Dus Transformation of Inverse Weibull Distribution: An Upside-Down Failure Rate Model

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Abstract

A new upside-down bathtub shaped failure rate distribution, DUS Inverse Weibull (DUS-IW) distribution is proposed and its properties are studied. The DUS-IW distribution has upside-down bathtub shaped and decreasing failure rate functions. Moments, moment generating function, characteristic function, quantiles, etc. are derived. Estimation of parameters of the distribution is performed via maximum likelihood method. Reliability of single component and multi component stress-strength models are derived. A simulation study is performed for validating the estimates of the model parameters. DUS-IW distribution is applied to two real data sets and found that DUS-IW distribution is a better fit than other well-known distributions.

Keywords: upside-down bathtub shaped failure rate, reliability, stress-strength

I. Introduction

The statistical analysis of lifetime data is of importance in various fields of applied science especially in reliability, biomedical, engineering, social sciences, etc. To explain reallife phenomenon, there are a lot of lifetime distributions. Among all of them, some frequently used distributions are Exponential, Gamma, Weibull, Lognormal, etc. Each has their own merits and demerits due to their flexibility of shapes and failure rates like increasing, decreasing or constant failure rate, depends on the nature of distributions. A comprehensive account of lifetime models and the methods for analyzing them are given in [15]. Weibull distribution is one of the most widely used distributions from the exponential family and is used in the reliability engineering, hydrological, energy studies, etc. There exist many variations of it using different transformations, such as Inverse Weibull (IW) distribution. Similar to Weibull distribution, IW distribution has its significance and role in real life phenomenon. The complementary Weibull and reciprocal Weibull (see [8] and [19]) are exactly the IW distribution. The IW distributionhas received some attention in the literature and is another lifetime probability distribution which can be used in the reliability engineering discipline. The IW distribution can be used to model a variety of failure characteristics such as infant mortality, useful life and wear-out periods. Extensive work has been doneon the IW distribution, see [12], [3], [4] and [16]. Along with these, IW distribution is used as an alternative to Weibull distribution to model wind speed data, [1].

Generalized Inverse Weibull distribution is another extension of Weibull distribution, [10]. Theoretical analysis of IW distribution is found to be quite interesting as well, [13].

In statistical literature, there are various methods to propose a new distribution by using some baseline distribution. A transformation gives a more accurate distribution with easy computation and interpretation as it contains no new parameters other than parameters involved in the baseline distribution. Exponential transformed Lindley distribution is applied to Yarn data, [18]. DUS transformation is a method which has been introducedto get a new distribution by using Exponential distribution as baseline distribution, [14], with application to survival data analysis. DUS transformation of Exponential distribution used for the problem of estimation of the parameter based on upper record values, [22]. An upside-down bathtub shaped failure rate model using DUS transformation on the Lomax distribution as baseline distribution is found to be a better fit than existing distributions, [7]. A new lifetime distribution based on the DUS transformation by using Weibull distribution as the baseline distribution is another choice of existing models, [11]. In this paper, an attempt has been made here to obtain a new distribution using the DUS transformation with IW distribution as baseline distribution, to study upside down bathtub data.

This paper is organized as follows. In Section II, the details of DUS Transformation are given. In section III, the probability density function, cumulative distribution function, survival function and failure rate function of the DUS transformation of IW distribution is given. Shapes of the probability density function andfailure rate function are discussed in this Section IV. Statistical properties including moments, moment generating function (mgf), characteristic function (cf) and quantile function of the proposed distribution are discussed in Section V. Stress-strength reliability evaluation is discussed in Section VI. Estimation of the parameters using method of maximum likelihood is discussed in SectionVII. The results are given in Section VIII which includes a simulation study that is conducted to validate the estimatesand a real data analysis which is used to illustrate the usefulness of the proposed distribution. Final conclusions and discussions are given in Section IX.

II. DUS Transformation

Let f(x) and F(x) be the probability density function (pdf) and cumulative distribution function (cdf) of some baseline distribution, then the pdf g(x) of the distribution obtained by DUS Transformation of the baseline distribution is given by

$$g(x) = \frac{1}{e-1} f(x) e^{F(x)}. \tag{1}$$

If the pdf g(x) of the distribution obtained by DUS Transformation is given by (1), then the corresponding cdf, survival function and failure rate function are given by

$$G(x) = \frac{1}{e-1}\left[e^{F(x)} - 1\right] \tag{2}$$

$$S(x) = \frac{1}{e-1}\left[e - e^{F(x)}\right] \tag{3}$$

and

$$h(x) = \frac{1}{e-e^{F(x)}} f(x) e^{F(x)} \tag{4}$$

respectively.

III. DUS-IW(α,β) Distribution

Let Y be a random variable from the two-parameter Weibull distribution with the shape parameter α and the scale parameter β. Its pdf is given by

$$f(y) = \frac{\alpha}{\beta}\left(\frac{y}{\beta}\right)^{\alpha-1} e^{-\left(\frac{y}{\beta}\right)^{\alpha}}, \ y>0,\ \alpha,\beta>0.$$

By using the reciprocal of the random variable Y, i.e., X=1/Y, the distribution of X is called Inverse Weibull distribution with the shape parameter α and the scale parameter β, denoted by IW(α,β).

The pdf and the cdf of the IW distribution are given below

$$f(x) = \frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}, \text{ x>0, } \alpha, \beta>0 \tag{5}$$

and

$$F(x) = e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}, \text{x>0, } \alpha, \beta>0 \tag{6}$$

respectively.

Let g(x) be the pdf obtained by DUS transformation (1), corresponding to the baseline pdf (5) and cdf (6), then

$$g(x) = \frac{1}{e-1}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{-(\alpha+1)} \exp\left\{-\left(\frac{x}{\beta}\right)^{-\alpha} + e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right\}, \text{ x>0, } \alpha, \beta>0. \tag{7}$$

For simplicity, we call the distribution having pdf (7) as DUS transformation of IW(α,β) distribution and denote it as DUS-IW(α,β)distribution.

The cdf of DUS-IW(α,β) distribution is obtained using (2) and is given by,

$$G(x) = \frac{1}{e-1}\left\{e^{e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}} - 1\right\}, \text{ x>0, } \alpha, \beta>0. \tag{8}$$

I. Survival function and Failure rate function

The survival function S(x), using (3), is given by,

$$S(x) = \frac{1}{e-1}\left\{e - e^{e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}\right\}, \text{ x>0, } \alpha, \beta>0. \tag{9}$$

The failure rate function of DUS-IW(α,β) distribution, using (4), is given by,

$$h(x) = \frac{1}{e - e^{e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}}\left(\frac{\alpha}{\beta}\right)\left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{\left\{-\left(\frac{x}{\beta}\right)^{-\alpha} + e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right\}}, \text{ x>0, } \alpha, \beta>0. \tag{10}$$

IV. Shapes

It can be seen that the pdf of DUS-IW(α, β) distribution has the shape properties, specifically,

$\alpha < 1$, β > 1 ⇒ g(x) is decreasing,
$\alpha < 1$, β < 1 ⇒ g(x) is unimodal,
$\alpha > 1$, β < 1 ⇒ g(x) is unimodal,
$\alpha > 1$, β > 1 ⇒ g(x) is unimodal.

Mode of the distribution can be found as a solution of the equation,

$$\frac{d}{dx}\log g(x) = 0.$$

By substituting the pdf, we get

$$\frac{d}{dx}\log\left[\frac{1}{e-1}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{-(\alpha+1)} \exp\left\{-\left(\frac{x}{\beta}\right)^{-\alpha} + e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right\}\right] = 0.$$

Simplifying the equation, we get,

$$\frac{-(\alpha+1)}{x} + \beta^{\alpha}\alpha x^{-(\alpha+1)}\left[1 + e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right] = 0. \tag{11}$$

On solving (11) numerically, we get the mode of the distribution. The plots of pdf and failure rate function of DUS-IW(α,β) distribution for different values of α and β are shown the Figures 1 and 2 respectively.

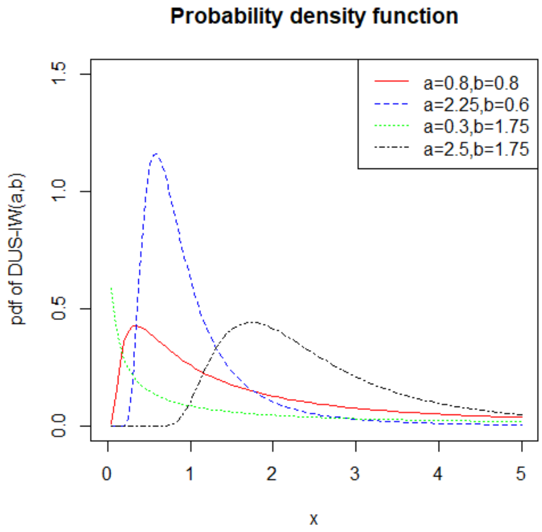

Figure 1: *Pdf of DUS-IW(α,β) for (0.8,0.8),(2.25,0.6), (0.3,1.75) and (2.5,1.75)*

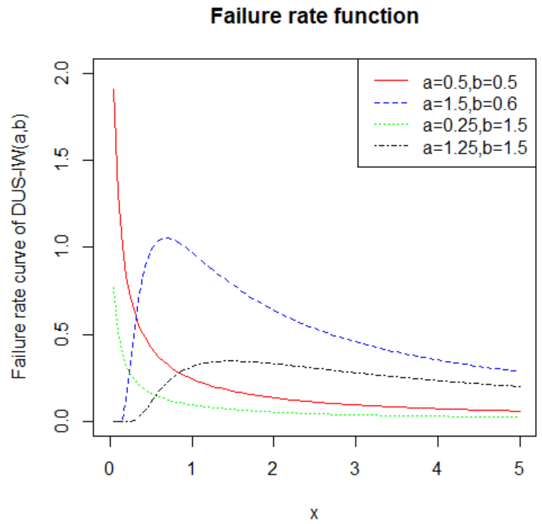

Figure 2: *Failure rate function of DUS-IW(α,β) for (0.5,0.5),(1.5,0.6),(0.25,1.5) and (1.25,1.5)*

Figure 2 shows the graph of failure rate function of DUS-IW(α, β) distribution for the parameter values (0.5,0.5), (1.5,0.6), (0.25,1.5) and (1.25,1.5). Failure rate function h(x) is monotonically decreasing for α <1 and upside-down for α >1.

V. Statistical Properties

I. Moments

Let X be a random variable with its pdf given by (7), then its rth raw moment is obtained by

$$\mu_r' = E(X^r)$$

$$= \int_0^\infty x^r \left[\frac{1}{e-1}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}\right] dx$$

$$= \frac{1}{e-1}\frac{\alpha}{\beta}\int_0^\infty x^r \left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}} \sum_{m=0}^{\infty}\frac{\left[e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right]^m}{m!} dx$$

$$= \sum_{m=0}^{\infty}\frac{1}{e-1}\frac{\alpha}{\beta}\frac{\beta^{\alpha+1}}{m!}\int_0^\infty x^r\,(x)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta}\right)^{-\alpha}} dx.$$

Put$u = x^{-\alpha}$
Then, $du = -\alpha x^{-(\alpha+1)}dx$,
implies,

$$\mu_r' = \sum_{m=0}^{\infty} \frac{\beta^\alpha}{m!\,(e-1)} \int_0^\infty \left(u^{-\frac{1}{\alpha}}\right)^r e^{-(m+1)(\beta)^\alpha u} du$$

$$= \sum_{m=0}^{\infty} \frac{\beta^\alpha}{m!\,(e-1)} \int_0^\infty u^{-\frac{r}{\alpha}+1-1} e^{-(m+1)(\beta)^\alpha u} du$$

$$= \sum_{m=0}^{\infty} \frac{\beta^\alpha}{m!\,(e-1)} \frac{\Gamma(-\frac{r}{\alpha}+1)}{((m+1)\beta^\alpha)^{-\frac{r}{\alpha}+1}}$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(-\frac{r}{\alpha}+1)}{(m+1)!\,(e-1)((m+1)\beta^\alpha)^{-\frac{r}{\alpha}}}$$

$$= \sum_{m=0}^{\infty} \frac{\Gamma(-\frac{r}{\alpha}+1)(m+1)^{\frac{r}{\alpha}}\beta^r}{(m+1)!\,(e-1)}.$$

Therefore,

$$\mu_r' = \sum_{m=0}^{\infty} \frac{\Gamma(-\frac{r}{\alpha}+1)(m+1)^{\frac{r}{\alpha}}\beta^r}{(m+1)!(e-1)} \tag{12}$$

Hence, the r[th] raw moment.
Putting r=1 in (12), we get the 1[st] raw moment (mean) and is given by,

$$\mu_1' = E(X)$$

$$\mu_1' = \sum_{m=0}^{\infty} \frac{\Gamma(-\frac{1}{\alpha}+1)(m+1)^{\frac{1}{\alpha}}\beta}{(m+1)!(e-1)}. \tag{13}$$

Putting r=2 in (12), we get the 2[nd] raw moment and is given by,

$$\mu_2' = E(X^2)$$

$$\mu_2' = \sum_{m=0}^{\infty} \frac{\Gamma(-\frac{2}{\alpha}+1)(m+1)^{\frac{2}{\alpha}}\beta^2}{(m+1)!\,(e-1)}.$$

Then, the variance of the random variable X is given by,

$$V(X) = E(X^2) - \left(E(X)\right)^2$$

$$V(X) = \sum_{m=0}^{\infty} \frac{\Gamma(-\frac{2}{\alpha}+1)(m+1)^{\frac{2}{\alpha}}\beta^2}{(m+1)!(e-1)} - \left(\sum_{m=0}^{\infty} \frac{\Gamma(-\frac{1}{\alpha}+1)(m+1)^{\frac{1}{\alpha}}\beta}{(m+1)!(e-1)}\right)^2. \tag{14}$$

II. Moment Generating Function

The mgf of DUS-IW(α,β) distribution is,

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx}\, g(x) dx$$

$$= \int_0^\infty e^{tx} \left[\frac{1}{e-1}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}\right] dx$$

$$= \frac{1}{e-1}\frac{\alpha}{\beta}\beta^{(\alpha+1)} \int_0^\infty e^{tx}\,(x)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}} \sum_{m=0}^{\infty} \frac{\left[e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right]^m}{m!} dx$$

$$= \sum_{m=0}^{\infty} \frac{1}{e-1}\frac{\alpha\beta^\alpha}{m!} \int_0^\infty (x)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta}\right)^{-\alpha}} \sum_{n=0}^{\infty} \frac{[tx]^n}{n!} dx$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\alpha\beta^\alpha}{(e-1)m!\,n!} \int_0^\infty (x)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta}\right)^{-\alpha}} (tx)^n dx.$$

Put$u = x^{-\alpha}$
Then, $du = -\alpha x^{-(\alpha+1)} dx$
implies,

$$M_X(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\beta^{\alpha}}{(e-1)m!\,n!} \int_0^{\infty} \left(\frac{1}{u}\right)^{\frac{n}{\alpha}} e^{-(m+1)(\beta)^{\alpha}u} (t)^n du$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\beta^{\alpha}(t)^n}{(e-1)m!\,n!} \int_0^{\infty} (u)^{-\frac{n}{\alpha}+1-1} e^{-(m+1)(\beta)^{\alpha}u} du$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(t)^n \Gamma(-\frac{n}{\alpha}+1)(m+1)^{\frac{n}{\alpha}}\beta^n}{(e-1)(m+1)!\,n!}.$$

Therefore,

$$M_X(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(t)^n \Gamma(-\frac{n}{\alpha}+1)(m+1)^{\frac{n}{\alpha}}\beta^n}{(e-1)(m+1)!n!}. \quad (15)$$

III. Characteristic Function

The cf, $\phi_X(t)$ is given by,

$$\phi_X(t) = E(e^{itX}) = \int_0^{\infty} e^{itx}\, g(x).\, dx$$

$$= \int_0^{\infty} e^{itx} \left[\frac{1}{e-1}\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}}\right] dx$$

$$= \frac{1}{e-1}\frac{\alpha}{\beta}\beta^{(\alpha+1)} \int_0^{\infty} e^{itx} (x)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta}\right)^{-\alpha}} \sum_{m=0}^{\infty} \frac{\left[e^{-\left(\frac{x}{\beta}\right)^{-\alpha}}\right]^m}{m!} dx$$

$$= \sum_{m=0}^{\infty} \frac{1}{e-1}\frac{\alpha\beta^{\alpha}}{m!} \int_0^{\infty} (x)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta}\right)^{-\alpha}} \sum_{n=0}^{\infty} \frac{[itx]^n}{n!} dx$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\alpha\beta^{\alpha}}{(e-1)m!\,n!} \int_0^{\infty} (x)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta}\right)^{-\alpha}} (itx)^n dx$$

Put$u = x^{-\alpha}$.
Then, $du = -\alpha x^{-(\alpha+1)} dx$,
implies,

$$\phi_X(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\beta^{\alpha}}{(e-1)m!\,n!} \int_0^{\infty} \left(\frac{1}{u}\right)^{\frac{n}{\alpha}} e^{-(m+1)(\beta)^{\alpha}u} (it)^n du$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{\beta^{\alpha}(it)^n}{(e-1)m!\,n!} \int_0^{\infty} (u)^{-\frac{n}{\alpha}+1-1} e^{-(m+1)(\beta)^{\alpha}u} du$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(it)^n \Gamma(-\frac{n}{\alpha}+1)(m+1)^{\frac{n}{\alpha}}\beta^n}{(e-1)(m+1)!\,n!}.$$

Therefore,

$$\phi_X(t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(it)^n \Gamma(-\frac{n}{\alpha}+1)(m+1)^{\frac{n}{\alpha}}\beta^n}{(e-1)(m+1)!n!}. \quad (16)$$

IV. Quantile Function

The p[th] quantile function, denoted by Q(p) of DUS-IW(α,β) distribution is obtained by solving F(Q(p))=p, where 0<p<1. That is,

$$\frac{1}{e-1}\left\{e^{e^{-\left(\frac{Q(p)}{\beta}\right)^{-\alpha}}} - 1\right\} = p$$

and the p[th] quantile function is,

$$Q(p) = \frac{-\beta}{\{\log(\log(1+p(e-1)))\}^{\frac{1}{\alpha}}}. \quad (17)$$

Median (2[nd] quartile) of DUS-IW(α,β) distribution is obtained by substituting p = 1/2 in (17), i.e.,

$$Q_2 = \frac{-\beta}{\left\{\log(\log(1+\frac{1}{2}(e-1)))\right\}^{\frac{1}{\alpha}}}. \quad (18)$$

Setting p = 1/4 in (17), we get the 1[st] quartile of DUS-IW(α,β) distribution as follows,

$$Q_1 = \frac{-\beta}{\left\{\log(\log(1+\frac{1}{4}(e-1)))\right\}^{\frac{1}{\alpha}}}. \quad (19)$$

Setting p = 3/4 in (17), we get the 3[rd] quartile of DUS-IW(α,β)distribution as follows,

$$Q_3 = \frac{-\beta}{\left\{\log(\log(1+\frac{3}{4}(e-1)))\right\}^{\frac{1}{\alpha}}}. \quad (20)$$

A random sample X with DUS-IW(α,β) distribution can be simulated using

$$X = \frac{-\beta}{\{\log(\log(1+U(e-1)))\}^{\frac{1}{\alpha}}}, \text{ where } U\sim U(0,1) \quad (21)$$

VI. Stress-Strength Reliability

Stress-Strength model has a significant role in reliability engineering. The stress-strength reliability is defined as the probability that the random strength greater than the random stress of a component or system, [6]. A study on point estimation of the stress-strength reliability parameter for parallel system with independent and non-identical components can be seen in literature, [20]. In this section, reliability estimation of single component stress-strength model (SSS) and multicomponent stress-strength model (MSS) are considered.

I. Single Component Stress-Strength Reliability

Here we consider the reliability of SSS based on two independent random variables X and Y, where X represents the 'strength' and Y represents the 'stress'. Suppose X and Y have the DUS-IW(α, β_1) and DUS-IW (α, β_2) distributions respectively, then the system reliability R = P(Y < X) is

$$R = P(Y < X)$$
$$= \int_0^\infty g_X(x) G_Y(x).\,dx$$
$$= \int_0^\infty \frac{1}{e-1}\frac{\alpha}{\beta_1}\left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}} \frac{1}{e-1}\left\{e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} - 1\right\} dx$$
$$= \frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^\infty \left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}} \left\{e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} - 1\right\}.dx$$
$$= \frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^\infty \left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} \left[\exp\left(-\left(\frac{x}{\beta_1}\right)^{-\alpha} + e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}} + e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}\right) - \exp\left(-\left(\frac{x}{\beta_1}\right)^{-\alpha} + e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}\right)\right] dx$$
$$= \frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^\infty \left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}} e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} dx - \frac{1}{e-1}$$
$$= I_1 - \frac{1}{e-1}$$

where $I_1 = \frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^\infty \left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}} e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} dx$

$$= \frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^\infty \left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}} \sum_{m=0}^{\infty}\frac{\left[e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}\right]^m}{m!} \sum_{n=0}^{\infty}\frac{\left[e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}\right]^n}{n!} dx$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!\,n!}\frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^{\infty}\left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta_1}\right)^{-\alpha}} e^{-n\left(\frac{x}{\beta_2}\right)^{-\alpha}} dx$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!\,n!}\frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^{\infty}\left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta_1}\right)^{-\alpha}} \sum_{p=0}^{\infty}\frac{(-1)^p n^p \left(\frac{x}{\beta_2}\right)^{-\alpha p}}{p!} dx$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p n^p}{m!\,n!\,p!}\frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^{\infty}\left(\frac{x}{\beta_1}\right)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta_1}\right)^{-\alpha}} \left(\frac{x}{\beta_2}\right)^{-\alpha p} dx$$

Put$u = x^{-\alpha}$

Then, $du = -\alpha x^{-(\alpha+1)}dx$,

implies,

$$I_1 = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p n^p}{m!\,n!\,p!}\frac{1}{(e-1)^2}\frac{\alpha}{\beta_1}\int_0^{\infty}\frac{{\beta_1}^{\alpha+1}{\beta_2}^{\alpha p}}{\alpha} e^{-(m+1){\beta_1}^{\alpha}u}.u^p.du$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p n^p}{m!\,n!\,p!}\frac{1}{(e-1)^2}\frac{1}{\beta_1}{\beta_1}^{\alpha+1}{\beta_2}^{\alpha p}\int_0^{\infty} e^{-(m+1){\beta_1}^{\alpha}u}.u^p.du$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p n^p}{m!\,n!\,p!}\frac{1}{(e-1)^2}{\beta_1}^{\alpha}{\beta_2}^{\alpha p}\frac{\Gamma(p+1)}{((m+1){\beta_1}^{\alpha})^{p+1}}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p n^p}{m!\,n!\,p!}\frac{1}{(e-1)^2}\frac{{\beta_1}^{\alpha}{\beta_2}^{\alpha p}}{{\beta_1}^{\alpha p}{\beta_1}^{\alpha}}\frac{\Gamma(p+1)}{((m+1))^{p+1}}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p n^p}{m!\,n!\,p!}\frac{1}{(e-1)^2}\left(\frac{\beta_2}{\beta_1}\right)^{\alpha p}\frac{\Gamma(p+1)}{((m+1))^{p+1}}$$

Therefore,

$$R = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^p n^p}{m!n!p!}\frac{1}{(e-1)^2}\left(\frac{\beta_2}{\beta_1}\right)^{\alpha p}\frac{\Gamma(p+1)}{((m+1))^{p+1}} - \frac{1}{e-1} \quad (22)$$

II. Multicomponent Stress-Strength Reliability

Here we consider the reliability of MSS based on the independent random variables $X_1, X_2, \ldots, X_k$ and Y, where $X_1, X_2, \ldots, X_k$ represents the k i.i.d. 'strength' components and Y represents the 'stress'. Suppose asystem with these X_i's functions if atleast s $(1 \le s \le k)$ components operate. Let, F(.) be the distribution function of X and G(.) be the distribution function of Y. Then the system reliability $R_{s,k}$ = Prob(atleast s of the Xi's exceed Y).

i.e.,

$$R_{s,k} = \sum_{i=s}^{k}\binom{k}{i}\int_{-\infty}^{\infty}[1-F(x)]^i[F(x)]^{k-i}dG(x) \quad (23)$$

Various works have been done in MSS (see for example, [2], [17] and [21]). Suppose X_i's and Y have the DUS-IW(α, β_1) and DUS-IW(α, β_2) distributions respectively, then the reliability in MSS using (23) is,

$$R_{s,k} = \sum_{i=s}^{k}\binom{k}{i}\int_0^{\infty}\left[1-\frac{1}{e-1}\left\{e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}-1\right\}\right]^i\left[\frac{1}{e-1}\left\{e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}-1\right\}\right]^{k-i}$$
$$\frac{1}{e-1}\frac{\alpha}{\beta_2}\left(\frac{x}{\beta_2}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} dx$$

$$= \sum_{i=s}^{k}\binom{k}{i}\int_0^{\infty}\left[1+\frac{1}{e-1}-\frac{1}{e-1}e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}\right]^i\left[-\frac{1}{e-1}\left\{1-e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}\right\}\right]^{k-i}$$
$$\frac{1}{e-1}\frac{\alpha}{\beta_2}\left(\frac{x}{\beta_2}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} dx$$

$$= \sum_{i=s}^{k}\binom{k}{i}\int_0^{\infty}\left[\frac{e}{e-1}\left\{1-e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}-1}\right\}\right]^i\left[(-1)\frac{1}{e-1}\left\{1-e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}\right\}\right]^{k-i}$$

$$\frac{1}{e-1}\frac{\alpha}{\beta_2}\left(\frac{x}{\beta_2}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} dx$$

$$= \sum_{i=s}^{k}\binom{k}{i}\left(\frac{e}{e-1}\right)^{i}(-1)^{k-i}\left(\frac{1}{e-1}\right)^{k-i}\frac{1}{e-1}\frac{\alpha}{\beta_2}\int_0^\infty\left[1-e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}-1}\right]^{i}\left[1-e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}\right]^{k-i}$$
$$\left(\frac{x}{\beta_2}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}} e^{e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}} dx$$

Using binomial expansion $(1-x)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k x^{n-k}$, we get,

$$= \sum_{i=s}^{k}\binom{k}{i}e^{i}(-1)^{k-i}\left(\frac{1}{e-1}\right)^{k+1}\frac{\alpha}{\beta_2}\int_0^\infty\sum_{j=0}^{i}\binom{i}{j}(-1)^j\left[e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}-1}\right]^{i-j}\sum_{p=0}^{k-i}\binom{k-i}{p}(-1)^p\left[e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}\right]^{k-i-p}$$
$$\left(\frac{x}{\beta_2}\right)^{-(\alpha+1)} e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}\sum_{m=0}^{\infty}\frac{\left[e^{-\left(\frac{x}{\beta_2}\right)^{-\alpha}}\right]^{m}}{m!}dx$$

$$= \sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{i}(-1)^{k-i}}{m!}\left(\frac{1}{e-1}\right)^{k+1}\frac{\alpha}{\beta_2}(-1)^{j+p}\int_0^\infty\left[e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}-1}\right]^{i-j}\left[e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}\right]^{k-i-p}$$
$$\left(\frac{x}{\beta_2}\right)^{-(\alpha+1)} e^{-(m+1)\left(\frac{x}{\beta_2}\right)^{-\alpha}} dx$$

$$= \sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{i}\alpha{\beta_2}^{\alpha}}{m!\,(e-1)^{k+1}}(-1)^{k-i+j+p}e^{-i+j}\int_0^\infty\left[e^{e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}}\right]^{k-j-p}$$
$$(x)^{-(\alpha+1)}e^{-(m+1)\left(\frac{x}{\beta_2}\right)^{-\alpha}}dx$$

$$= \sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{j}{\beta_2}^{\alpha}}{m!\,(e-1)^{k+1}}(-1)^{k-i+j+p}\int_0^\infty\alpha(x)^{-(\alpha+1)}$$
$$\sum_{n=0}^{\infty}\frac{\left[(k-j-p)e^{-\left(\frac{x}{\beta_1}\right)^{-\alpha}}\right]^{n}}{n!}e^{-(m+1)\left(\frac{x}{\beta_2}\right)^{-\alpha}}dx$$

$$= \sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{j}{\beta_2}^{\alpha}(k-j-p)^{n}}{m!\,(e-1)^{k+1}n!}(-1)^{k-i+j+p}\int_0^\infty\alpha(x)^{-(\alpha+1)}$$
$$e^{-n\left(\frac{x}{\beta_1}\right)^{-\alpha}}e^{-(m+1)\left(\frac{x}{\beta_2}\right)^{-\alpha}}dx.$$

Put $u = x^{-\alpha}$

Then, $du = -\alpha x^{-(\alpha+1)}dx$,

implies,

$$R_{s,k} = \sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{j}{\beta_2}^{\alpha}(k-j-p)^{n}}{m!\,(e-1)^{k+1}n!}(-1)^{k-i+j+p}\int_0^\infty e^{-n{\beta_1}^{\alpha}u}e^{-(m+1){\beta_2}^{\alpha}u}du$$

$$= \sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{j}{\beta_2}^{\alpha}(k-j-p)^{n}}{m!\,(e-1)^{k+1}n!}(-1)^{k-i+j+p}\int_0^\infty e^{-u\left(n{\beta_1}^{\alpha}+(m+1){\beta_2}^{\alpha}\right)}du$$

$$= \sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{j}{\beta_2}^{\alpha}(k-j-p)^{n}}{m!\,(e-1)^{k+1}n!}(-1)^{k-i+j+p}\left[\frac{e^{-u\left(n{\beta_1}^{\alpha}+(m+1){\beta_2}^{\alpha}\right)}}{-\left(n{\beta_1}^{\alpha}+(m+1){\beta_2}^{\alpha}\right)}\right]_0^\infty$$

$$= \sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{j}{\beta_2}^{\alpha}(k-j-p)^{n}(-1)^{k-i+j+p}}{m!\,(e-1)^{k+1}n!\,\left(n{\beta_1}^{\alpha}+(m+1){\beta_2}^{\alpha}\right)}.$$

Therefore,

$$R_{s,k} = \sum_{i=s}^{k}\sum_{j=0}^{i}\sum_{p=0}^{k-i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{k}{i}\binom{i}{j}\binom{k-i}{p}\frac{e^{j}{\beta_2}^{\alpha}(k-j-p)^{n}(-1)^{k-i+j+p}}{m!(e-1)^{k+1}n!\left(n{\beta_1}^{\alpha}+(m+1){\beta_2}^{\alpha}\right)}. \quad (24)$$

VII. Maximum Likelihood Estimation

In this section, we discuss method of maximum likelihood for the estimation of parameters α and β. Let $X_1, X_2, \ldots, X_n$ be an observed random sample from DUS-IW(α,β) distribution with unknown parameters α and β. The maximum likelihood estimator(MLE)s of the parameters of the DUS-IW(α, β) distribution are derived as below. The likelihood function is

$$L(x) = \prod_{i=1}^{n} f(x_i, \alpha, \beta)$$

i.e.,

$$L(x) = \left(\frac{1}{e-1}\right)^n \frac{\alpha^n}{\beta^n} \prod_{i=1}^{n} \left(\frac{x_i}{\beta}\right)^{-(\alpha+1)} e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha}} e^{\sum_{i=1}^{n} e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}}}$$

So that the log-likelihood function becomes

$$\log L = -\, n\log(e-1) + n\log\alpha - n\log\beta - (\alpha+1)\sum_{i=1}^{n}[\log x_i - \log\beta] - \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha} + \sum_{i=1}^{n} e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}} \quad (25)$$

The partial derivatives of log L in (25) with respect to unknown parameters α and β are,

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}[\log x_i - \log\beta] + \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha} \log \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right) + \sum_{i=1}^{n} e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}} \left(\frac{x_i}{\beta}\right)^{-\alpha} \log\left(\frac{x_i}{\beta}\right) \quad (26)$$

$$\frac{\partial \log L}{\partial \beta} = -\frac{n}{\beta} + (\alpha+1)\frac{n}{\beta} - \alpha\sum_{i=1}^{n}(x_i)^{-\alpha}\beta^{\alpha-1} - \sum_{i=1}^{n} e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}} \alpha\sum_{i=1}^{n}(x_i)^{-\alpha}\beta^{\alpha-1} \quad (27)$$

Setting the left side of above two equations to zero, we get the likelihood equations as a system of two non-linear equations in α and β.

$$\frac{n}{\alpha} - \sum_{i=1}^{n}[\log x_i - \log\beta] + \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{-\alpha} \log \sum_{i=1}^{n}\left(\frac{x_i}{\beta}\right) + \sum_{i=1}^{n} e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}} \left(\frac{x_i}{\beta}\right)^{-\alpha} \log\left(\frac{x_i}{\beta}\right) = 0 \quad (28)$$

$$-\frac{n}{\beta} + (\alpha+1)\frac{n}{\beta} - \alpha\sum_{i=1}^{n}(x_i)^{-\alpha}\beta^{\alpha-1} - \sum_{i=1}^{n} e^{-\left(\frac{x_i}{\beta}\right)^{-\alpha}} \alpha\sum_{i=1}^{n}(x_i)^{-\alpha}\beta^{\alpha-1} = 0 \quad (29)$$

Solving these systems, (28) and (29), in α and β gives the MLEs of α and β. These equations cannot be solved analytically and statistical software R can be used to solve them numerically, by taking initial value arbitrarily.

VIII. Results

I. Simulation Study

In statistics, simulation is used to assess the performance of the model. It is a numerical technique for conducting experiments on the computer. There are certain simulation techniques for generating and analyzing like Monte-Carlo simulation. With considered (21), here we take different combinations of parameters α and β with samples of sizes n=25, 50, 100,500 and 1000 and the samples are generated from the DUS-IW(α,β) model. The bias and the mean square error (MSE) of the parameter estimates are calculated using the equations,

$$\text{Bias} = \frac{1}{n}\sum_{i=1}^{n}(\hat{\varepsilon}_i - \varepsilon_i) \quad (30)$$

and

$$\text{MSE} = \frac{1}{n}\sum_{i=1}^{n}(\hat{\varepsilon}_i - \varepsilon_i)^2. \quad (31)$$

The simulation is conducted for three different cases using different true parameter values. The selected true parameter values are α=0.2 and β=0.8, α=1 and β=1.5 and α=1 and β=1 for the first, second, and third cases, respectively. As the sample size increases, MSE decreases for all selected parameter values as in Table 1, 2 and 3. Also, the bias is nearer to zero when the sample size increases. Thus, the estimates tend to the true parameter values as the sample size increases.

Table 1: *Simulation study at α=0.2 and β=0.8*

n	Estimated value of Parameters	Bias	MSE
25	$\hat{\alpha}$=0.2112962	0.01129618	0.001374719
	$\hat{\beta}$=1.871778	1.071778	13.29915
50	$\hat{\alpha}$=0.204941	0.004941044	0.0005514951
	$\hat{\beta}$=1.185226	0.3852255	2.116492
100	$\hat{\alpha}$=0.2020091	0.002009123	0.0002558455
	$\hat{\beta}$=0.9837115	0.1837115	0.4104516
500	$\hat{\alpha}$=0.1988177	-0.001182251	$4.310696\text{x}10^{-05}$
	$\hat{\beta}$=0.8201508	0.02015084	0.04473991
1000	$\hat{\alpha}$=0.1982047	-0.0017953	$2.387253\text{x}10^{-05}$
	$\hat{\beta}$=0.8006457	0.0006456743	0.01871488

Table 2: *Simulation study at α=1 and β=1.5*

n	Estimated value of Parameters	Bias	MSE
25	$\hat{\alpha}$=1.056977	0.0569774	0.03443659
	$\hat{\beta}$=1.594545	0.09454492	0.1449253
50	$\hat{\alpha}$=1.024739	0.02473883	0.01378851
	$\hat{\beta}$=1.536913	0.03691277	0.06391387
100	$\hat{\alpha}$=1.010164	0.01016389	0.006394628
	$\hat{\beta}$=1.522311	0.02231105	0.03083641
500	$\hat{\alpha}$=0.9940904	-0.005909631	0.001077642
	$\hat{\beta}$=1.499751	-0.0002490474	0.005798989
1000	$\hat{\alpha}$=0.9910253	-0.008974667	0.0005967668
	$\hat{\beta}$=1.496801	-0.003199132	0.002582339

Table 3: *Simulation study at α=1 and β=1*

n	Estimated value of Parameters	Bias	MSE
25	$\hat{\alpha}$= 1.056977	0.05697739	0.03443657
	$\hat{\beta}$= 1.06303	0.06302988	0.06441129
50	$\hat{\alpha}$=1.024739	0.02473881	0.0137885
	$\hat{\beta}$= 1.024608	0.02460846	0.02840616
100	$\hat{\alpha}$= 1.010164	0.0101639	0.00639462
	$\hat{\beta}$= 1.014874	0.01487402	0.0137051
500	$\hat{\alpha}$= 0.9940904	-0.005909626	0.001077644
	$\hat{\beta}$= 0.9998339	-0.0001661259	0.002577332
1000	$\hat{\alpha}$= 0.9910254	-0.008974635	0.0005967686
	$\hat{\beta}$= 0.9978672	-0.002132811	0.001147696

Hence, DUS-IW(α, β) distribution possesses least bias and MSE values as the sample size increases.

II. Data Analysis

In this section, we illustrate the use of DUS-IW(α, β) distribution using two real data sets. We fit DUS-IW(α, β) distribution to these two data sets and compare the results with IW distribution, DUS-Lomax (DUS-L) distribution, Lomax distribution, Gompertz Lomax (GL) distribution, DUS-Exponential(DUS-E) distribution and Inverse Lindley (IL) distribution. The first data-sets, considered here, represent the survival times of two groups of patients suffering from head and neck cancer disease. The data here considered is of the patients belonging to one group who were treated using a combined radiotherapy and chemotherapy (CT + RT) ([9]). Another data is of 46 observations reported on active repair times (hours) for an airborne communication transceiver ([5]). The data sets are given below:

Table 4: *Survival times of patients treated using RT+CT:*

12.2	23.56	23.74	25.87	31.98	37	41.35	47.38	55.46	58.36
63.47	68.46	78.26	74.47	81.43	84	92	94	110	112
119	127	130	133	140	146	155	159	173	179
194	195	209	249	281	319	339	432	469	519
633	725	817	1776						

Table 5: *Repair Time:*

0.2	0.3	0.5	0.5	0.5	0.5	0.6	0.6	0.7	0.7
0.7	0.8	0.8	1.0	1.0	1.0	1.0	1.1	1.3	1.5
1.5	1.5	1.5	2.0	2.0	2.2	2.5	2.7	3.0	3.0
3.3	3.3	4.0	4.0	4.5	4.7	5.0	5.4	5.4	7.0
7.5	8.8	9.0	10.3	22.0	24.5				

Using the R software, the analysis is carried out. The tables 6 and 7 gives the estimates of the model parameters, AIC (Akaike information criterion) and the BIC (Bayesian information criterion) values, where,

$$AIC = -2l + 2k \tag{32}$$

$$BIC = -2l + k \log n \tag{33}$$

where l denotes the log-likelihood function, k is the number of parameters and n is the sample size. Also, using the Kolmogorov-Smirnov(K-S) test, the perfection of competing models is tested.
The K-S statistic is given by,

$$KS = max\left\{\frac{i}{m} - z_i, z_i - \frac{i-1}{m}\right\} \tag{34}$$

where, z_i is the cumulative distribution of x_i, x_i's being the ordered observations and m is the number of classes. Both the KS-statistic and p-value are given in tables 6 and 7 as well.

Table 6: *Estimates of the parameters, AIC, BIC and KS statistic of the fitted model in data set 1 (Table 4):*

Model	Estimates	AIC	BIC	KS-statistic	p-value
DUS-IW	$\hat{\alpha}$=1.119 $\hat{\beta}$=57.556	561.83	565.40	0.087	0.868
IW	$\hat{\alpha}$= 1.013 $\hat{\beta}$=76.227	563.14	566.71	0.093	0.811
DUS-L	$\hat{\alpha}$=3.165 $\hat{\beta}$=0.003	563.81	567.38	0.093	0.806
Lomax	$\hat{\alpha}$=4.40 $\hat{\beta}$=0.001	564.91	568.48	0.104	0.695
GL	$\hat{\theta}$=0.0185 $\hat{\alpha}$=0.467 $\hat{\beta}$=0.719 $\hat{\gamma}$=1.99	571.54	578.68	0.129	0.414
DUS-E	$\hat{\theta}$=0.006	569.82	571.60	0.198	0.021
IL	$\hat{\theta}$=77.68	561.16	562.94	29.01	5.551×10^{-16}

Table 6 and Table 7 show that, DUS-IW(α, β) has lowest AIC, BIC, KS-Statistic, and largest p-value based on KS-Statistic. The second lowest AIC, BIC, KS-Statistic and second largest p-value are obtained by the IW distribution. The proposed distribution, DUS-IW(α, β) can be used when failure rate pattern of lifetime distribution is upside-down shaped. In Data set 1 and 2 it seems that DUS-IW(α,β) is more appropriate than other distributions (IW distribution, DUS-L distribution, Lomax distribution, GL distribution, DUS-E distribution and IL distribution). So DUS-IW(α, β) is better alternative in the situations in which upside-down distributions arises.

Table 7: *Estimates of the parameters, AIC, BIC and KS statistic of the fitted model in data set 2 (Table 5):*

Model	Estimates	AIC	BIC	KS-statistic	p-value
DUS-IW	$\hat{\alpha}$=1.109 $\hat{\beta}$=0.857	204.68	208.34	0.078	0.942
IW	$\hat{\alpha}$=1.013 $\hat{\beta}$=1.130	205.38	209.04	0.081	0.926
DUS-L	$\hat{\alpha}$=2.610 $\hat{\beta}$=0.227	209.40	213.06	0.118	0.548
Lomax	$\hat{\alpha}$=3.549 $\hat{\beta}$=0.108	209.91	213.57	0.127	0.446
GL	$\hat{\theta}$=1.776 $\hat{\alpha}$=1.165 $\hat{\beta}$=0.189 $\hat{\gamma}$=0.245	213.96	221.27	0.129	0.432
DUS-E	$\hat{\theta}$=0.344	217.31	219.14	0.211	0.033
IL	$\hat{\theta}$=1.577	204.34	206.17	0.883	2.2×10^{-16}

IX. Discussion

A new distribution, DUS-IW(α, β) distribution, is proposed and its properties are studied. The DUS-IW(α, β) has an upside-down shaped and decreasing failure rate functions. We derived the moments, moment generating function, characteristic function, quantiles, etc. of the proposed distribution. Estimation of parameters of the distribution is performed via maximum likelihood method. Reliability of single component and multi component stress-strength models are derived. A simulation study is performed to validate the estimates of the model parameters. DUS-IW(α, β) distribution is applied to two real data sets and shown that DUS-IW(α, β) distribution is a better fit than other well-known distributions. Thus DUS-IW(α, β) distribution can be used in real data analysis as a better alternative to the existing distributions.

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