat{\theta} = 0.0056$	-283.91	569.82	571.60	0.198	0.0208
IL	$\hat{\theta} = 77.68$	-279.58	561.16	562.94	29.01	5.551×10^{-16}

Lomax distribution for the Data set 2 is computed as

$$= \begin{pmatrix} 41.1184 & -0.05016 \\ -0.05016 & 6.0923 \times 10^{-5} \end{pmatrix}.$$

Thus, the variances of the MLE of α and β is $\text{Var}(\hat{\alpha}) = 41.118$ and $\text{Var}(\hat{\beta}) = 6.0923 \times 10^{-5}$. Therefore, 95% confidence intervals for α and β are [-7.382, 13.713] and [-0.0101, 0.0156] respectively. Histogram and Empirical cdf of DUS-Lomax (α, β) are given in Figure 4.4.

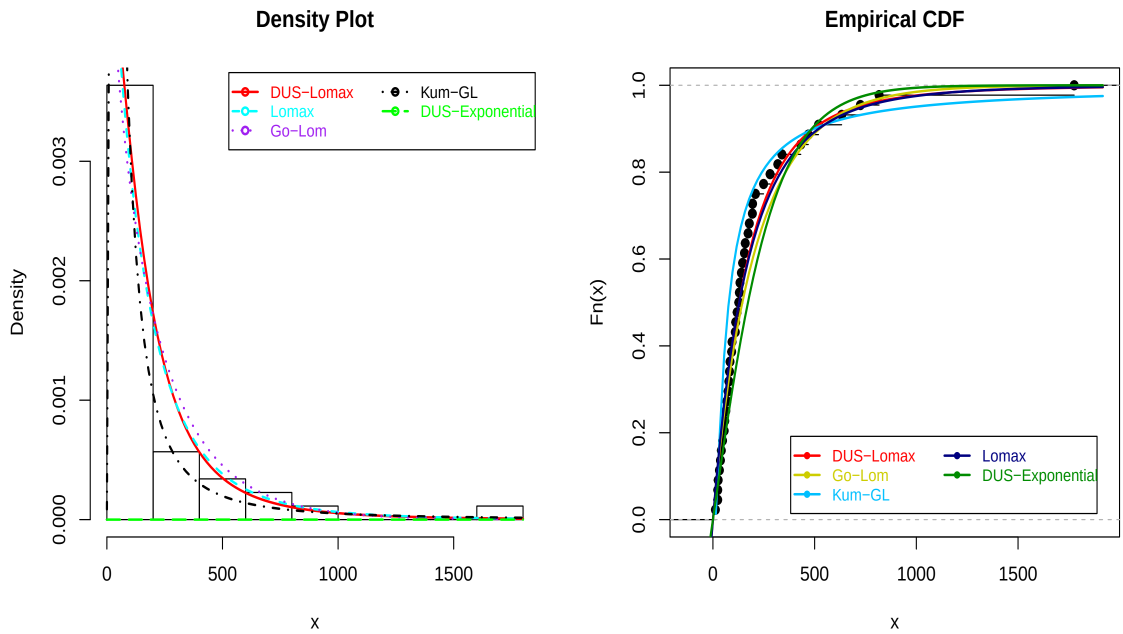

Figure 4.4: Histogram with fitted pdf (left) and Empirical cdf with fitted cdf (right) for the Data set 2.

4.9.3 Complete Data Repair Time

The data set is given below:

Table 4.9: Repair Time:

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6
0.7	0.7	0.7	0.8	0.8	1.0	1.0	1.0
1.0	1.1	1.3	1.5	1.5	1.5	1.5	2.0
2.0	2.2	2.5	2.7	3.0	3.0	3.3	3.3
4.0	4.0	4.5	4.7	5.0	5.4	5.4	7.0
7.5	8.8	9.0	10.3	22.0	24.5		

The values of the AIC, BIC and K-S Statistic are listed in Table 4.10. The variance

Table 4.10: MLEs of the parameters, Log-likelihoods, AIC, BIC, K-S Statistics of the fitted models in Data set 3.

Model	MLEs	$\log \mathbf{L}$	AIC	BIC	KS-Statistic	p-value
DUS-Lomax	$\hat{\alpha} = 2.610$ $\hat{\beta} = 0.227$	-102.70	209.40	213.06	0.118	0.548
Lomax	$\hat{\alpha} = 3.549$ $\hat{\beta} = 0.108$	-102.95	209.91	213.57	0.127	0.446
GoL	$\hat{\theta} = 1.776$ $\hat{\alpha} = 1.165$ $\hat{\beta} = 0.189$ $\hat{\gamma} = 0.245$	-102.95	213.96	221.27	0.129	0.432
DUS-E(θ)	$\hat{\theta} = 0.344$	-107.66	217.31	219.14	0.211	0.033
ILD	$\hat{\theta} = 1.577$	-101.17	204.34	206.17	0.883	2.2×10^{-16}

covariance matrix of the MLEs under the DUS-Lomax distribution for the Data set 3 is computed as

$$= \begin{pmatrix} 1.3693 & -0.15959 \\ -0.15959 & 0.0203 \end{pmatrix}.$$

Thus, the variances of the MLE of α and β is $\mathrm{Var}(\hat{\alpha}) = 1.369$ and $\mathrm{Var}(\hat{\beta}) = 0.0203$. Therefore, 95% confidence intervals for α and β are [0.6855, 4.535] and [-0.00767, 0.4610] respectively. Histogram and Empirical cdf of DUS-Lomax (α, β) are given in Figure 4.5.

Table 4.6, Table 4.8 and Table 4.10 show that, DUS-Lomax (α, β) has lowest AIC, BIC, KS-Statistic, and largest Log-likelihood value and p-value based on K-S Statistic. The second lowest AIC, BIC, K-S Statistic and second largest log-likelihood value and p value are obtained by the Lomax distribution. The

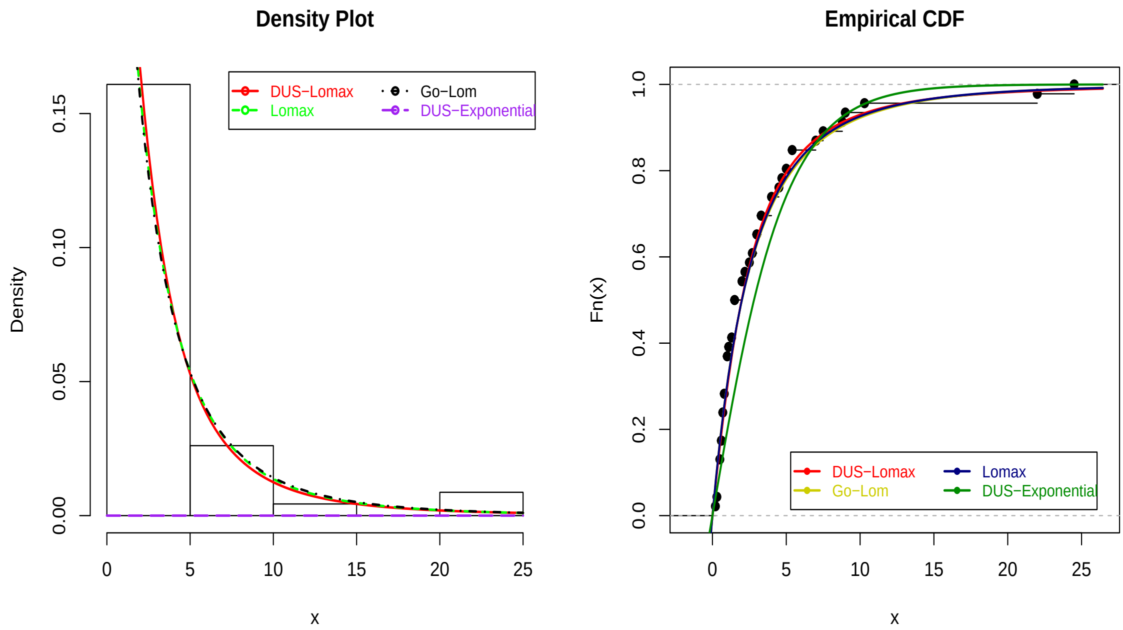

Figure 4.5: Histogram with fitted pdfs (left) and Empirical cdf with fitted cdfs (right) for the Data set 3.

proposed distribution, DUS-Lomax (α, β) can be used when failure rate pattern of lifetime distribution is upside-down bathtub shaped. In Data set 1, 2 and 3 seems that DUS-Lomax (α, β) is more appropriate than Lomax distribution, GoL distribution, KL distribution, DUS-Exponential distribution and IL distribution. So DUS-Lomax (α, β) is better alternative in the situations in which upside-down bathtub distributions arises.

4.10 Summary

DUS-transformation is a kind of parsimonious distribution. That is, we can do computation and interpretation very easily even without changing the parameters. Then if we apply this transformation into Lomax, its failure rate behaviour is changing into an upside down bathtub one. A new distribution, DUS-Lomax (α, β) distribution, is proposed and its properties are studied. The DUS-Lomax (α, β) has UBFR function. We derived the moments, moment generating function, characteristic function, quantiles, entropy etc., of the proposed distribution. Distributions of minimum and maximum are obtained. Estimation of parameters of the distribution is performed via maximum likelihood method. Reliability of stress-strength models is derived. A simulation study is performed for validate the MLE. DUS-Lomax (α, β) distribution is applied to three real data sets and shows that DUS-Lomax (α, β) distribution is a better fit than other well-known distributions.