Deepthi K.S. “Modelling and analysis of reliability data using bathtub shaped failure rate distributions.” Thesis. Research & Post Graduate Department of Statistics, St. Thomas’ College (Autonomous), University of Calicut, 2020.

CHAPTER 6

TOTAL TIME ON TEST TRANSFORM AND ORDERING OF LIFE DISTRIBUTIONS

6.1 Introduction

[1] The concept of total time on test (TTT) transform was studied in the early 1970s (see Barlow and Campo (1975)). When several units are tested for studying their life lengths, some of the units would fail while others may survive the test duration. The sum of all observed and incomplete life lengths is generally visualized as the TTT statistic. When the number of items placed on test tends to infinity, the limit of this statistic is called the TTT transform. The plots provided information about the identification of failure rate model of the lifetime r.v. Incomplete censored data can be analyzed using TTT transform.

[1] Some contents of this chapter are based on Deepthi and Chacko (2021).

Many research papers on TTT concentrate on its engineering applications. Aarset (1985) derived the exact distribution of TTT transform under the null hypothesis of exponentiality. Gupta and Michalek (1985) developed an explicit method to determine the reliability function by using the TTT transform. Abouammoh and Khalique (1997) investigated the properties of scaled TTT for some test statistics for testing exponentially against all these mean residual life criteria. Bergman (1977) studied the exact and asymptotic distributions of the number of crossings are given under the hypothesis of exponentiality.

Recently, Vera and Lynch (2005) introduced higher-order TTT transforms by applying definition of TTT recursively to the transformed distributions. Nair et al. (2008) studied the properties of TTT transform of order n and examined their applications in reliability analysis. Nair and Sankaran (2013) listed some known characterizations of common aging notions in terms of the TTT transform function. Franco-Pereira and Shaked (2013) derived two characterizations of the decreasing percentile residual life ($DPRL(\alpha)$) of order and aging notion in terms of the TTT function.

TTT transform provide the central value of censored data. In order to get the dispersion values in the censored situation, we need the distributions of the increasing convex (concave) functions of lifetime random variables. The problem of fitting an appropriate distribution for the function of r.v can be addressed through the identification of failure rate model. The problem of identification of failure rate behavior of increasing convex (concave) function of r.v based on distributional properties of the lifetime variable is also an unexplored one. So, we consider TTT transform of increasing convex (concave) function of r.v and study its properties. The behavior of TTT transform of increasing convex (concave) function with the behavior of failure rate function of the lifetime r.v need to be undergo more investigation.

In this chapter, we considered increasing convex (concave) total time on test (ICXTTT (ICVTTT)) transform of a lifetime r.v and its properties. In section 6.2, ordering of life distribution is discussed. In section 6.3, the concept of the TTT processes is discussed. In section 6.4, we defined increasing convex (concave) TTT (ICXTTT (ICVTTT)) transform of the random variable. Some results about the ageing patterns are given in section 6.5. In section 6.6, we defined ICXTTT (ICVTTT) transform order and obtained its relationship with stochastic ordering. Illustrative examples are given in section 6.7.

6.2 Ordering of life distributions

By the ageing of a mathematical unit, component or some other physical or biological system, we mean the phenomenon by which an older system has a shorter remaining lifetime, in some stochastic sense, than a newer or younger one. Many criteria of ageing have been developed in the literature. The stochastic comparison of distributions has been an important area of research in many diverse areas of statistics and probability. We are comparing two lifetime variables X and Y in terms of their failure rates $r_F(t)$ and $r_G(t)$, density functions $f(t)$ and $g(t)$, survival functions $\bar{F}(t)$ and $\bar{G}(t)$, mean residual lives $\mu_F(t)$ and $\mu_G(t)$ or other ageing characteristics. Ageing classes can often be characterized by some partial ordering. For example, in Barlow and Proschan (1975), IFR and IFRA classes are characterized by convex ordering and star-shaped ordering respectively. Many different types of stochastic orders have been studied in the literature; for example Deshpande et al. (1986) and a comprehensive discussion of ordering is available in Shaked and Shanthikumar (2007). It is often easy to make value judgements when such ordering exist. Stochastic ordering between two probability distributions, if it holds, is more informative than simply comparing their means or medians only. Similarly, if one wishes to compare the dispersion or spread between two distributions, the simplest way would to be to compare their standard deviations or some such other measures of dispersion.

6.2.1 Stochastic order

The r.v X is stochastically larger than the random variable Y, written $X \geq_{\text{st}} Y$, if

$$P(X > a) \geq P(Y > a), \qquad \forall\ a. \tag{6.2.1}$$

If X and Y have distributions F and G respectively, then (6.2.1) is equivalent to

$$\bar{F}(a) \geq \bar{F}(a), \qquad \forall\ a$$

denoted by $X \leq_{\text{st}} Y$.

6.2.2 Hazard rate order

Let X and Y be two nonnegative r.v's with absolutely continuous distribution functions and with failure rate functions r and q, respectively, such that

$$r(t) \geq q(t), \quad t \in \mathbb{R}. \tag{6.2.2}$$

Then X is said to be smaller than Y in the hazard rate order (denoted as $X \leq_{\text{hr}} Y$).

The hazard rate order can be trivially (but beneficially) used to characterize IFR random variables.

6.2.3 Convex (Concave) order

If two variables have the same mean, they can still be compared by how spread out the distributions are. This is captured to limit extend by the variance, but more fully by a range of stochastic orders. Convex order is a special kind of variability order.

Definition 6.2.1. *Let X and Y be two random variables such that*

$$E[\phi(X)] \leq E[\phi(Y)] \text{ for all increasing convex [concave] functions } \phi : \mathbb{R} \to \mathbb{R}, \tag{6.2.3}$$

provided the expectations exist. Then X is said to be smaller than Y in the increasing convex [concave] order (denoted by $X \leq_{icx} Y[X \leq_{icv} Y]$).

Roughly speaking, if $X \leq_{\text{icx}} Y$ then X is both smaller and less variable than Y in some stochastic sense. Similarly, $X \leq_{\text{icv}} Y$ then X is both smaller and more variable than Y in some stochastic sense. Decreasing convex (concave) order by requiring (6.2.1) to hold for all decreasing convex (concave) functions ϕ (denoted as $X \leq_{\text{dcx}} Y[X \leq_{\text{dcv}} Y]$). The term decreasing convex and decreasing concave are counter intuitive in the sense that if X is smaller than Y in the sense of either of these two orders then X is larger than Y in some stochastic sense.

6.3 Total Time on Test Transform

The concept of the TTT transform processes was first defined by Barlow and Campo (1975). Given a sample of size n from the non-negative r.v X having distribution F, let $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(k)} \leq \ldots \leq X_{(n)}$ be the order statistics corresponding to the sample. Total time test to the r^{th} failure is,

$$\begin{aligned} T(X_{(r)}) &= nX_{(1)} + (n-1)(X_{(2)} - X_{(1)}) + \ldots + (n-r+1)(X_{(r)} - X_{(r-1)}) \\ &= \sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}. \end{aligned}$$

Let $H_n^{-1}(\frac{r}{n}) = \frac{1}{n}T(X_{(r)})$

$$\text{i.e., } H_n^{-1}(\frac{r}{n}) = \int_0^{F_n^{-1}(\frac{r}{n})} (1 - F_n(u))du.$$

The empirical distribution function defined in terms of the order statistics is

$$F_n(u) = \begin{cases} 0, & u < X_{(i)} \\ \frac{i}{n}, & X_{(i)} \leq u < X_{(i+1)} \\ 1, & X_{(n)} > u. \end{cases}$$

If there exist an inverse function $F_n^{-1}(x) = \inf\{x : F_n(x) \geq u\}$, the fact that $F_n(u) \xrightarrow{a.s.} F(u)$ implies, by Glivenko Candelli theorem,

$$\lim_{\substack{\frac{r}{n}\to t \\ n\to\infty}} \int_0^{F_n^{-1}(\frac{r}{n})} (1 - F_n(u))\, du = \int_0^{F^{-1}(t)} (1 - F(u))\, du \tag{6.3.1}$$

uniformly in $t \in [0, 1]$. Barlow and Campo (1975) defined TTT transform of F as

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} (1 - F(u))\, du \quad t \in [0, 1]. \tag{6.3.2}$$

There is a one to one correspondence between distribution F and their transform H_F^{-1}. Suppose F has density f, then

$$\begin{aligned} \frac{d}{dt} H_F^{-1}(t) &= \frac{d}{dt} \int_0^{F^{-1}(t)} (1 - F(u))\, du \\ &= (1 - t) \frac{d}{dt} F^{-1}(t). \end{aligned}$$

So that

$$\frac{d}{dt} F^{-1}(t) = \frac{\frac{d}{dt} H_F^{-1}(t)}{(1 - t)}.$$

Note that H_F is a distribution with support on $[0, \mu]$, where μ is the mean of F, since

$$\begin{aligned} H_F^{-1}(1) &= \int_0^{F^{-1}(1)} (1 - F(u))\, du \\ &= \mu, \qquad \text{when } \bar{F}(0) = 0. \end{aligned}$$

It is easy to verify that the scaled TTT transform is

$$\phi(t) = \frac{H_F^{-1}(t)}{H_F^{-1}(1)} = \frac{H_F^{-1}(t)}{\mu}$$

is continuous increasing function on $[0, 1]$ which is 0 at $t = 0$ and 1 at $t = 1$.

The curve $\phi(t)$ versus $0 \leq t \leq 1$ is called the scaled TTT transform curve. Using the scaled TTT transform curve, the shape of the failure rate function of the distribution can be classified as one of the following.

- If the scaled TTT transform curve is concave above the 45^o line, the failure rate is increasing.
- If the scaled TTT transform curve is convex below the 45^o line, then the failure rate is decreasing.
- If the scaled TTT transform curve is first convex below the 45^o line then concave above the line the shape of the failure rate is a bathtub shaped.
- The shape of the failure rate will be unimodal shaped if the scaled TTT transform curve is first concave above the 45^o line followed by convex below the 45^o line.

Figure 6.1 summarizes the different shapes of the scaled TTT transform curve for distributions with increasing, decreasing, bathtub and unimodal failure rate functions. For an ordered sample $x_{0:n}, x_{1:n}, x_{2:n}, \ldots, x_{n:n}$, the total time one test

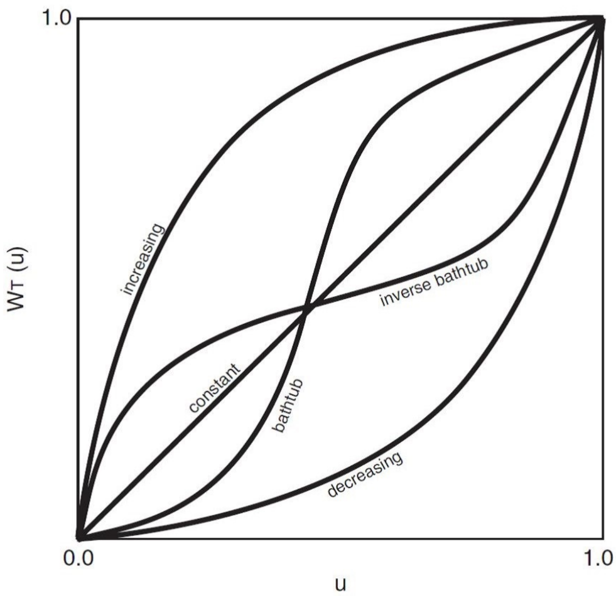

Figure 6.1: Theoretical aspects of TTT plots.

statistics is given by

$$TTT_r = \sum_{i=1}^{r} (n-i+1)(x_{i:n} - x_{i-1:n}), \quad r = 1, 2, \ldots, n.$$

The empirical scaled TTT transform is

$$TTT_r^* = \frac{TTT_r}{TTT_n},$$

where $0 \leq TTT_n \leq 1$. The TTT-plot can be drawn by plotting $(\frac{r}{n})$ against TTT_r^*.

6.4 Increasing Convex (Concave) TTT transform

Let $g(x)$ be an increasing convex (concave) function of X. Let $G(x)$ be the distribution function of $g(X)$. Total observed values of transformed variables $g(X)$ under type 2 censored scheme is

$$\begin{aligned} Tg(X_{(r)}) &= n\ g(X_{(1)}) + \ldots + (n-r+1)\big(g(X_{(r)}) - g(X_{(r-1)})\big) \\ &= \sum_{i=1}^{r} g(X_{(i)}) + (n-r)g(X_{(r)}). \end{aligned}$$

For $g(x)$, define

$$(H_n^{-1})g\left(\frac{r}{n}\right) = \int_0^{(H_n^{-1}(\frac{r}{n}))} \big(1 - H_n(w)\big)dw = \int_0^{g(F_n^{-1}(\frac{r}{n}))} \big(1 - F_n(u)\big)du$$

where

$$H_n(u) = \begin{cases} 0, & g(u) < g(X_{(i)}) \\ \frac{i}{n}, & g(X_{(i)}) \le g(u) < g(X_{(i+1)}) \\ 1, & g(X_{(n)}) > g(u). \end{cases}$$

$H_n^{-1}(x) = \inf\{x : H_n(x) \ge g(u)\}$ and the fact that $F_n(u) \xrightarrow{a.s.} F(u)$ implies, $g(F_n^{-1}(u)) \xrightarrow{a.s.} g(F^{-1}(u))$, then by Glivenko Candelli theorem,

$$\lim_{\substack{\frac{r}{n}\to t \\ n\to\infty}} \int_0^{g(F_n^{-1}(\frac{r}{n}))} (1 - F_n(u))\ du = \int_0^{g(F^{-1}(t))} (1 - F(u))\ du.$$

We define TTT transform of increasing convex (concave) function $g(X)$ as

$$(H_F^{-1})g(t) = \int_0^{g(F^{-1}(t))} (1 - F(u))\ du \quad t \in [0, 1]. \tag{6.4.1}$$

But, $\frac{\frac{d}{dt} H_F^{-1}(t)}{1-t} = \frac{d}{dt} F^{-1}(t)$ and $\frac{d}{dt} H_F^{-1}(t)|_{t=F(x)} = \frac{1}{r(x)}$. Then

$$\begin{aligned}
\frac{d}{dt}(H_F^{-1})g(t) &= \frac{d}{dt} \int_0^{g(F^{-1}(t))} (1 - F(u))\ du \\
&= \left[1 - \int_0^{g(F^{-1}(t))} f(u)\ du\right] g'(F^{-1}(t)) \frac{d}{dt} F^{-1}(t) \\
&= \left[1 - \int_0^{g(F^{-1}(t))} f(u)\ du\right] g'(F^{-1}(t)) \frac{\frac{d}{dt} H_F^{-1}(t)}{1-t} \\
\frac{d}{dt}(H_F^{-1})g(t)|_{t=F(x)} &= \left[1 - \int_0^{g(x)} f(u)\ du\right] \frac{g'(x)}{\bar{F}(x) r(x)}.
\end{aligned}$$

That is,

$$\frac{d}{dt}(H_F^{-1})g(t)|_{t=F(x)} = \frac{\bar{F}(g(x))}{\bar{F}(x)} . \frac{g'(x)}{r(x)}. \tag{6.4.2}$$

Note that $H_F g(.)$ (the inverse of $H_F^{-1} g(.)$) is a distribution with support on $[0, \mu]$,

$$(H_F^{-1})g(1) = \int_0^{g(F^{-1}(1))} (1 - F(u))\ du = \mu.$$

It is easy to verify that the scaled transform $\frac{(H_F^{-1})g(t)}{(H_F^{-1})g(1)}$ is continuous increasing function on $[0, 1]$.

Example 6.4.1. Let $g(x) = x^2$, then

$$(H_F^{-1})^2(t) = \int_0^{(F^{-1}(t))^2} (1 - F(u))\ du \quad t \in [0,1]. \tag{6.4.3}$$

$$\text{Then,}\ \frac{d}{dt}(H_F^{-1})^2(t) = \frac{d}{dt}\int_0^{(F^{-1}(t))^2} (1 - F(u))\ du$$

$$= \left[1 - \int_0^{(F^{-1}(t))^2} f(u)\ du\right] 2\ F^{-1}(t)\ \frac{\frac{d}{dt}H_F^{-1}(t)}{1-t}$$

$$\frac{d}{dt}(H_F^{-1})^2(t)|_{t=F(x)} = \left[1 - \int_0^{x^2} f(u)\ du\right] \frac{2x}{\bar{F}(x)r(x)}.$$

That is,

$$\frac{d}{dt}(H_F^{-1})^2(t)|_{t=F(x)} = \frac{\bar{F}(x^2)}{\bar{F}(x)}.\frac{2x}{r(x)}.$$

Note that (H_F^2) (the inverse of $(H_F^{-1})^2$) is a distribution with support on $[0,\mu]$.

$$(H_F^{-1})^2(1) = \int_0^{(F^{-1}(1))^2} (1 - F(u))\ du = \mu.$$

It is easy to verify that the scaled TTT transform is $\frac{(H_F^{-1})^2(t)}{(H_F^{-1})^2(1)}$ is continuous increasing function on $[0,1]$.

Example 6.4.2. Let $F(x) = 1 - e^{-x/\theta},\ x > 0,\ \theta > 0$ be the distribution function of Exponential distribution with mean θ. Then

$$(H_F^{-1})^2(t) = \int_0^{(F^{-1}(t))^2} (1 - F(x))\ dx$$

$$
\begin{aligned}
&= \int_0^{(F^{-1}(t))^2} e^{-x/\theta}\ dx \\
&= \int_0^{(F^{-1}(t))^2} \theta\ dF(x) \\
(H_F^{-1})^2(t)|_{t=F(x)} &= \theta F((F^{-1}(t))^2) \\
\therefore \frac{(H_F^{-1})^2(t)}{(H_F^{-1})^2(1)} &= F((F^{-1}(t))^2).
\end{aligned}
$$

Now, we consider simulated data and plot TTT transform of Exponential distribution and its convex transform $g(x) = x^2$. From Figure 6.2, the TTT transform plot of Exponential data set indicates constant failure rate, but ICXTTT transform plot indicates that the transformed data follows the decreasing failure rate pattern.

So that, square of Exponential r.v follows some decreasing failure rate model. Thus we can choose any DFR model to square of Exponential data.

6.5 Ageing Properties

We prove some general results about the ageing patterns of function $g(X)$ using $(H_F^{-1})g(t)/(H_F^{-1})g(1)$, which is based on the failure rate function $r(x)$ of X having distribution F.

Proposition 6.5.1. G is IFR if rate of increase of $g'(x)\frac{\bar{F}(g(x))}{\bar{F}(x)}$ is smaller than the rate of increase of $r(x)$. G is DFR if $r(x)$ is decreasing in $x \geq 0$.

Proof. Clearly $\frac{d}{dt}(H_F^{-1})g(t)$ is decreasing in $t \in [0, 1]$, if rate of increase of $g'(x)\frac{\bar{F}(g(x))}{\bar{F}(x)}$

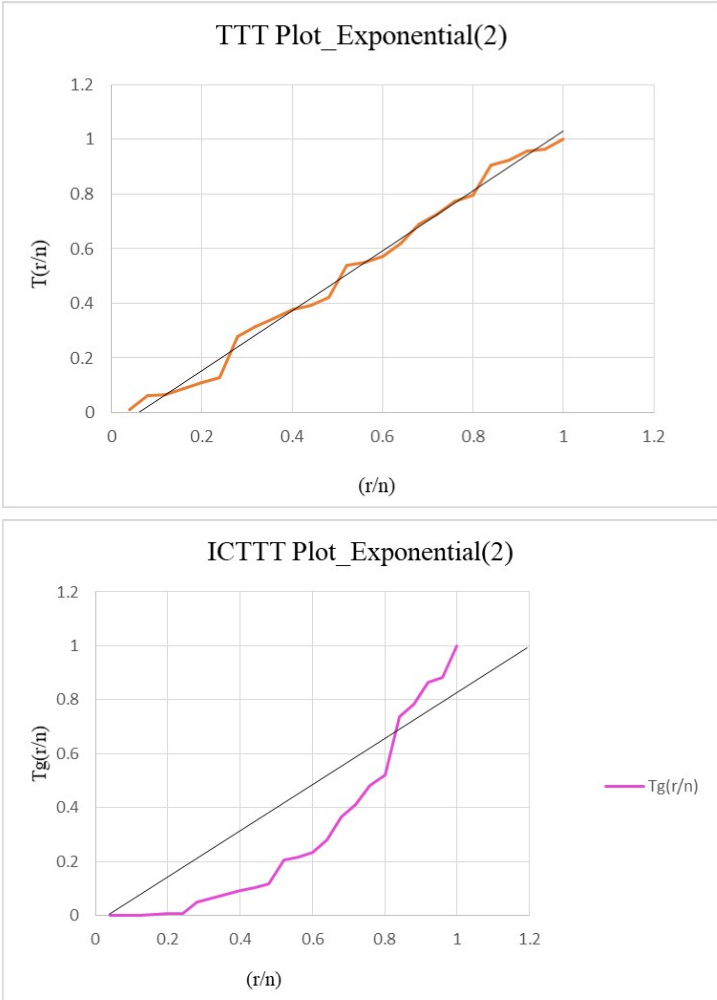

Figure 6.2: TTT plot (top) and ICXTTT plot (bottom) for the Exponential Simulated data with parameter θ=2.

is smaller than the rate of increase of $r(x) \Rightarrow \frac{(H_F^{-1})g(t)}{(H_F^{-1})g(1)}$ is concave in $t \in [0,1]$, if rate of increase of $g'(x)\frac{\bar{F}(g(x))}{\bar{F}(x)}$ is smaller than the rate of increase of $r(x)$. Hence G is IFR, if rate of increase of $g'(x)\frac{\bar{F}(g(x))}{\bar{F}(x)}$ is smaller than the rate of increase of $r(x)$.

Similarly,

$\frac{d}{dt}(H_F^{-1})g(t)$ is increasing in $t \in [0,1]$, if $r(x)$ is decreasing in $x \geq 0$.

That is, $\frac{(H_F^{-1})g(t)}{(H_F^{-1})g(1)}$ is convex in $t \in [0,1]$, if $r(x)$ is decreasing in $x \geq 0$. G is DFR, if $r(x)$ is decreasing in $x \geq 0$. □

Proposition 6.5.2. Let X has distribution F and $Y = g(X)$ has distribution $G(y)$. G is IFRA (DFRA) $\Rightarrow$ $\frac{(H_F^{-1})g(t)}{F(g(F^{-1}(t)))\ (H_F^{-1})g(1)}$ is decreasing (increasing) in $t \in [0,1]$.

Proof. Let $Y = g(X)$ and $G(y)$ be the distribution function of Y. G has IFRA $\Rightarrow \frac{1}{y}\int_0^y r(u)\ du$ is increasing in $y \geq 0$.

Let $T(y) = \int_0^y \bar{G}(u)\ du$. $\frac{T(y)}{y}$ is decreasing in $y \geq 0$, since it is an average of the decreasing function $\bar{G}(y)$.
Then, $\frac{\int_0^y r(u)\ dT(u)}{T(y)}$ is increasing in $y \geq 0$. Hence

$$\frac{G(y)}{\int_0^y \bar{G}(u)\ du} \text{ is increasing in } y \geq 0$$

and

$$\frac{\int_0^y \bar{G}(u)\ du}{G(y)} \text{ is decreasing in } y \geq 0.$$

Then,

$$\frac{\int_0^y \bar{G}(u)\ du}{G(y)} = \frac{\int_0^{g(x)} \bar{F}(w)\ dw}{F(g(x))} \text{ is decreasing in } x \geq 0$$

since $\bar{G}(u) = P(g(X) > u) = P(X > w) = \bar{F}(w)$ for $w = g^{-1}(u)$ corresponding to u.

Now make the change of variables $t = F(x)$ and $x = F^{-1}(t)$ and finally we have

$$\frac{\int_0^{g(F^{-1}(t))} \bar{F}(w)\ dw}{F(g(F^{-1}(t)))} = \frac{(H_F^{-1})g(t)}{F(g(F^{-1}(t)))} \text{ is decreasing in } t \in [0,1].$$

$$\Rightarrow \frac{(H_F^{-1})g(t)}{F(g(F^{-1}(t)))\ (H_F^{-1})g(1)} \text{ is decreasing in } t \in [0,1].$$

Similarly, for G is DFRA,

$$\frac{(H_F^{-1})g(t)}{F(g(F^{-1}(t)))} \text{ is increasing in } t \in [0,1].$$

$$\Rightarrow \frac{(H_F^{-1})g(t)}{F(g(F^{-1}(t)))\ (H_F^{-1})g(1)} \text{ is increasing in } t \in [0,1].$$

□

6.6 Increasing Convex (Concave) TTT transform order

In this section we defined the increasing convex (concave) TTT transform order. Let X and Y be two nonnegative random variables with distributions F and H respectively. If

$$\int_0^{F^{-1}(t)} (1 - F(u))\ du \leq \int_0^{H^{-1}(t)} (1 - G(u))\ du \quad t \in [0,1] \tag{6.6.1}$$

then X is said to be smaller than Y in the TTT order (denoted by $X \leq_{\text{ttt}} Y$). A sufficient condition for the order $\leq_{\text{ttt}}$ is the usual stochastic order:

$$X \leq_{\text{st}} Y \implies X \leq_{\text{ttt}} Y. \tag{6.6.2}$$

In order to verify (6.6.2) one may just notice that if $X \leq_{\text{st}} Y$, then $F^{-1}(u) \leq G^{-1}(u)$ for all $u \in (0,1)$ (see, Shaked and Shanthikumar (2007)). By letting $u \to 1$ in (6.6.2) it is seen that

$$X \leq_{\text{ttt}} Y \implies E(X) \leq E(Y). \qquad (6.6.3)$$

Let X and Y be two random variables such that $Tg(X_{(n)}) \leq Tg(Y_{(n)})$ for all increasing convex (concave) functions $g : \mathbb{R} \to \mathbb{R}$ and all samples of size n. Then X is smaller than Y in some stochastic sense, since $\frac{1}{n}Tg(X_{(n)})$ is average of total observed increasing convex (concave) transformed time of a test.

Definition 6.6.1. *Let X and Y be two non-negative random variables with absolutely continuous distribution functions F and H respectively. If*

$$(H_F^{-1})g(t) \leq (H_H^{-1})g(t) \quad \forall\ t \in [0,1]$$

where g is an increasing convex (concave) function, then X is smaller than Y in increasing convex (concave) TTT transform order (denoted as $X \leq_{icxttt} Y$ ($X \leq_{icvttt} Y$)).

Now we prove the relationship of ICXTTT (ICVTTT) transform orders to stochastic orders.

Theorem 6.6.1. *Let X and Y be two non-negative random variables having absolutely continuous distribution functions F and G respectively. Then*

$$X \leq_{st} Y \implies X \leq_{icxttt} Y.$$

Proof. Let g be the increasing convex (concave) function $g : \mathbb{R} \to \mathbb{R}$. Since, $X \leq_{\text{st}} Y$, $g(F^{-1}(t)) \leq g(G^{-1}(t))$ for all $t \in [0, 1]$.

Hence,

$$\int_0^{g(F^{-1}(t))} (1 - F(u))\, du \leq \int_0^{g(G^{-1}(t))} (1 - G(u))\, du, \ \forall t \in [0, 1]$$

then $X \leq_{\text{icxttt}} Y$. □

6.7 Examples

Usually the TTT transform plot is drawn by plotting $T(\frac{r}{n}) = \frac{\sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}}{\sum_{i=1}^{r} X_{(i)}}$ against $(\frac{r}{n})$, where $i = 1, 2, \ldots, r$ and $r = 1, 2, \ldots, n$. A TTT transform curve may be concave (convex) if corresponding distribution is IFR (DFR) distribution. A TTT transform curve is straight line if the distribution is exponential. If the shape of TTT transform is concave (convex) and then convex (concave), then the distribution has a bathtub (upside down bathtub) shaped failure rate function.

Then the ICXTTT (ICVTTT) transform plot is drawn by plotting $Tg(\frac{r}{n}) = \frac{\sum_{i=1}^{r} g(X_{(i)}) + (n-r)g(X_{(r)})}{\sum_{i=1}^{r} g(X_{(i)})}$ against $(\frac{r}{n})$, where $i = 1, 2, \ldots, r$ and $r = 1, 2, \ldots, n$,

Figure 6.3 and 6.4 shows scaled TTT transforms and scaled ICXTTT transforms of bathtub shaped failure rate data (Aarset data (Aarset (1987))) and Weibull simulated data respectively. From Figure 6.3, the failure rate pattern of transformed data still shows bathtub shape. It means that, even after transformation, we may be able to identify the failure rate pattern. It is useful because, selection of statistical distribution to the transformed variables is a problem for

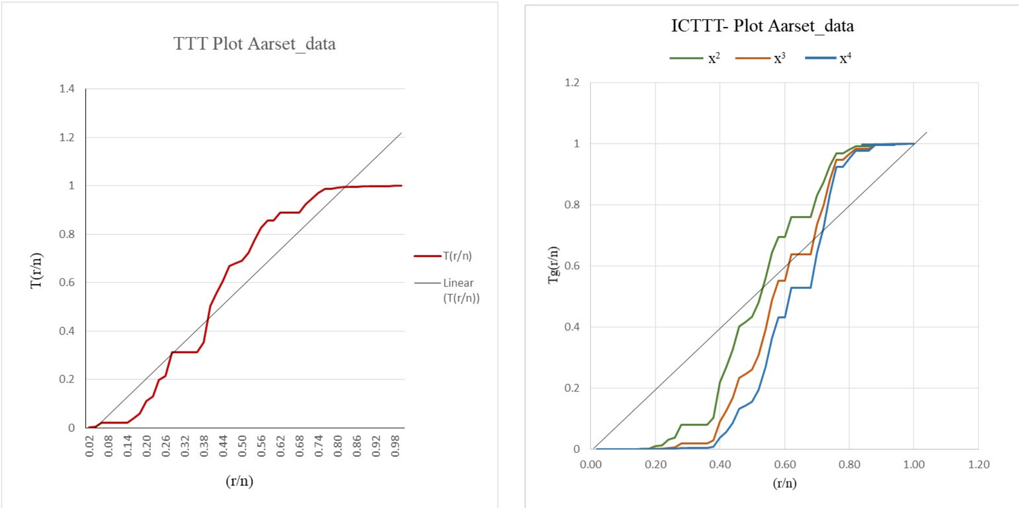

Figure 6.3: TTT plot (left) and ICXTTT plot (right) for the Aarset data.

many researchers. The researchers need only to search for a particular class of distribution, if they could identify the failure rate pattern using ICXTTT (ICVTTT) transform.

Another advantage of defining ICXTTT (ICVTTT) transform is that, the transform statistic can be used for estimating the dispersion parameters, variance etc of censored data.

$$(H_n^{-1})g\left(\frac{r}{n}\right) = \int_0^{(H_n^{-1}(\frac{r}{n}))} \big(1 - H_n(w)\big)dw = \int_0^{g(F_n^{-1}(\frac{r}{n}))} \big(1 - F_n(u)\big)du$$

is actually mean of censored-transformed data from F. This can be used for the purpose of estimation and testing the parameters of distribution of transformed data. Figure 6.4, shows that the TTT transform plot of Weibull($\alpha = 1.5, \lambda = 1$) data set indicates IFR behavior, but ICXTTT transform plot for Weibull($\alpha = 1.5, \lambda = 1$) indicates an upside down BFR pattern for the failure rate. The TTT

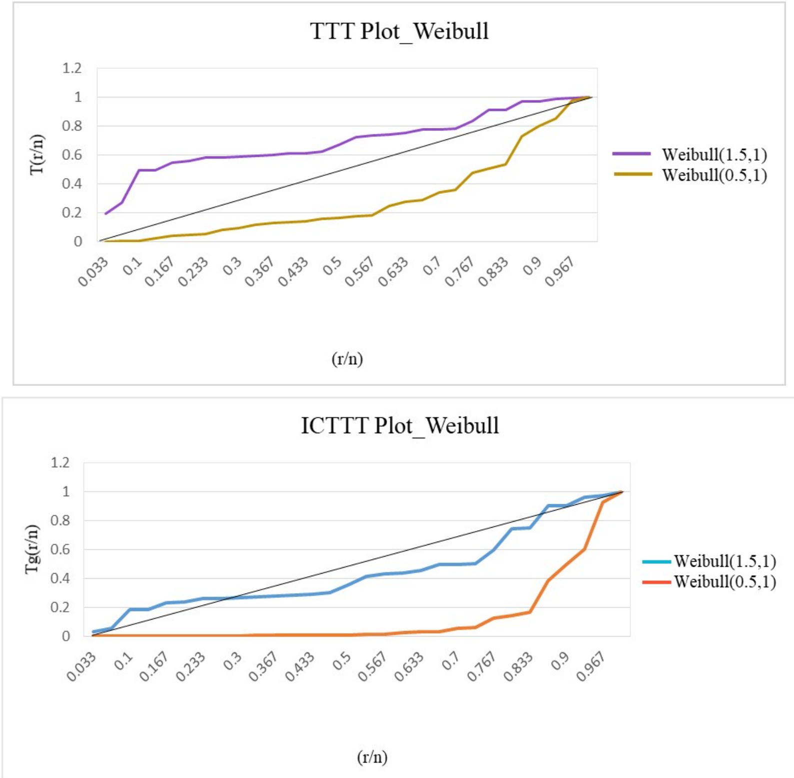

Figure 6.4: TTT plot (top) and ICXTTT plot (bottom) for the Weibull Simulated data with parameter $\alpha = 1.5, 0.5$ and λ=1 respectively.

transform plot of Weibull($\alpha = 0.5, \lambda = 1$) shows $F(t)$ has decreasing failure rate (DFR) while ICXTTT transform plot based on a Weibull($\alpha = 0.5, \lambda = 1$) shows decreasing behavior for failure rate.

6.8 Summary

We defined increasing convex (concave) TTT transform. The procedure of identification of the failure rate model of functions of random variables, using failure rate function of random variable is discussed. IFR (DFR) and IFRA (DFRA) properties of distribution of increasing convex (concave) transformations of the variable are explained. Illustrative examples are provided.