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CHAPTER 5

STRESS-STRENGTH RELIABILITY

5.1 Introduction

[1] The estimation of stress-strength reliability is a very common problem in statistical literature. This model is used in many applications of physics and engineering such as strength failure and the system collapse. In many practical situations, the components of a system are of different structure so that the assumption of identical strength distributions may not be quite realistic.

The term stress is defined as a failure inducing variable. It is defined as stress (load) which tends to produce a failure of a component or of a device of a material. The term load may be defined as mechanical load, environment, temperature and electric current etc.

[1] Some contents of this chapter are based on Deepthi and Chacko (2020).

The term strength is defined as it is failure resisting variable. The ability of component, device or a material to accomplish its required function (mission) satisfactorily without failure when subjected to the external loading and environment.

In reliability and survival analysis, the stress-strength model describes the probabilistic behavior of life of a component that has a random strength X and is subjected to random stress Y. The system fails if and only if the stress is greater than strength at any time. The reliability parameter, for a single component stress-strength (SSS) model, is

$$R = P(Y < X) = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(x, y)\ dy\ dx,$$

where $f(x, y)$ is the joint pdf of X and Y. If the r.v's X and Y are independent, then $f(x, y) = f(x)\ g(y)$, where $f(x)$ and $g(y)$ are the marginal pdfs of X and Y, so that

$$R = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f(x)\ g(y)\ dy\ dx.$$

Let $G_y(x) = \int_{-\infty}^{x} g(y)\ dy$, then R becomes

$$R = \int_{-\infty}^{\infty} G_y(x) f(x)\ dx.$$

The survival probability of a SSS model has been considered by several authors for different distributions. Birnbaum (1956) introduced the stress-strength model and proposed a non-parametric estimator of R. Guttman et al. (1988) and Weerahandi and Johnson (1992) considered the estimation of R, and also obtained the associated confidence interval of R, when both stress and strength depend on some known covariates. Sun et al. (1998) obtained a Bayesian approach for estimating stress-strength reliability.

Raqab and Kundu (2005) studied the estimation of stress-strength reliability, when Y and X two independent scaled Burr type X distribution. Kundu and Gupta (2005) studied stress-strength reliability based on independent generalized exponential distributions with different shape parameters but having the same scale parameters. Kundu and Gupta (2006) studied the estimation of R based od Weibull distribution. Baklizi and Eidous (2006) proposed an estimator of stress-strength reliability based on kernel estimators. Raqab et al. (2008) discussed estimation of R based on three-parameter generalized Exponential distribution. Zhou (2008) illustrated estimation of stress-strength reliability using bootstrap method. Jing et al. (2009) estimated stress-strength reliability using empirical likelihood method. Kundu and Raqab (2009) proposed estimation of R based on three-parameter Weibull distribution. Rezaei et al. (2010) studied the estimation of stress-strength reliability based on two independent generalized Pareto random variables. Baklizi (2012) studied inference on stress-strength reliability in the two-parameter Weibull model.

Recently Jose et al. (2019) and Xavier and Jose (2020) studied the stress-strength reliability estimation of single and multi-component systems using various generalizations of half logistic distribution. Joby et al. (2020) studied estimation of stress-strength reliability of single and multi-component systems based on discrete phase type distribution. Domma et al. (2019) proposed the stress-strength reliability based on the m-generalized order statistics and the corresponding concomitant. Krishna et al. (2019) studied estimation of R using inverse Weibull distribution based on progressive first failure censoring. Kohansal and Nadarajah (2019) considered estimation of R using Kumaraswamy distribution based on Type-II hybrid progressive censored samples. Musleh et al. (2019) studied inference on R in bivariate Lomax model.

Bai et al. (2018) considered reliability inference of stress-strength model under progressively Type-II censored samples when stress and strength have truncated proportional hazard rate distributions. Asgharzadeh et al. (2017) considered estimation of stress-strength reliability based on the generalized exponential distribution. Bi and Gui (2017) studied Bayesian estimation of R using inverse Weibull distribution. Estimation of stress-strength parameter using record values from proportional hazard model was considered by Basirat et al. (2016). Two-parameter bathtub shaped life time distribution based on upper record values was presented by Tarvirdizade and Ahmadpour (2016). Ghitany et al. (2014) discussed inference on stress-strength reliability based on Power Lindley distributions. Sharma (2014) proposed an upside-down bathtub shape distribution and estimate of stress-strength reliability of inverse Lindley distribution.

But, in reality, many of the system consist of two or more components. The reliability analysis of multi-component system having various lifetime distributions for the strength of its components is important for the researchers and engineers. The multi-component stress-strength (MSS) reliability modeling is quite desirable in various real life situations. Bhattacharyya and Johnson (1974) observed the performance of a system depends on more than one component and these components have their own strength. For example, an aircraft generally contains more than one engines (k) and assume that for take off at least $s(1 \leq s \leq k)$ engines are needed. So, the aircraft will take off smoothly, if s out of k engines work. In engineering, a power system powering a manufacturing unit has k fuse cut-outs arranged in a parallel way. The power system will keep powering the manufacturing unit as long as at least $s(1 \leq s \leq k)$ fuse cut-outs are working. In suspension bridges, the deck is supported by a series of vertical cables hung from the towers. Suppose a suspension bridge consists of k number of vertical cable pairs. The bridge will only survive if a minimum s number of vertical cables through the deck are not damaged when subjected to stresses due to wind loading, heavy traffic, corrosion etc.

To find the reliability of a k component system, let the random samples $Y, X_1, X_2, \ldots, X_k$ be independent, $G(y)$ be the continuous distribution function of stress Y and $F(x)$ be the common continuous distribution function of strength $X_1, X_2, \ldots, X_k$ of components $1, 2, \ldots, k$ respectively. The reliability in a MSS model developed by Bhattacharyya and Johnson (1974) is given by

$$\begin{aligned} R_{s,k} &= P[\text{at least } s \text{ of the } (X_1, X_2, \ldots, X_k) \text{ exceed } Y] \\ &= \sum_{i=s}^{k} \binom{k}{i} \int_{-\infty}^{\infty} [1 - F(y)]^i [F(y)]^{k-i} dG(y), \quad s = 1,\, 2, \ldots, k \end{aligned}$$

where $X_1, X_2, \ldots, X_k$ independent and identically distributed (iid) with common distribution function $F(x)$ and subjected to the common random stress Y. Several researchers developed inferential procedures for the reliability of MSS model. Mokhlis and Khames (2011) studied the reliability of some parallel and series MSS model using multivariate Marshall-Olkin Exponential distribution. Rao (2012) developed the MSS reliability based on Generalized Exponential distribution.

Recently, Rao et al. (2015) discussed the MSS reliability with the Burr-XII distribution. Dey et al. (2016) studied the estimation of reliability of a MSS model based on Kumaraswamy distribution. The MSS model using Lindley distribution, when the system consists of k components experiencing a random stress is considered by Khalil (2017). Kohansal (2017) investigated the estimation of MSS reliability by assuming the Kumaraswamy distribution based on progressively Type-II censored samples. Abouelmagd et al. (2018) studied the estimation of reliability of a MSS model based on both classical and Bayesian approaches assuming that the components follow power Lindley model. Hassan and Alohal (2018) studied estimation of MSS reliability based on generalized linear failure rate distribution.

Pandit and Joshi (2018) studied estimation of MSS reliability based on generalized Pareto distribution. Fatma (2019) studied estimation of MSS reliability using Topp-Leone distribution. Jamal et al. (2019) studied estimation of MSS reliability using Pareto distribution based on upper record values. Pak et al. (2019) investigated Bayesian estimation of the reliability of an MSS system for the bathtub-shaped distribution when the available data are reported in terms of record values. Jha et al. (2020) investigated the Bayesian estimation of MSS reliability under progressive Type II censoring when stress and strength variables follow unit Gompertz distributions. Hassan et al. (2020) studied Bayesian estimation of the reliability of a MSS system with Weibull distribution based on upper record values.

In this chapter, we consider the two different cases for stress-strength reliability

- Case 1: If $X \sim \text{TPGL}(\alpha, \beta_1, \lambda_1)$ and $Y \sim \text{TPGL}(\alpha, \beta_2, \lambda_2)$.
- Case 2: If $X \sim \text{TPGL}(\alpha, \beta, \lambda_1)$ and $Y \sim \text{PL}(\alpha, \lambda_2)$.

The procedure of estimating reliability of SSS model is considered in section 5.2. In section 5.3, estimating reliability of MSS model is considered. In section 5.4, a simulation study to investigate the merits of the proposed methods is given. Real data sets are analyzed in section 5.5. Conclusions are given in section in 5.6.

The aim of this chapter is to develop the inferential procedure for estimating the stress-strength reliability $R = P[X > Y]$, where X represents the strength and Y denotes the stress. It is further assumed that X and Y are independent Three Parameter Generalized Lindley (TPGL) and Power Lindley (PL) random variables, having bathtub shaped failure rate function. Stress-strength reliability plays a very important role in the reliability analysis, and has nice probabilistic interpretation. The reliability $R = P[X > Y]$ is the probability that failure will occur a high stress. Many authors developed the estimation procedures for estimating the stress-strength reliability from various lifetime models. In this chapter we discuss stress-strength reliability analysis of bathtub shaped failure rate models.

5.2 Estimation of SSS Reliability

In this section, the procedure of estimating reliability of SSS model using two different cases. That is, when $X \sim \text{TPGL}(\alpha, \beta_1, \lambda_1)$, $Y \sim \text{TPGL}(\alpha, \beta_2, \lambda_2)$ and $X \sim \text{TPGL}(\alpha, \beta, \lambda_1)$, $Y \sim \text{PL}(\alpha, \lambda_2)$. The system fails if and only if the applied stress is greater than its strength. In section 5.2.1 and 5.2.2 obtain the MLE of R in both cases and obtain its asymptotic distribution in both cases in section 5.2.3. The asymptotic distribution has been used to construct an asymptotic confidence interval.

Case 1: Suppose X and Y are random variables independently distributed as $X \sim \text{TPGL}(\alpha, \beta_1, \lambda_1)$ and $Y \sim \text{TPGL}(\alpha, \beta_2, \lambda_2)$.

Nosakhare and Opone (2018) introduced a TPGL distribution, which exhibits bathtub shape for its failure rate function. These distributions are generated using the exponentiation and power transformations to the Lindley distribution. Reliability estimation of SSS and MSS model using TPGL distribution is an unexplored problem.

The pdf of TPGL distribution is

$$f(x; \alpha, \beta, \lambda) = \frac{\alpha\lambda^2}{1+\lambda\beta}(\beta + x^\alpha)x^{\alpha-1}e^{-\lambda x^\alpha}, \quad x > 0,\ \alpha > 0, \beta > 0, \lambda > 0.$$

Here β and λ are scale parameters, and α is the shape parameter. The cdf is given by

$$F(x; \alpha, \beta, \lambda) = 1 - \left(\frac{1 + \beta\lambda + \lambda x^\alpha}{1 + \beta\lambda}\right) e^{-\lambda x^\alpha}, \quad x > 0,\ \alpha > 0, \beta > 0, \lambda > 0$$

and the reliability function is given by

$$R(x; \alpha, \beta, \lambda) = \left(\frac{1 + \beta\lambda + \lambda x^\alpha}{1 + \beta\lambda}\right) e^{-\lambda x^\alpha}, \quad x > 0,\ \alpha > 0, \beta > 0, \lambda > 0.$$

The failure rate function of TPGL distribution is

$$r(x;\alpha,\beta,\lambda)=\frac{\alpha\lambda^2(\beta+x^\alpha)x^{\alpha-1}}{1+\beta\lambda+\lambda x^\alpha},\quad x>0,\ \alpha>0,\beta>0,\lambda>0.$$

Then, SSS reliability is

$$\begin{aligned}R=P(Y<X)&=\int_0^\infty f(x)F_y(x)\ dx\\&=\frac{\alpha\lambda_1^2}{1+\beta_1\lambda_1}\int_0^\infty\left(\beta_1x^{\alpha-1}e^{-\lambda_1x^\alpha}+x^{2\alpha-1}e^{-\lambda_1x^\alpha}\right)\left[1-\left(1+\frac{\lambda_2x^\alpha}{1+\beta_2\lambda_2}\right)e^{\lambda_2x^\alpha}\right]dx\\&=\frac{\lambda_1^2\lambda_2}{(1+\beta_1\lambda_1)(1+\beta_2\lambda_2)(\lambda_1+\lambda_2)}\left[(1+\beta_1)\left(1+\frac{1}{\lambda_1+\lambda_2}\right)+\frac{2}{(\lambda_1+\lambda_2)^2}\right].\end{aligned}\tag{5.2.1}$$

Case 2: Suppose X follows TPGL distribution with parameters (α,β,λ_1) and Y follows PL distribution with parameters (α,λ_2) and they are independent random variables. The PL distribution proposed by Ghitany et al. (2013) an extension of the Lindley distribution. The pdf of PL distribution is

$$f(x;\alpha,\lambda)=\frac{\alpha\lambda^2}{1+\lambda}(1+x^\alpha)x^{\alpha-1}e^{-\lambda x^\alpha},\quad x>0,\ \alpha>0,\ \lambda>0.\tag{5.2.2}$$

Here λ and α are scale and shape parameters. The corresponding cdf is given by

$$F(x;\alpha,\lambda)=1-\left(1+\frac{\lambda x^\alpha}{1+\lambda}\right)e^{-\lambda x^\alpha},\quad x>0,\ \alpha>0,\ \lambda>0.\tag{5.2.3}$$

Suppose that X represent the strength of a component exposed to Y stress, then the single component stress-strength reliability is obtained as follows,

$$\begin{aligned}
R &= P(X>Y) = \int_0^\infty P[X>Y|Y=y]f_y(y)\ dy \\
&= \int_0^\infty \frac{\alpha\lambda_2^2}{1+\lambda_2}\left\{\int_0^x (1+y^\alpha)y^{\alpha-1}e^{\lambda_2 y^\alpha}\ dy\right\}\frac{\alpha\lambda_1^2}{1+\beta\lambda_1}(\beta+x^\alpha)x^{\alpha-1}e^{-\lambda_1 x^\alpha}\ dx \\
&= \frac{\alpha\lambda_2^2}{1+\lambda_2}\frac{\alpha\lambda_1^2}{1+\beta\lambda_1}\int_0^\infty \left\{\frac{e^{-\lambda_2 x^\alpha}}{\alpha\lambda_2}+\frac{x^\alpha e^{-\lambda_2 x^\alpha}}{\alpha\lambda_2}-\frac{e^{-\lambda_2 x^\alpha}}{\alpha\lambda_2^2}\right\}\left((\beta+x^\alpha)x^{\alpha-1}e^{-\lambda_1 x^\alpha}\right)\ dx \\
&= \frac{\alpha^2\lambda_1^2\lambda_2^2}{(1+\beta\lambda_1)(1+\lambda_2)}\left\{\frac{\beta}{\alpha\lambda_2}\int_0^\infty x^{\alpha-1}e^{-(\lambda_1+\lambda_2)x^\alpha}\ dx+\frac{1}{\alpha\lambda_2}\int_0^\infty x^{2\alpha-1}e^{-(\lambda_1+\lambda_2)x^\alpha}\ dx\right. \\
&+\frac{\beta}{\alpha\lambda_2}\int_0^\infty x^{2\alpha-1}e^{-(\lambda_1+\lambda_2)x^\alpha}\ dx+\frac{1}{\alpha\lambda_2}\int_0^\infty x^{3\alpha-1}e^{-(\lambda_1+\lambda_2)x^\alpha}\ dx \\
&\left.-\frac{\beta}{\alpha\lambda_2^2}\int_0^\infty x^{\alpha-1}e^{-(\lambda_1+\lambda_2)x^\alpha}\ dx-\frac{1}{\alpha\lambda_2^2}\int_0^\infty x^{2\alpha-1}e^{-(\lambda_1+\lambda_2)x^\alpha}\ dx\right\} \\
&= \frac{\lambda_1^2\lambda_2}{(1+\beta\lambda_1)(1+\lambda_2)(\lambda_1+\lambda_2)}\left\{\left(2+\beta-\frac{1}{\lambda_2}\right)\left(1+\frac{1}{\lambda_1+\lambda_2}\right)+\frac{2}{(\lambda_1+\lambda_2)^2}\right\}.
\end{aligned} \tag{5.2.4}$$

Remark 5.2.1.

- The stress-strength reliability parameter R in (5.2) and (5.2.4) does not depend on the common shape parameter α.
- If $R = 0.5$ which means there is an equal chance that strength X is greater than stress Y.
- If $R > 0.5$ which means there is a small chance that X is greater than Y.
- If $R < 0.5$ which means there is a high chance that X is greater than Y.

5.2.1 Maximum Likelihood Estimation of R (Case 1:)

Let X be the strength r.v following TPGL $(\alpha, \beta_1, \lambda_1)$ distribution and Y be the stress r.v. following TPGL$(\alpha, \beta_2, \lambda_2)$ distribution. Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ be two ordered random samples of size n, m respectively, taken from TPGL distribution. Then the likelihood function based on the combined random sample is given by

$$L = \prod_{i=1}^{n} \frac{\alpha\lambda_1^2}{1+\beta_1\lambda_1}(\beta_1 + x_i^{\alpha})x_i^{\alpha-1}e^{-\lambda_1 x_i^{\alpha}} \prod_{j=1}^{m} \frac{\alpha\lambda_2^2}{1+\beta_2\lambda_2}(\beta_2 + y_j^{\alpha})y_j^{\alpha-1}e^{-\lambda_2 y_j^{\alpha}}.$$

The log-likelihood function is

$$\begin{aligned} l = \log L = (n+m)\log\alpha + 2n\log\lambda_1 - n\log(1+\beta_1\lambda_1) + \sum_{i=1}^{n}\log(\beta_1 + x_i^{\alpha}) \\ + (\alpha-1)\sum_{i=1}^{n}\log x_i - \lambda_1\sum_{i=1}^{n} x_i^{\alpha} + 2m\log\lambda_2 - m\log(1+\beta_2\lambda_2) \\ + \sum_{j=1}^{m}\log(\beta_2 + y_j^{\alpha}) + (\alpha-1)\sum_{j=1}^{m}\log y_j - \lambda_2\sum_{j=1}^{m} y_j^{\alpha}. \end{aligned}$$

The MLE of the parameters is the solution of following non-linear equations

$$\frac{\partial l}{\partial \beta_1} = -\frac{n\lambda_1}{1+\beta_1\lambda_1} + \sum_{i=1}^{n}\frac{1}{\beta_1 + x_i^{\alpha}} \tag{5.2.5}$$

$$\frac{\partial l}{\partial \beta_2} = -\frac{m\lambda_2}{1+\beta_2\lambda_2} + \sum_{j=1}^{m}\frac{1}{\beta_2 + y_j^{\alpha}} \tag{5.2.6}$$

$$\frac{\partial l}{\partial \lambda_1} = \frac{2n}{\lambda_1} - \frac{n\beta_1}{1+\beta_1\lambda_1} - \sum_{i=1}^{n} x_i^{\alpha} \tag{5.2.7}$$

$$\frac{\partial l}{\partial \lambda_2} = \frac{2m}{\lambda_2} - \frac{m\beta_2}{1+\beta_2\lambda_2} - \sum_{j=1}^{m} y_j^{\alpha} \tag{5.2.8}$$

and
$$\frac{\partial l}{\partial \alpha} = \frac{n+m}{\alpha} + \sum_{i=1}^{n} \frac{x_i^{\alpha}\log x_i}{(\beta_1 + x_i^{\alpha})} + \sum_{i=1}^{n} \log x_i - \lambda_1 \sum_{i=1}^{n} x_i^{\alpha} \log x_i$$
$$+ \sum_{j=1}^{m} \frac{y_j^{\alpha}\log y_j}{(\beta_2 + y_j^{\alpha})} + \sum_{j=1}^{m} \log y_j - \lambda_2 \sum_{j=1}^{m} y_j^{\alpha} \log y_j. \tag{5.2.9}$$

The second partial derivatives are

$$\frac{\partial^2 l}{\partial \beta_1^2} = \frac{n\lambda_1^2}{(1+\beta_1\lambda_1^2)^2} - \sum_{i=1}^{n}\frac{1}{(\beta_1+x_i^{\alpha})^2}, \quad \frac{\partial^2 l}{\partial \beta_1 \partial \alpha} = -\sum_{i=1}^{n}\frac{x_i^{\alpha}\log x_i}{(\beta_1+x_i^{\alpha})^2},$$
$$\frac{\partial^2 l}{\partial \beta_2^2} = \frac{m\lambda_2^2}{(1+\beta_2\lambda_2^2)^2} - \sum_{j=1}^{m}\frac{1}{(\beta_2+y_j^{\alpha})^2}, \quad \frac{\partial^2 l}{\partial \beta_2 \partial \alpha} = -\sum_{j=1}^{m}\frac{y_j^{\alpha}\log y_j}{(\beta_2+y_j^{\alpha})^2},$$
$$\frac{\partial^2 l}{\partial \beta_1 \partial \beta_2} = \frac{\partial^2 l}{\partial \beta_1 \partial \lambda_2} = \frac{\partial^2 l}{\partial \lambda_1 \partial \lambda_1} = 0,$$
$$\frac{\partial^2 l}{\partial \lambda_1^2} = \frac{n\beta_1^2}{(1+\beta_1\lambda_1)^2} - \frac{2n}{\lambda_1^2}, \quad \frac{\partial^2 l}{\partial \beta_1 \partial \lambda_1} = -\frac{n}{(1+\beta_1\lambda_1^2)^2}, \quad \frac{\partial^2 l}{\partial \lambda_1 \partial \alpha} = -\sum_{i=1}^{n} x_i^{\alpha}\log x_i,$$
$$\frac{\partial^2 l}{\partial \lambda_2^2} = \frac{m\beta_2^2}{(1+\beta_2\lambda_2)^2} - \frac{2m}{\lambda_2^2}, \quad \frac{\partial^2 l}{\partial \beta_2 \partial \lambda_2} = -\frac{m}{(1+\beta_2\lambda_2^2)^2}, \quad \frac{\partial^2 l}{\partial \lambda_2 \partial \alpha} = -\sum_{j=1}^{m} y_j^{\alpha}\log y_j$$

and

$$\frac{\partial^2 l}{\partial \alpha^2} = -\frac{n+m}{\alpha^2} + \sum_{i=1}^{n}\left\{\frac{x_i^{\alpha}\log^2(x_i)}{\beta_1+x_i^{\alpha}} - \frac{x_i^{2\alpha}\log^2 x_i}{(\beta_1+x_i^{\alpha})^2}\right\}$$
$$-\lambda_1\sum_{i=1}^{n} x_i^{\alpha}\log^2(x_i) - \lambda_2\sum_{j=1}^{m} y_j^{\alpha}\log^2(y_j) + \sum_{j=1}^{m}\left\{\frac{y_j^{\alpha}\log^2(y_j)}{\beta_2+y_j^{\alpha}} - \frac{y_j^{2\alpha}\log^2 y_j}{(\beta_2+y_j^{\alpha})^2}\right\}.$$

The MLE of SSS reliability R is obtained by

$$\hat{R}^{ML} = \frac{\hat{\lambda_1}^2\hat{\lambda_2}}{(1+\hat{\beta}_1\hat{\lambda_1})(1+\hat{\beta}_2\hat{\lambda_2})(\hat{\lambda_1}+\hat{\lambda_2})}\left[(1+\hat{\beta}_1)\left(1+\frac{1}{\hat{\lambda_1}+\hat{\lambda_2}}\right)+\frac{2}{(\hat{\lambda_1}+\hat{\lambda_2})^2}\right]. \tag{5.2.10}$$

5.2.2 Maximum Likelihood Estimation of R (Case 2:)

Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n from TPGL$(\alpha, \beta, \lambda_1)$ and $Y_1, Y_2, \ldots, Y_m$ is a random sample of size m from PL(α, λ_2). Then the likelihood function is given by

$$L = \prod_{i=1}^{n} \frac{\alpha\lambda_1^2}{1+\beta\lambda_1}(\beta + x_i^\alpha)x_i^{\alpha-1}e^{-\lambda_1 x_i^\alpha} \prod_{j=1}^{m} \frac{\alpha\lambda_2^2}{1+\lambda_2}(1+y_j^\alpha)y_j^{\alpha-1}e^{-\lambda_2 y_j^\alpha}.$$

Then the log-likelihood function is

$$\begin{aligned} l = \log L = &(n+m)\log\alpha + 2n\log\lambda_1 - n\log(1+\beta\lambda_1) + \sum_{i=1}^{n}\log(\beta + x_i^\alpha) \\ &+ (\alpha-1)\sum_{i=1}^{n}\log x_i - \lambda_1\sum_{i=1}^{n}x_i^\alpha + 2m\log\lambda_2 - m\log(1+\lambda_2) \\ &+ \sum_{j=1}^{m}\log(1+y_j^\alpha) + (\alpha-1)\sum_{j=1}^{m}\log y_j - \lambda_2\sum_{j=1}^{m}y_j^\alpha. \end{aligned} \tag{5.2.11}$$

The MLE of the parameters is the solution of non-linear equations as follows

$$\frac{\partial l}{\partial \beta} = -\frac{n\lambda_1}{1+\beta\lambda_1} + \sum_{i=1}^{n}\frac{1}{\beta + x_i^\alpha} \tag{5.2.12}$$

$$\frac{\partial l}{\partial \lambda_1} = \frac{2n}{\lambda_1} - \frac{n\beta}{1+\beta\lambda_1} - \sum_{i=1}^{n} x_i^{\alpha} \tag{5.2.13}$$

$$\frac{\partial l}{\partial \lambda_2} = \frac{2m}{\lambda_2} - \frac{m}{1+\lambda_2} - \sum_{j=1}^{m} y_j^{\alpha} \tag{5.2.14}$$

and

$$\begin{aligned}\frac{\partial l}{\partial \alpha} &= \frac{n+m}{\alpha} + \sum_{i=1}^{n} \frac{x_i^{\alpha}\log x_i}{(\beta + x_i^{\alpha})} + \sum_{i=1}^{n} \log x_i - \lambda_1 \sum_{i=1}^{n} x_i^{\alpha} \log x_i \\ &+ \sum_{j=1}^{m} \frac{y_j^{\alpha}\log y_j}{(1+y_j^{\alpha})} + \sum_{j=1}^{m} \log y_j - \lambda_2 \sum_{j=1}^{m} y_j^{\alpha}\log y_j. \end{aligned} \tag{5.2.15}$$

The second partial derivatives are

$$\frac{\partial^2 l}{\partial \beta^2} = \frac{n\lambda_1^2}{(1+\beta\lambda_1^2)^2} - \sum_{i=1}^{n} \frac{1}{(\beta + x_i^{\alpha})^2}, \quad \frac{\partial^2 l}{\partial \beta \partial \alpha} = -\sum_{i=1}^{n} \frac{x_i^{\alpha}\log x_i}{(\beta + x_i^{\alpha})^2},$$

$$\frac{\partial^2 l}{\partial \lambda_1^2} = \frac{n\beta^2}{(1+\beta\lambda_1)^2} - \frac{2n}{\lambda_1^2}, \quad \frac{\partial^2 l}{\partial \beta \partial \lambda_1} = -\frac{n}{(1+\beta\lambda_1^2)^2}, \quad \frac{\partial^2 l}{\partial \lambda_1 \partial \alpha} = -\sum_{i=1}^{n} x_i^{\alpha}\log x_i,$$

$$\frac{\partial^2 l}{\partial \lambda_2^2} = \frac{m}{(1+\lambda_2)^2} - \frac{2m}{\lambda_2^2}, \quad \frac{\partial^2 l}{\partial \beta \partial \lambda_2} = 0, \quad \frac{\partial^2 l}{\partial \lambda_2 \partial \alpha} = -\sum_{j=1}^{m} y_j^{\alpha}\log y_j$$

and

$$\begin{aligned}\frac{\partial^2 l}{\partial \alpha^2} &= -\frac{n+m}{\alpha^2} + \sum_{i=1}^{n}\left\{\frac{x_i^{\alpha}\log^2(x_i)}{\beta + x_i^{\alpha}} - \frac{x_i^{2\alpha}\log^2 x_i}{(\beta + x_i^{\alpha})^2}\right\} \\ &- \lambda_1 \sum_{i=1}^{n} x_i^{\alpha}\log^2(x_i) - \lambda_2 \sum_{j=1}^{m} y_j^{\alpha}\log^2(y_j) + \sum_{j=1}^{m}\left\{\frac{y_j^{\alpha}\log^2(y_j)}{1+y_j^{\alpha}} - \frac{y_j^{2\alpha}\log^2 y_j}{(1+y_j^{\alpha})^2}\right\}.\end{aligned}$$

The MLE of single component stress-strength reliability R is obtained by

$$\hat{R}^{ML} = \frac{\hat{\lambda_1}^2\hat{\lambda_2}}{(1+\hat{\beta}\hat{\lambda_1})(1+\hat{\lambda_2})(\hat{\lambda_1}+\hat{\lambda_2})}\left\{\left(2+\hat{\beta}-\frac{1}{\hat{\lambda_2}}\right)\left(1+\frac{1}{\hat{\lambda_1}+\hat{\lambda_2}}\right)+\frac{2}{(\hat{\lambda_1}+\hat{\lambda_2})^2}\right\}. \tag{5.2.16}$$

5.2.3 Asymptotic distribution and Confidence Intervals

In this section, the asymptotic distribution and confidence interval of the MLE of R is obtained from case 1. To find an asymptotic variance of the $\hat{R}^{ML}$ in (5.2.10), let us denote the Fisher information matrix of $\theta = (\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$ as $I(\theta) = [I_{ij}(\theta);\ i,j = 1,2,\ldots,5]$, i.e.,

$$I(\theta) = E\begin{bmatrix} -\frac{\partial^2 l}{\partial\alpha^2} & -\frac{\partial^2 l}{\partial\alpha\partial\beta_1} & -\frac{\partial^2 l}{\partial\alpha\partial\beta_2} & -\frac{\partial^2 l}{\partial\alpha\partial\lambda_1} & -\frac{\partial^2 l}{\partial\alpha\partial\lambda_2} \\ -\frac{\partial^2 l}{\partial\beta_1\partial\alpha} & -\frac{\partial^2 l}{\partial\beta_1^2} & -\frac{\partial^2 l}{\partial\beta_1\partial\beta_2} & -\frac{\partial^2 l}{\partial\beta_1\partial\lambda_1} & -\frac{\partial^2 l}{\partial\beta_1\partial\lambda_2} \\ -\frac{\partial^2 l}{\partial\beta_2\partial\alpha} & -\frac{\partial^2 l}{\partial\beta_2\partial\beta_1} & -\frac{\partial^2 l}{\partial\beta_2^2} & -\frac{\partial^2 l}{\partial\beta_2\partial\lambda_1} & -\frac{\partial^2 l}{\partial\beta_2\partial\lambda_2} \\ -\frac{\partial^2 l}{\partial\lambda_1\partial\alpha} & -\frac{\partial^2 l}{\partial\lambda_1\partial\beta_1} & -\frac{\partial^2 l}{\partial\lambda_1\partial\beta_2} & -\frac{\partial^2 l}{\partial\lambda_1^2} & -\frac{\partial^2 l}{\partial\lambda_1\partial\lambda_2} \\ -\frac{\partial^2 l}{\partial\lambda_2\partial\alpha} & -\frac{\partial^2 l}{\partial\lambda_2\partial\beta_1} & -\frac{\partial^2 l}{\partial\lambda_2\partial\beta_2} & -\frac{\partial^2 l}{\partial\lambda_2\partial\lambda_1} & -\frac{\partial^2 l}{\partial\lambda_2^2} \end{bmatrix}.$$

In order to establish the asymptotic Normality of R, we further define

$$d(\theta) = \left(\frac{\partial R}{\partial\alpha}, \frac{\partial R}{\partial\beta_1}, \frac{\partial R}{\partial\beta_2}, \frac{\partial R}{\partial\lambda_1}, \frac{\partial R}{\partial\lambda_2}\right)' = (d_1, d_2, d_3, d_4, d_5)',$$

where

$$\frac{\partial R}{\partial\alpha} = 0, \quad \frac{\partial R}{\partial\beta_1} = -\frac{\lambda_1^2\lambda_2\left((\lambda_1-1)\lambda_2^2 + (2\lambda_1^2-\lambda_1-1)\lambda_2 + \lambda_1^3 + \lambda_1\right)}{(1+\beta_1\lambda_1)^2(1+\beta_2\lambda_2)(\lambda_1+\lambda_2)^3},$$

$$\frac{\partial R}{\partial \beta_2} = -\frac{\lambda_1^2\lambda_2^2}{(1+\beta_1\lambda_1)(1+\beta_2\lambda_2)^2(\lambda_1+\lambda_2)}\left[(1+\beta_1)\left(1+\frac{1}{\lambda_1+\lambda_2}\right)+\frac{2}{(\lambda_1+\lambda_2)^2}\right],$$

$$\begin{aligned}\frac{\partial R}{\partial \lambda_1} = \lambda_1\lambda_2 \Bigg[& \frac{\left((\beta_1^2+\beta_1)\lambda_2-\beta_1^2+1\right)\lambda_1^3+\left((2\beta_1^2+2\beta_1)\lambda_2^2+(4\beta_1+4)\lambda_2-4\beta_1\right)\lambda_1^2}{(1+\beta_1\lambda_1)^2(1+\beta_2\lambda_2)(\lambda_1+\lambda_2)^4} \\ & + \frac{\left((\beta_1^2+\beta_1)\lambda_2^3+(\beta_1^2+6\beta_1+5)\lambda_2^2+(4\beta_1+2)\lambda_2-2\right)\lambda_1}{(1+\beta_1\lambda_1)^2(1+\beta_2\lambda_2)(\lambda_1+\lambda_2)^4} \\ & + \frac{(2\beta_1+2)\lambda_2^3+(2\beta_1+2)\lambda_2^2+4\lambda_2}{(1+\beta_1\lambda_1)^2(1+\beta_2\lambda_2)(\lambda_1+\lambda_2)^4}\Bigg]\end{aligned}$$

and

$$\begin{aligned}\frac{\partial R}{\partial \lambda_2} = -\lambda_1^2 \Bigg[& \frac{(1+\beta_1)\beta_2\lambda_2^4+\left((2\beta_1+2)\beta_2\lambda_1+(2\beta_1+2)\beta_2\right)\lambda_2^3}{(1+\beta_1\lambda_1)(1+\beta_2\lambda_2)^2(\lambda_1+\lambda_2)^4} \\ & + \frac{((-2\beta_1-2)\lambda_1^2+4)\lambda_2-2\lambda_1}{(1+\beta_1\lambda_1)(1+\beta_2\lambda_2)^2(\lambda_1+\lambda_2)^4} \\ & + \frac{((1+\beta_1)\beta_2\lambda_1^2+((2\beta_1+2)\beta_2-\beta_1-1)\lambda_1+6\beta_2+\beta_1+1)\lambda_2^2}{(1+\beta_1\lambda_1)(1+\beta_2\lambda_2)^2(\lambda_1+\lambda_2)^4} \\ & + \frac{(-\beta_1-1)\lambda_1^3+(-\beta_1-1)\lambda_1^2}{(1+\beta_1\lambda_1)(1+\beta_2\lambda_2)^2(\lambda_1+\lambda_2)^4}\Bigg].\end{aligned}$$

We obtain the asymptotic distribution of $\hat{R}^{ML}$ as

$$\sqrt{n+m}(\hat{R}^{ML}-R) \xrightarrow{d} N(0, d'(\theta)I^{-1}(\theta)d(\theta)).$$

Asymptotic variance of $\hat{R}^{ML}$ is obtained as

$$AV(\hat{R}^{ML}) = \frac{1}{n+m}d'(\theta)I^{-1}(\theta)d(\theta)$$

$$AV(\hat{R}^{ML}) = V(\hat{\alpha})d_1^2 + V(\hat{\beta_1})d_2^2 + V(\hat{\beta_2})d_3^2 + V(\hat{\lambda_1})d_4^2 + V(\hat{\lambda_2})d_5^2$$
$$+ 2d_1d_2Cov(\hat{\alpha}, \hat{\beta_1}) + 2d_1d_3Cov(\hat{\alpha}, \hat{\beta_2}) + \ldots + 2d_4d_5Cov(\hat{\lambda_1}, \lambda_2). \tag{5.2.17}$$

Asymptotic $100(1-\gamma)\%$ confidence interval for R can be obtained as

$$\hat{R}^{ML} \pm Z_{\frac{\gamma}{2}}\sqrt{AV(\hat{R}^{ML})},$$

where $Z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}$ quantile of the standard Normal distribution.

Case 2: To find an asymptotic variance of $\hat{R}^{ML}$ in (5.2.16). Let us denote the Fisher information matrix of $\theta = (\alpha, \beta, \lambda_1, \lambda_2)$ as $I(\theta) = I_{ij}(\theta);\ i, j = 1, 2, 3, 4.$

$$I(\theta) = E\begin{bmatrix} -\frac{\partial^2 l}{\partial\alpha^2} & -\frac{\partial^2 l}{\partial\alpha\partial\beta} & -\frac{\partial^2 l}{\partial\alpha\partial\lambda_1} & -\frac{\partial^2 l}{\partial\alpha\partial\lambda_2} \\ -\frac{\partial^2 l}{\partial\beta\partial\alpha} & -\frac{\partial^2 l}{\partial\beta^2} & -\frac{\partial^2 l}{\partial\beta\partial\lambda_1} & -\frac{\partial^2 l}{\partial\beta\partial\lambda_2} \\ -\frac{\partial^2 l}{\partial\lambda_1\partial\alpha} & -\frac{\partial^2 l}{\partial\lambda_1\partial\beta} & -\frac{\partial^2 l}{\partial\lambda_1^2} & -\frac{\partial^2 l}{\partial\lambda_1\partial\lambda_2} \\ -\frac{\partial^2 l}{\partial\lambda_2\partial\alpha} & -\frac{\partial^2 l}{\partial\lambda_2\partial\beta} & -\frac{\partial^2 l}{\partial\lambda_2\partial\lambda_1} & -\frac{\partial^2 l}{\partial\lambda_2^2} \end{bmatrix}.$$

In order to establish the asymptotic normality of R, we further define

$$d(\theta) = \left(\frac{\partial R}{\partial\alpha}, \frac{\partial R}{\partial\beta}, \frac{\partial R}{\partial\lambda_1}, \frac{\partial R}{\partial\lambda_2}\right)' = (d_1, d_2, d_3, d_4)',$$

where $\frac{\partial R}{\partial\alpha} = 0,$

$$\frac{\partial R}{\partial\beta} = \frac{-\lambda_1^2\left((2\lambda_1-1)\lambda_2^3 + (4\lambda_1-\lambda_1-1)\lambda_2^2 + (2\lambda_1^3-\lambda_1^2)\lambda_2 - \lambda_1^3 - \lambda_1^2\right)}{(1+\lambda_2)(\lambda_1+\lambda_2)^3(1+\beta\lambda_1)^2},$$

$$\begin{aligned}\frac{\partial R}{\partial \lambda_1} &= \lambda_1 \Bigg[\frac{\left((\beta^2+2\beta)\lambda_2^2 + (-\beta^2-2\beta+2)\lambda_2 + \beta - 1\right)\lambda_1^3}{(1+\lambda_2)(\lambda_1+\lambda_2)^4(1+\beta\lambda_1)^2} \\ &+ \frac{\left((2\beta^2+4\beta)\lambda_2^3 + (2\beta+8)\lambda_2^2 + (-4\beta-4)\lambda_2\right)\lambda_1^2}{(1+\lambda_2)(\lambda_1+\lambda_2)^4(1+\beta\lambda_1)^2} \\ &+ \frac{\left((\beta^2+2\beta)\lambda_2^4 + (\beta^2+6\beta+10)\lambda_2^3 + (3\beta-1)\lambda_2^2 - 4\lambda_2\right)\lambda_1}{(1+\lambda_2)(\lambda_1+\lambda_2)^4(1+\beta\lambda_1)^2} \\ &+ \frac{(2\beta+4)\lambda_2^4 + (2\beta+2)\lambda_2^3 + 2\lambda_2^2}{(1+\lambda_2)(\lambda_1+\lambda_2)^4(1+\beta\lambda_1)^2} \Bigg],\end{aligned}$$

$$\begin{aligned}\frac{\partial R}{\partial \lambda_2} &= -\lambda_1^2 \Bigg[\frac{(\beta+2)\lambda_2^4 + ((2\beta+4)\lambda_1 + 2\beta+2)\lambda_2^3}{(1+\beta\lambda_1)(1+\lambda_2)^2(\lambda_1+\lambda_2)^4} \\ &+ \frac{((\beta+2)\lambda_1^2 + (\beta-3)\lambda_1 + \beta+4)\lambda_2^2}{(1+\beta\lambda_1)(1+\lambda_2)^2(\lambda_1+\lambda_2)^4} \\ &+ \frac{((-2\beta-8)\lambda_1^2 - 6\lambda_1 + 2)\lambda_2 + (-\beta-3)\lambda_1^3 + (-\beta-4)\lambda_1^4 - 4\lambda_1}{(1+\beta\lambda_1)(1+\lambda_2)^2(\lambda_1+\lambda_2)^4} \Bigg].\end{aligned}$$

The asymptotic variance of $\hat{R}^{ML}$ is obtained as

$$\begin{aligned} AV(\hat{R}^{ML}) &= \frac{1}{n+m} d'(\theta) I^{-1}(\theta) d(\theta) \\ &= V(\hat{\alpha})d_1^2 + V(\hat{\beta})d_2^2 + V(\hat{\lambda_1})d_3^2 + V(\hat{\lambda_2})d_4^2 \\ &+ 2d_1d_2Cov(\hat{\alpha},\hat{\beta}) + \ldots + 2d_3d_4Cov(\hat{\lambda_1},\lambda_2). \end{aligned}$$

Hence, an asymptotic $100(1-\eta)\%$ confidence interval for R can be obtained as

$$\hat{R}^{ML} \pm Z_{\frac{\eta}{2}} \sqrt{AV(\hat{R}^{ML})},$$

where $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}$ quantile of the standard Normal distribution.

5.3 Estimation of Reliability in MSS Model

Suppose that $Y, X_1, X_2, \ldots, X_k$ are independent, $G(y)$ is the cdf of Y and $F(x)$ is the common cdf of $X_1, X_2, \ldots, X_k$. The reliability of MSS model with TPGL distribution is

$$R_{s,k} = \frac{\alpha\lambda_2^2}{1+\beta_2\lambda_2}\sum_{i=s}^{k}\binom{k}{i}\int_0^\infty \left[\left(1+\frac{\lambda_1 x^\alpha}{1+\beta_1\lambda_1}\right)e^{-\lambda_1 x^\alpha}\right]^i \left[1-\left(1+\frac{\lambda_1 x^\alpha}{1+\beta_1\lambda_1}\right)e^{-\lambda_1 x^\alpha}\right]^{k-i}(\beta_2+x^\alpha)x^{\alpha-1}e^{-\lambda_2 x^\alpha}\,dx.$$

Expanding the terms inside the integral, we get

$$= \frac{\alpha\lambda_2^2}{1+\beta_2\lambda_2}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta_1\lambda_1}\right)^{l_1+l_3} \int_0^\infty \beta_2 x^{\alpha(l_1+l_3+1)-1}e^{-x^\alpha[\lambda_1(i+l_2)+\lambda_2]}\,dx + \int_0^\infty x^{\alpha(l_1+l_3+1)-1}e^{-x^\alpha[\lambda_1(i+l_2)+\lambda_2]}\,dx.$$

After the simplification, we get

$$R_{s,k} = \frac{\alpha\lambda_2^2}{1+\beta_2\lambda_2}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta_1\lambda_1}\right)^{l_1+l_3} \left\{\frac{\beta_2(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}}+\frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}\right\}. \quad (5.3.1)$$

By using the invariance property, MLE of MSS reliability $R_{s,k}$, $s = 1, 2, \ldots, k$, is obtained by

$$\hat{R}^{ML}_{s,k} = \frac{\hat{\alpha}\hat{\lambda_2}^2}{1+\hat{\beta}_2\hat{\lambda_2}} \sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2} \binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\hat{\lambda_1}}{1+\hat{\beta}_1\hat{\lambda_1}}\right)^{l_1+l_3}$$
$$\left\{\frac{\hat{\beta}_2(l_1+l_3)!}{\hat{\alpha}\left[\hat{\lambda_1}(i+l_2)+\hat{\lambda_2}\right]^{l_1+l_3+1}} + \frac{(l_1+l_3+1)!}{\hat{\alpha}\left[\hat{\lambda_1}(i+l_2)+\hat{\lambda_2}\right]^{l_1+l_3+2}}\right\}. \quad (5.3.2)$$

In order to establish the asymptotic Normality of $R_{s,k}$, $1 \le s \le k$, we further define

$$d(\theta) = \left(\frac{\partial R_{s,k}}{\partial \alpha}, \frac{\partial R_{s,k}}{\partial \beta_1}, \frac{\partial R_{s,k}}{\partial \beta_2}, \frac{\partial R_{s,k}}{\partial \lambda_1}, \frac{\partial R_{s,k}}{\partial \lambda_2}\right)' = (d_1, d_2, d_3, d_4, d_5)',$$

where, $\frac{\partial R_{s,k}}{\partial \alpha} = 0$,

$$\frac{\partial R_{s,k}}{\partial \beta_1} = \frac{\alpha\lambda_2^2}{1+\beta_2\lambda_2} \sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2} \binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta_1\lambda_1}\right)^{l_1+l_3+1}$$
$$(-(l_1+l_3))\left\{\frac{\beta_2(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}} + \frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}\right\},$$

$$\frac{\partial R_{s,k}}{\partial \beta_2} = \alpha \sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2} \binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta_1\lambda_1}\right)^{l_1+l_3}$$
$$\left[\frac{-\lambda_2}{(1+\beta_2\lambda_2)}\left\{\frac{\beta_2(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}} + \frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}\right\}\right.$$
$$\left. + \frac{\lambda_2^2}{(1+\beta_2\lambda_2)}\frac{(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}}\right],$$

$$\frac{\partial R_{s,k}}{\partial \lambda_1} = \frac{\alpha\lambda_2^2}{1+\beta_2\lambda_2} \sum_{i=s}^{k} \sum_{l_1=0}^{i} \sum_{l_2=0}^{k-i} \sum_{l_3=0}^{l_2} \binom{k}{i} \binom{i}{l_1} \binom{k-i}{l_2} \binom{l_2}{l_3} (-1)^{l_2} \left(\frac{\lambda_1}{1+\beta_1\lambda_1}\right)^{l_1+l_3}$$
$$\left[\frac{(l_1+l_3)(\beta_1-1)}{\lambda_1(1+\beta_1\lambda_1)} \left\{ \frac{\beta_2(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}} + \frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}} \right\}\right.$$
$$\left. + (i+l_2) \left\{ \frac{\beta_2(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}} + \frac{(l_1+l_3+2)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+3}} \right\}\right],$$

$$\frac{\partial R_{s,k}}{\partial \lambda_2} = \alpha\frac{\lambda_2^2+2\lambda_2}{1+\beta_2\lambda_2} \sum_{i=s}^{k} \sum_{l_1=0}^{i} \sum_{l_2=0}^{k-i} \sum_{l_3=0}^{l_2} \binom{k}{i} \binom{i}{l_1} \binom{k-i}{l_2} \binom{l_2}{l_3} (-1)^{l_2} \left(\frac{\lambda_1}{1+\beta_1\lambda_1}\right)^{l_1+l_3}$$
$$\left[(\beta_2-1) \left\{ \frac{\beta_2(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}} + \frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}} \right\}\right.$$
$$\left. + \left\{ \frac{\beta_2(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}} + \frac{(l_1+l_3+2)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+3}} \right\}\right].$$

Asymptotic variance of $\hat{R}_{s,k}^{ML}$ is

$$AV(\hat{R}_{s,k}^{ML}) = \sum_{i=1}^{5} \sum_{j=1}^{5} \frac{\partial R_{s,k}}{\partial \theta_i} \frac{\partial R_{s,k}}{\partial \theta_j} I^{-1}(\theta) \tag{5.3.3}$$

where $\theta = (\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$ and $I^{-1}(\theta)$ is the Fisher Information Matrix. Therefore, asymptotic $100(1-\nu)\%$ confidence interval for $R_{s,k}$ can be obtained as $\hat{R}_{s,k}^{ML} \pm Z_{\frac{\nu}{2}} \sqrt{AV(\hat{R}_{s,k}^{ML})}$ where $Z_{\frac{\nu}{2}}$ is the upper $\frac{\nu}{2}-$ quantile of standard Normal distribution.

Case 2: Suppose that $Y, X_1, X_2, \ldots, X_k$ are independent, $G(y)$ is the cumulative function of Y and $F(x)$ is the common cumulative function of $X_1, X_2, \ldots, X_k$. Here, $X \sim$ TPGL$(\alpha, \beta, \lambda_1)$ and $Y \sim$ PL(α, λ_2). The reliability in multi-component stress strength is

$$R_{s,k} = \frac{\alpha\lambda_2^2}{1+\lambda_2}\sum_{i=s}^{k}\binom{k}{i}\int_0^\infty \left[\left(1+\frac{\lambda_1 x^\alpha}{1+\beta\lambda_1}\right)e^{-\lambda_1 x^\alpha}\right]^i \left[1-\left(1+\frac{\lambda_1 x^\alpha}{1+\beta\lambda_1}\right)e^{-\lambda_1 x^\alpha}\right]^{k-i}(1+x^\alpha)x^{\alpha-1}e^{-\lambda_2 x^\alpha}\,dx.$$

Expanding the terms inside the integral, we get

$$\begin{aligned}R_{s,k} &= \frac{\alpha\lambda_2^2}{1+\lambda_2}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta\lambda_1}\right)^{l_1+l_3}\\ &\quad\int_0^\infty x^{\alpha(l_1+l_3+1)-1}e^{-x^\alpha[\lambda_1(i+l_2)+\lambda_2]}\,dx+\int_0^\infty x^{\alpha(l_1+l_3+1)-1}e^{-x^\alpha[\lambda_1(i+l_2)+\lambda_2]}\,dx\\ &= \frac{\alpha\lambda_2^2}{1+\lambda_2}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta\lambda_1}\right)^{l_1+l_3}\\ &\quad\left\{\frac{(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}}+\frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}\right\}. \end{aligned} \tag{5.3.4}$$

By using the invariance property, MLE of $R_{s,k}$ is obtained by

$$\begin{aligned}\hat{R}_{s,k}^{ML} &= \frac{\hat{\alpha}\hat{\lambda_2}^2}{1+\hat{\lambda_2}}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\hat{\lambda_1}}{1+\hat{\beta}\hat{\lambda_1}}\right)^{l_1+l_3}\\ &\quad\left\{\frac{(l_1+l_3)!}{\hat{\alpha}\left[\hat{\lambda_1}(i+l_2)+\hat{\lambda_2}\right]^{l_1+l_3+1}}+\frac{(l_1+l_3+1)!}{\hat{\alpha}\left[\hat{\lambda_1}(i+l_2)+\hat{\lambda_2}\right]^{l_1+l_3+2}}\right\}. \end{aligned} \tag{5.3.5}$$

In order to establish the asymptotic Normality of $R_{s,k}$, we further define

$$d(\theta) = \left(\frac{\partial R_{s,k}}{\partial\alpha}, \frac{\partial R_{s,k}}{\partial\beta}, \frac{\partial R_{s,k}}{\partial\lambda_1}, \frac{\partial R_{s,k}}{\partial\lambda_2}\right)' = (d_1, d_2, d_3, d_4)',$$

where, $\frac{\partial R_{s,k}}{\partial \alpha} = 0$

$$\frac{\partial R_{s,k}}{\partial \beta} = \frac{\alpha\lambda_2^2}{1+\lambda_2}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta\lambda_1}\right)^{l_1+l_3+1}$$
$$(-(l_1+l_3))\left\{\frac{(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}}+\frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}\right\}$$

$$\frac{\partial R_{s,k}}{\partial \lambda_1} = \frac{\alpha\lambda_2^2}{1+\lambda_2}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta\lambda_1}\right)^{l_1+l_3}$$
$$\left[\frac{(l_1+l_3)(\beta-1)}{\lambda_1(1+\beta\lambda_1)}\left\{\frac{(l_1+l_3)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+1}}+\frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}\right\}\right.$$
$$\left.+(i+l_2)\left\{\frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}+\frac{(l_1+l_3+2)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+3}}\right\}\right]$$

$$\frac{\partial R_{s,k}}{\partial \lambda_2} = \alpha\frac{\lambda_2^2+2\lambda_2}{1+\lambda_2}\sum_{i=s}^{k}\sum_{l_1=0}^{i}\sum_{l_2=0}^{k-i}\sum_{l_3=0}^{l_2}\binom{k}{i}\binom{i}{l_1}\binom{k-i}{l_2}\binom{l_2}{l_3}(-1)^{l_2}\left(\frac{\lambda_1}{1+\beta\lambda_1}\right)^{l_1+l_3}$$
$$\left\{\frac{(l_1+l_3+1)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+2}}+\frac{(l_1+l_3+2)!}{\alpha\left[\lambda_1(i+l_2)+\lambda_2\right]^{l_1+l_3+3}}\right\}.$$

The asymptotic variance of $\hat{R}_{s,k}^{ML}$ is

$$AV(\hat{R}_{s,k}^{ML}) = \sum_{i=1}^{4}\sum_{j=1}^{4}\frac{\partial R_{s,k}}{\partial \theta_i}\frac{\partial R_{s,k}}{\partial \theta_j}I^{-1}(\theta) \tag{5.3.6}$$

where $\theta = (\alpha, \beta, \lambda_1, \lambda_2)$ and $I^{-1}(\theta)$ is the Fisher Information Matrix. Therefore, an asymptotic $100(1-\zeta)\%$ confidence interval for $R_{s,k}$ can be obtained as $\hat{R}_{s,k}^{ML} \pm Z_{\frac{\zeta}{2}}\sqrt{AV(\hat{R}_{s,k}^{ML})}$ where $Z_{\frac{\zeta}{2}}$ is the upper $\frac{\zeta}{2}-$ quantile of standard Normal distribution.

5.4 Simulation Study

This section consists a simulation study to compare the performances of the estimators proposed in the previous sections. Here, we studied the behavior of the estimators of parameters, R and $R_{s,k}$ on the basis of simulated sample with varying sample size and various combinations of the parameters. All the results are based on 1000 replications.

For this purpose, we need a simulation algorithm for generating a random sample from TPGL and PL distributions. The simplest method used for this purpose is inverse cdf method that utilizes probability integral transformation. Since the probability integral transformation under TPGLD and PLD cannot be applied explicitly, one can apply either Newton's method to solve the the Lambert W function as suggested by (Jorda (2010)). First, we perform the simulation study when $X \sim TPGL(\alpha, \beta_1, \lambda_1)$ and $Y \sim TPGL(\alpha, \beta_2, \lambda_2)$ distributions (case 1) in section 5.4.1. Second, we perform the simulation study when $X \sim TPGL(\alpha, \beta, \lambda_1)$ and $Y \sim PL(\alpha, \lambda_2)$ distributions (case 2) in section 5.4.2.

5.4.1 MLE of R and $R_{s,k}$ (Case 1)

In this section, we perform simulation study R and $R_{s,k}$ when (s, k) = (1,3) and (2,4) respectively, when $X \sim TPGL(\alpha, \beta_1, \lambda_1)$ and $Y \sim TPGL(\alpha, \beta_2, \lambda_2)$. Now to study the behavior of $\hat{R}^{ML}$ and $\hat{R}^{ML}_{s,k}$ we use the following algorithm.

Algorithm

1. For given values of $(\alpha, \beta_1, \beta_2, \lambda_1, \lambda_2)$ compute R, $R_{1,3}$ and $R_{2,4}$ from (5.2.1) and (5.3.1).

2. Using Newton-Raphson formula to generate 1000 random sample.

3. Compute $\hat{\beta}_1, \hat{\beta}_2, \hat{\lambda}_1, \hat{\lambda}_2$ and $\hat{\alpha}$ from (5.2.5) to (5.2.9).

4. Compute $\hat{R}^{ML}$ and $\hat{R}^{ML}_{s,k}$.

5. Compute the average bias, average MSE and asymptotic 95% confidence interval of R, $R_{1,3}$ and $R_{2,4}$. $\text{Bias}_1 = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML} - R)$, $\text{Bias}_2 = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML}_{s,k} - R_{s,k})$, $\text{MSE}_1 = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML} - R)^2$, $\text{MSE}_2 = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML}_{s,k} - R_{s,k})^2$ when (s,k) = (1,3) and (2,4). Compute asymptotic 95% confidence interval of R and $R_{s,k}$.

We considered two sets of parameter values $(\alpha, \beta_1, \lambda_1)$ and $(\alpha, \beta_2, \lambda_2)$: (3,5.5,1.25) and (3,0.5,2.5), (2.5,4.5,1.75) and (2.5,3,1), (2,2.5,1.5) and (2,0.5,1.5), (1.5,6,1) and (1.5,0.05,2), and different choice of sample sizes (n,m)= (10,10), (15,15), (15,25), (25,25), (25,30), (30,50), (50,50). From each samples, we compute the estimates of $(\alpha, \beta_1, \lambda_1, \beta_2, \lambda_2)$ using ML estimation. Once we estimate $(\alpha, \beta_1, \lambda_1, \beta_2, \lambda_2)$, we obtain the estimates of R by substituting in (5.2.1). Also obtain the estimates of $R_{s,k}$ by substituting in (5.3.1) for (s,k)=(1,3) and (2,4) respectively. These parameter values correspond to the R values 0.492 (moderate), 0.244 (small), 0.662 (high) and 0.827 (high), respectively. When (s,k) = (1,3), corresponding $R_{s,k}$ values are 0.858 (high), 0.540 (moderate), 0.662 (high) and 0.799 (high) respectively. When $(s,k) = $ (2,4), corresponding $R_{s,k}$ values are 0.73 (high), 0.386 (small), 0.489 (small) and 0.631 (high) respectively.

5.4.2 MLE of R and $R_{s,k}$ (Case 2)

In this section, we perform our simulation study of R and $R_{s,k}$ when $(s,k) = $ (1,3) and (2,4) respectively, when $X \sim TPGL(\alpha, \beta, \lambda_1)$ and $Y \sim PL(\alpha, \lambda_2)$. To study the behavior of $\hat{R}^{ML}$ and $\hat{R}^{ML}_{s,k}$ we use the following algorithm.

Algorithm

1. For given values of $(\alpha, \beta, \lambda_1, \lambda_2)$ compute R, $R_{1,3}$ and $R_{2,4}$ from (5.2.4) and (5.3.4).

2. Using Newton-Raphson formula to generate 1000 random sample.

3. Compute $\hat{\beta}, \hat{\lambda_1}, \hat{\lambda_2}$ and $\hat{\alpha}$ from (5.2.12) to (5.2.15)

4. Compute $\hat{R}^{ML}$ and $\hat{R}^{ML}_{s,k}$.

5. Compute the average bias, average MSE and asymptotic 95% confidence interval of R, $R_{1,3}$ and $R_{2,4}$. Where, $\text{Bias}_{1,3} = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML} - R)$, $\text{Bias}_{2,4} = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML}_{s,k} - R_{s,k})$, and $\text{MSE}_{1,3} = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML} - R)^2$, $\text{MSE}_{2,4} = \frac{1}{N}\sum_{i=1}^{N}(\hat{R}^{ML}_{s,k} - R_{s,k})^2$ when $(s,k) = $ (1,3) and (2,4), and asymptotic 95% confidence interval of R and $R_{s,k}$.

We considered two sets of parameter values $(\alpha, \beta, \lambda_1)$ and (α, λ_2): (2.5,5,1.5) and (2.5,0.5), (3,0.5,2) and (3,1.5), (3.5,2,1.25) and (3.5,1.75), (2.75,1,2) and (2.75,2.25), and different choice of sample sizes (n,m)= (10,10), (15,15), (15,25), (25,25), (25,30), (30,50), (50,50). From each samples, we compute the estimates of $(\alpha, \beta, \lambda_1, \lambda_2)$ using ML estimation. Using the estimate of $(\alpha, \beta, \lambda_1, \lambda_2)$, we obtain the estimates of R by substituting in (5.2.4). Also obtain the estimates of $R_{s,k}$ by substituting in (5.3.4) for (s, k) = (1,3) and (2,4) respectively. These parameter values correspond to the R values 0.353 (small), 0.864 (high), 0.454 (moderate) and 0.710 (high), respectively. When (s, k) = (1,3), corresponding $R_{s,k}$ values are 0.257 (small), 0.682 (high), 0.823 (high) and 0.788 (high) respectively. When (s, k) = (2,4), corresponding $R_{s,k}$ values are 0.153 (small), 0.530 (moderate), 0.687 (high) and 0.646 (high) respectively.

In Tables 5.1-5.16, the average biases, MSE and confidence intervals of the estimates of R and $R_{s,k}$ based on MLE method are given.

Table 5.1: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(3, 5.5, 1.25)$ and $Y \sim TPGL(3, 0.5, 2.5)$

(n, m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.6893	0.000197	0.0000388	(0.6459, 0.7327)
(15,15)	0.4669	-0.0000255	0.0000065	(0.4464, 0.4874)
(15,25)	0.6679	0.000176	0.0000308	(0.6371, 0.6987)
(25,25)	0.6455	0.000153	0.0000234	(0.6248, 0.6662)
(25,30)	0.9421	0.000450	0.000202	(0.9259, 0.9583)
(30,50)	0.3931	-0.0000993	0.0000099	(0.3720, 0.4142)
(50,50)	0.7441	0.0000252	0.0000634	(0.7136, 0.7746)

From the simulation results, it is observed that as the sample size (n, m) increases, the biases and the MSEs decreases. That means when the sample size increases, then the estimated reliability reaches nearest to true value. Thus the consistency properties of all the methods are verified.

Table 5.2: MLE of $\hat{R}^{ML}_{s,k}$, Bias and MSE using $X \sim TPGL(3, 5.5, 1.25)$ and $Y \sim TPGL(3, 0.5, 2.5)$

(s,k)	(n,m)	$\hat{R}^{ML}_{s,k}$	Bias	MSE	95% ACI
(1,3)	(10,10)	0.7101	-0.000148	0.0000219	(0.6875, 0.7327)
	(15,15)	0.9426	0.0000845	0.00000714	(0.9135, 0.9717)
	(15,25)	0.9447	0.0000866	0.00000749	(0.9287, 0.9607)
	(25,25)	0.8788	0.0000207	0.00000043	(0.8511, 0.9065)
	(25,30)	0.9563	0.0000982	0.00000964	(0.9315, 0.9811)
	(30,50)	0.8861	0.000028	0.00000078	(0.8499, 0.9223)
	(50,50)	0.8824	0.0000243	0.00000059	(0.8463, 0.9185)
(2,4)	(10,10)	0.5592	-0.000171	0.0000292	(0.5366, 0.5818)
	(15,15)	0.8707	0.000141	0.0000198	(0.8548, 0.8866)
	(15,25)	0.8780	0.000148	0.0000219	(0.8609, 0.8951)
	(25,25)	0.7664	0.0000364	0.00000133	(0.7315, 0.8013)
	(25,30)	0.8914	0.000162	0.0000261	(0.8731, 0.9097)
	(30,50)	0.7730	0.000043	0.00000185	(0.7371, 0.8089)
	(50,50)	0.7707	0.0000407	0.00000165	(0.7586, 0.7828)

Table 5.3: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(2.5, 4.5, 1.75)$ and $Y \sim TPGL(2.5, 3, 1)$

(n,m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.6649	0.000421	0.000178	(0.6431, 0.6867)
(15,15)	0.4039	0.000160	0.0000257	(0.3824, 0.4254)
(15,25)	0.2758	0.0000322	0.00000104	(0.2613, 0.2903)
(25,25)	0.4273	0.000184	0.0000338	(0.4079, 0.4467)
(25,30)	0.4380	0.000915	0.0000378	(0.4188, 0.4572)
(30,50)	0.7933	0.000550	0.000302	(0.7844, 0.8022)
(50,50)	0.2079	-0.0000357	0.00000127	(0.1919, 0.2239)

Table 5.4: MLE of $\hat{R}^{ML}_{s,k}$, Bias and MSE using $X \sim TPGL(2.5, 4.5, 1.75)$ and $Y \sim TPGL(2.5, 3, 1)$

(s,k)	(n,m)	$\hat{R}^{ML}_{s,k}$	Bias	MSE	95% ACI
(1,3)	(10,10)	0.6012	0.0000608	0.0000037	(0.5855, 0.6169)
	(15,15)	0.5200	-0.0000204	0.00000042	(0.5054, 0.5346)
	(15,25)	0.6060	0.0000657	0.00000431	(0.5912, 0.6208)
	(25,25)	0.4636	-0.0000768	0.00000589	(0.4477, 0.4795)
	(25,30)	0.6754	0.000135	0.0000182	(0.6432, 0.7076)
	(30,50)	0.5294	-0.0000110	0.000000121	(0.5066, 0.5522)
	(50,50)	0.5534	0.0000131	0.000000171	(0.5433, 0.5635)
(2,4)	(10,10)	0.4405	0.0000543	0.00000295	(0.4200, 0.4610)
	(15,15)	0.3851	-0.0000011	0.0000000011	(0.3532, 0.4170)
	(15,25)	0.4491	0.0000629	0.00000396	(0.4282, 0.4700)
	(25,25)	0.3212	-0.0000650	0.00000422	(0.3076, 0.3348)
	(25,30)	0.5194	0.000133	0.0000178	(0.4947, 0.5441)
	(30,50)	0.3566	-0.0000296	0.000000879	(0.3373, 0.3759)
	(50,50)	0.4000	0.0000138	0.000000191	(0.3839, 0.4161)

Table 5.5: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(2, 2.5, 1.5)$ and $Y \sim TPGL(2, 0.5, 1.5)$

(n,m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.6598	-0.0000019	0.0000000036	(0.6347, 0.6849)
(15,15)	0.6370	-0.0000247	0.00000061	(0.6199, 0.6541)
(15,25)	0.4395	-0.000222	0.0000494	(0.4109, 0.4681)
(25,25)	0.6120	-0.0000497	0.00000247	(0.6004, 0.6236)
(25,30)	0.7054	0.0000437	0.00000191	(0.6893, 0.7215)
(30,50)	0.5911	-0.0000706	0.00000498	(0.5799, 0.6023)
(50,50)	0.7032	0.0000415	0.00000172	(0.6871, 0.7193)

5.5 Data Analysis

In this section, we consider two real data sets of the breaking strengths of jute fiber at two different gauge lengths (see Xia et al. (2009)). Two sets of real data

Table 5.6: MLE of $\hat{R}^{ML}_{s,k}$, Bias and MSE using $X \sim TPGL(2, 2.5, 1.5)$ and $Y \sim TPGL(2, 0.5, 1.5)$

(s,k)	(n,m)	$\hat{R}^{ML}_{s,k}$	Bias	MSE	95% ACI
(1,3)	(10,10)	0.4161	-0.000246	0.0000603	(0.3896, 0.4426)
	(15,15)	0.5462	-0.000115	0.0000133	(0.5202, 0.5722)
	(15,25)	0.6214	-0.0000402	0.00000162	(0.6098, 0.6330)
	(25,25)	0.4636	-0.0000768	0.00000589	(0.4358, 0.4914)
	(25,30)	0.5353	-0.000126	0.0000160	(0.5180, 0.5526)
	(30,50)	0.6482	-0.0000135	0.000000182	(0.6205, 0.6759)
	(50,50)	0.7806	0.000119	0.0000142	(0.7754, 0.7858)
(2,4)	(10,10)	0.3018	-0.000188	0.0000352	(0.2928, 0.3108)
	(15,15)	0.3706	-0.000119	0.0000141	(0.3662, 0.3750)
	(15,25)	0.4569	-0.0000324	0.00000105	(0.4129, 0.5009)
	(25,25)	0.4839	-0.0000054	0.000000029	(0.4665, 0.5013)
	(25,30)	0.3693	-0.000120	0.0000144	(0.3544, 0.3842)
	(30,50)	0.5052	0.000016	0.00000025	(0.4907, 0.5197)
	(50,50)	0.6299	0.000141	0.0000198	(0.6069, 0.6529)

Table 5.7: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(1.5, 6, 1)$ and $Y \sim TPGL(1.5, 0.05, 2)$

(n,m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.3659	-0.000462	0.000213	(0.3422, 0.3896)
(15,15)	0.4616	-0.000366	0.000134	(0.4404, 0.4828)
(15,25)	0.5579	-0.000269	0.0000726	(0.5484, 0.5674)
(25,25)	0.8876	0.0000603	0.00000363	(0.8610, 0.9146)
(25,30)	0.7866	-0.0000407	0.00000166	(0.7780, 0.7952)
(30,50)	0.6834	-0.000144	0.0000206	(0.6726, 0.6942)
(50,50)	0.9782	0.0001508	0.0000228	(0.9632, 0.9932)

are shown as follows:

Data set I: Breaking strength of jute fiber length 10 mm (variable X). 693.73, 704.66, 323.83, 778.17, 123.06, 637.66, 383.43, 151.48, 108.94, 50.16,

Table 5.8: MLE of $\hat{R}^{ML}_{s,k}$, Bias and MSE using $X \sim TPGL(1.5, 6, 1)$ and $Y \sim TPGL(1.5, 0.05, 2)$

(s,k)	(n,m)	$\hat{R}^{ML}_{s,k}$	Bias	MSE	95% ACI
(1,3)	(10,10)	0.6560	-0.000143	0.0000206	(0.6405, 0.6715)
	(15,15)	0.8787	0.0000793	0.0000063	(0.8544, 0.9030)
	(15,25)	0.8064	0.0000070	0.00000079	(0.7975, 0.8153)
	(25,25)	0.7580	-0.0000414	0.00000171	(0.7479, 0.7681)
	(25,30)	0.9255	0.000126	0.0000159	(0.9190, 0.9320)
	(30,50)	0.8522	0.0000528	0.00000278	(0.8387, 0.8657)
	(50,50)	0.8405	0.0000411	0.00000169	(0.8293, 0.8517)
(2,4)	(10,10)	0.4984	-0.000133	0.0000176	(0.4844, 0.5124)
	(15,15)	0.7533	0.000122	0.0000149	(0.7324, 0.7742)
	(15,25)	0.6674	0.0000363	0.00000132	(0.6331, 0.7017)
	(25,25)	0.5816	-0.0000496	0.00000256	(0.5618, 0.6014)
	(25,30)	0.8290	0.000198	0.0000391	(0.8131, 0.8449)
	(30,50)	0.7182	0.0000870	0.00000757	(0.7005, 0.7359)
	(50,50)	0.7021	0.0000709	0.00000503	(0.6902, 0.7140)

Table 5.9: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(2.5, 5, 1.5)$ and $Y \sim PL(2.5, 0.5)$

(n,m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.1381	-0.000215	0.0000462	(0.0119, 0.2643)
(15,15)	0.2229	-0.00013	0.0000169	(0.1768, 0.2690)
(15,25)	0.3503	-0.0000026	0.0000000068	(0.1614, 0.5392)
(25,25)	0.3240	-0.0000289	0.00000084	(0.1139, 0.5342)
(25,30)	0.4003	0.0000474	0.00000225	(0.0956, 0.7050)
(30,50)	0.3607	0.00000772	0.000000060	(0.2117, 0.5097)
(50,50)	0.3663	0.0000134	0.000000179	(0.0601, 0.6726)

671.49, 183.16, 257.44, 727.23, 291.27, 101.15, 376.42, 163.40, 141.38, 700.74, 262.90, 353.24, 422.11, 43.93, 590.48, 212.13, 303.90, 506.60, 530.55, 177.25.

Table 5.10: MLE of $\hat{R}^{ML}_{s,k}$, Bias and MSE using $X \sim TPGL(2.5, 5, 1.5)$ and $Y \sim PL(2.5, 0.5)$

(s,k)	(n,m)	$\hat{R}^{ML}_{s,k}$	Bias	MSE	95% ACI
(1,3)	(10,10)	0.1339	-0.000123	0.0000151	(0.1187, 0.1491)
	(15,15)	0.1509	-0.000106	0.0000112	(0.1268, 0.1750)
	(15,25)	0.2363	-0.0000206	0.000000422	(0.2014, 0.2712)
	(25,25)	0.2274	-0.0000294	0.000000863	(0.1860, 0.2688)
	(25,30)	0.3359	0.0000791	0.00000626	(0.2917, 0.3801)
	(30,50)	0.3061	0.0000493	0.00000243	(0.2445, 0.3677)
	(50,50)	0.3110	0.0000632	0.00000399	(0.2780, 0.3440)
(2,4)	(10,10)	0.0747	-0.0000785	0.00000617	(0.0015, 0.1479)
	(15,15)	0.0884	-0.0000648	0.0000042	(0.0361, 0.1407)
	(15,25)	0.1418	-0.0000114	0.00000013	(0.0463, 0.2327)
	(25,25)	0.1367	-0.0000165	0.000000272	(0.1110, 0.1624)
	(25,30)	0.2238	0.0000706	0.00000498	(0.2037, 0.2439)
	(30,50)	0.1949	0.0000417	0.00000174	(0.1589, 0.2309)
	(50,50)	0.2152	0.0000620	0.00000384	(0.1717, 0.2587)

Table 5.11: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(3, 0.5, 2)$ and $Y \sim PL(3, 1.5)$

(n,m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.7027	-0.000162	0.0000261	(0.6816, 0.7238)
(15,15)	0.6747	-0.000190	0.0000359	(0.6259, 0.7235)
(15,25)	0.5274	-0.000337	0.000113	(0.5078, 0.5470)
(25,25)	0.5548	-0.000309	0.0000957	(0.5271, 0.5825)
(25,30)	0.6049	-0.000259	0.0000672	(0.5886, 0.6212)
(30,50)	0.7467	-0.000117	0.0000138	(0.7061, 0.7873)
(50,50)	0.9789	0.000115	0.0000132	(0.9624, 0.9954)

Data set II: Breaking strength of jute fiber length 20 mm (variable Y). 71.46, 419.02, 284.64, 585.57, 456.60, 113.85, 187.85, 688.16, 662.66, 45.58, 578.62, 756.70, 594.29, 166.49, 99.72, 707.36, 765.14, 187.13, 145.96, 350.70,

Table 5.12: MLE of $\hat{R}^{ML}_{s,k}$, Bias and MSE using $X \sim TPGL(3, 0.5, 2)$ and $Y \sim PL(3, 1.5)$

(s,k)	(n,m)	$\hat{R}^{ML}_{s,k}$	Bias	MSE	95% ACI
(1,3)	(10,10)	0.6085	-0.0000733	0.00000538	(0.5736, 0.6434)
	(15,15)	0.7038	0.0000221	0.000000486	(0.6903, 0.7173)
	(15,25)	0.6388	-0.000043	0.00000185	(0.6128, 0.6648)
	(25,25)	0.7109	0.0000292	0.00000085	(0.6849, 0.7369)
	(25,30)	0.6632	-0.0000185	0.000000344	(0.6313, 0.6951)
	(30,50)	0.6898	0.00000801	0.000000064	(0.6505, 0.7291)
	(50,50)	0.6575	-0.0000242	0.000000587	(0.6372, 0.6778)
(2,4)	(10,10)	0.4486	-0.0000815	0.00000664	(0.4006, 0.4966)
	(15,15)	0.5475	0.0000174	0.000000304	(0.5134, 0.5816)
	(15,25)	0.4765	-0.0000536	0.00000287	(0.4370, 0.5160)
	(25,25)	0.5522	0.0000221	0.000000486	(0.5340, 0.5704)
	(25,30)	0.5011	-0.0000290	0.000000843	(0.4810, 0.5212)
	(30,50)	0.5375	0.00000744	0.000000055	(0.5077, 0.5673)
	(50,50)	0.5085	-0.0000216	0.000000466	(0.4774, 0.5396)

Table 5.13: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(3.5, 2, 1.25)$ and $Y \sim PL(3.5, 1.75)$

(n,m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.4557	0.0000018	0.0000000031	(0.4194, 0.4920)
(15, 15)	0.3193	-0.000135	0.0000181	(0.0512, 0.5875)
(15,25)	0.4487	-0.00000527	0.000000028	(0.4160, 0.4814)
(25,25)	0.3450	-0.000109	0.0000119	(0.3287, 0.3613)
(25,30)	0.4061	-0.0000478	0.00000229	(0.3851, 0.4271)
(30,50)	0.5237	0.0000698	0.00000487	(0.5186, 0.5288)
(50,50)	0.4074	-0.0000466	0.00000217	(0.0337, 0.7810)

547.44, 116.99, 375.81, 581.60, 119.86, 48.01, 200.16, 36.75, 244.53, 83.55.

These data were first used by Xia et al. (2009) and later by Saracoglu et al.

Table 5.14: MLE of $\hat{R}^{ML}_{s,k}$, Bias and MSE using $X \sim TPGL(3.5, 2, 1.25)$ and $Y \sim PL(3.5, 1.75)$

(s,k)	(n,m)	$\hat{R}^{ML}_{s,k}$	Bias	MSE	95% ACI
(1,3)	(10,10)	0.9414	0.000118	0.0000140	(0.9183, 0.9645)
	(15, 15)	0.8443	0.0000212	0.000000448	(0.8050, 0.8836)
	(15, 25)	0.8390	0.0000159	0.000000252	(0.7921, 0.8859)
	(25, 25)	0.8435	0.0000204	0.00000042	(0.8019, 0.8851)
	(25,30)	0.9196	0.0000965	0.00000932	(0.8867, 0.9525)
	(30,50)	0.7758	-0.0000473	0.00000224	(0.7120, 0.8396)
	(50,50)	0.7610	-0.0000621	0.00000385	(0.7289, 0.7931)
(2,4)	(10, 10)	0.8723	0.000185	0.0000343	(0.8376, 0.9070)
	(15,15)	0.7118	0.0000248	0.00000061	(0.6949, 0.7287)
	(15,25)	0.7091	0.0000221	0.000000488	(0.6899, 0.7283)
	(25,25)	0.7118	0.0000247	0.00000061	(0.7003, 0.7233)
	(25,30)	0.8327	0.000146	0.0000212	(0.8131, 0.8523)
	(30,50)	0.6278	-0.0000592	0.00000351	(0.5919, 0.6637)
	(50,50)	0.6059	-0.0000811	0.00000659	(0.5855, 0.6263)

Table 5.15: MLE of $\hat{R}^{ML}$, Bias and MSE using $X \sim TPGL(2.75, 1, 2)$ and $Y \sim PL(2.75, 2.25)$

(n,m)	$\hat{R}^{ML}$	Bias	MSE	95% ACI
(10,10)	0.4675	-0.000242	0.0000587	(0.2107, 0.7243)
(15, 15)	0.6207	-0.0000890	0.00000793	(0.4591, 0.7823)
(15, 25)	0.7498	0.0000401	0.00000161	(0.7164, 0.7832)
(25, 25)	0.7738	0.0000641	0.00000411	(0.5773, 0.9703)
(25, 30)	0.5643	-0.000145	0.0000212	(0.4451, 0.6835)
(30, 50)	0.8633	0.000154	0.0000236	(0.8399, 0.8867)
(50, 50)	0.7907	0.0000810	0.00000655	(0.7292, 0.8522)

(2012). Shahsanaei and Daneshkhah (2013) used the data to study the estimation of stress-strength parameter for generalized linear failure rate (GLFR) distribution under progressive type-II censoring and studied the validity of GLFR for both data

Table 5.16: MLE of $\hat{R}^{ML}_{s,k}$, Bias and MSE using $X \sim TPGL(2.75, 1, 2)$ and $Y \sim PL(2.75, 2.25)$

(s,k)	(n,m)	$\hat{R}^{ML}_{s,k}$	Bias	MSE	95% ACI
(1,3)	(10, 10)	0.9482	0.000160	0.0000255	(0.9136, 0.9828)
	(15,15)	0.8412	0.0000528	0.00000279	(0.7914, 0.8910)
	(15,25)	0.8527	0.0000643	0.00000413	(0.8170, 0.8884)
	(25,25)	0.7268	-0.0000616	0.00000379	(0.7095, 0.7441)
	(25,30)	0.7535	-0.0000349	0.00000122	(0.7165, 0.7905)
	(30,50)	0.7748	-0.0000136	0.000000185	(0.7291, 0.8205)
	(50,50)	0.7402	-0.0000482	0.00000233	(0.7008, 0.7796)
(2,4)	(10,10)	0.8830	0.000237	0.0000560	(0.8079, 0.9581)
	(15,15)	0.7142	0.0000679	0.00000461	(0.6931, 0.7353)
	(15,25)	0.7349	0.0000886	0.00000785	(0.6990, 0.7708)
	(25,25)	0.5745	-0.0000718	0.00000516	(0.5246, 0.6244)
	(25,30)	0.6007	-0.0000456	0.00000208	(0.5895,0.6119)
	(30,50)	0.6312	-0.0000151	0.000000229	(0.6018, 0.6606)
	(50,50)	0.5897	-0.0000567	0.00000321	(0.5425, 0.6369)

sets.

In Table 5.17 we provided the MLEs of the parameters of TPGL and PL, i.e., α, β, λ as well as the results of K-S and A-D goodness of fit tests.

The unknown parameters of case 1 are $\hat{\alpha}$=0.928, $\hat{\beta}_1$=1.492, $\hat{\beta}_2$=3.495, $\hat{\lambda}_1$=0.0085 and $\hat{\lambda}_2$=0.00895. The MLE of R becomes $\hat{R}$=0.2388 and the 95% interval of R is (0.2110, 0.2576). The MLEs and 95% confidence interval of $R_{s,k}$ are provided in Table 5.18.

The unknown parameters of case 2 are $\hat{\alpha}$=0.9232, $\hat{\beta}$=2.112, $\hat{\lambda}_1$=0.00878 and $\hat{\lambda}_2$=0.00933. The MLE of R becomes $\hat{R}$=0.0117 and the 95% interval of R is (0.0078, 0.0156). The MLEs and 95% confidence interval of $R_{s,k}$ are provided in

Table 5.18.

Table 5.17: MLEs and K-S and A-D tests

Plane	Model	MLEs	K-S	p-value	A-D	p-value
length 10 mm (X)	**TPGL**	$\hat{\alpha} = 0.954$ $\hat{\beta} = 0.9845$ $\hat{\lambda} = 0.0073$	0.1005	0.8928	0.4573	0.7893
	PL	$\hat{\alpha} = 0.954$ $\hat{\lambda} = 0.0073$	0.1005	0.8931	0.4573	0.7893
length 20 mm (Y)	**TPGL**	$\hat{\alpha} = 0.889$ $\hat{\beta} = 0.985$ $\hat{\lambda} = 0.0115$	0.1523	0.4456	0.7577	0.5114
	PL	$\hat{\alpha} = 0.889$ $\hat{\lambda} = 0.011$	0.1523	0.4457	0.7578	0.5112

Table 5.18: Estimates of $R_{s,k}$

	Case 1		Case 2	
(s, k)	$\hat{R}_{s,k}^{ML}$	95% ACI	$\hat{R}_{s,k}^{ML}$	95% ACI
(1, 3)	0.7705	(0.7103, 0.8307)	0.7712	(0.7042, 0.8382)
(2, 4)	0.6257	(0.6067, 0.6497)	0.6253	(0.5957, 0.6549)
(3, 5)	0.5268	(0.4758, 0.5778)	0.5253	(0.4188, 0.6318)

5.6 Summary

We estimated $R = P(Y < X)$ in two cases. First, when Y and X both follow TPGL distribution. Second, when Y and X follows PL distribution and TPGL distribution, respectively. We provided MLE to estimate the unknown parameters and used this to estimate of $R$and $R_{s,k}$. Also obtained asymptotic $100(1-\nu)\%$ CI for the reliability parameter. Also obtain asymptotic CI for the reliability parameter. The simulation results indicate that, when increasing the sample sizes, MSE caused by the estimates are nearer to zero. The MLE of R is 0.2388 in case 1, which means there is a small chance that strength is greater than stress. Then the SSS reliability in case 2 is comparatively low than in case 1.

The MLE of MSS reliability is 0.771 in both cases for (1,3) component system, which means there is a high chance that strength is greater than stress. The MLE of MSS reliability is 0.527 in case 1 and 0.525 in case 2 for (3,5) component system, which means there is an equal chance that strength is greater than stress.