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Exponential-Gamma $(3, \theta)$ Distribution and its Applications

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Abstract

Lifetime distributions for many components usually have a bathtub shape for its failure rate function in practice. However, there are a very few distribution have bathtub shaped failure rate function. Models with bathtub-shaped failure rate functions are useful in reliability analysis, particularly in reliability related decision making, cost analysis and burn-in analysis. When considering a failure mechanism, the failure of units in system may be due to random failure occurred by change in temperature, voltage, jurking etc or due to ageing. This paper study on a distribution, which is a mixture of Exponential and Gamma (3) distribution, which have bathtub shaped failure rate function. Moments, skewness, kurtosis, moment generating function, characteristic function are derived. Renyi entroy, Lorenz curve and Gini index are obtained. Reliability of stress-strength model is derived. Distribution of maximum and minimum order statistics are obtained. We have obtained maximum likelihood estimators. A simulation study is conducted to illustrate the performance of the accuracy of the estimation method used. Application is illustrated using real data.

Keywords: Reliability, Bathtub shaped failure rate, Moments, Entropy, Maximum Likelihood estimator.

I. Introduction

Modeling and analysis of lifetime data has a prominent role in many applied sciences such as medicine, engineering and finance. Various lifetime data have been modeled using distributions such as Exponential, Weibull, Gamma, Rayleigh distributions and their generalizations. It is proved that Exponential distribution (ED) have constant failure function and Rayleigh distribution have monotone increasing failure functions. Two parameter generalized Exponential distribution is introduced by Gupta and Kundu [6] and proved that it has monotone failure functions, depending on its shape parameter. Generalized Rayleigh distribution has an increasing or bathtub shaped failure function, see Surles and Padgett [13]. A new distribution with probability density function

$$f(x,\theta)=\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}, x>0,\ \theta>0.$$

is proposed by Lindley [8] in the context of Bayesian statistics. Ghitany et al. [5] studied the properties and application of the Lindley distribution. They highlighted that the Lindley distribution is a better model than one based on the exponential distribution. Ghitany et al. [3] showed that the Lindley distribution can be written as a mixture of a Exponential distribution and a Gamma distribution with shape parameter 2. Sankaran [12] proposed the discrete Poisson-Lindley distribution as a combination of the Poisson and Lindley distributions. An Upside-down Bathtub Shaped failure rate model using DUS Transformation of Lomax Distribution is discussed by Deepthi and Chacko (2020). When considering a failure mechanism, the failure of units in system may be due to random failure occurred by change in temperature, voltage, jurking etc or due to ageing. In such situations, we need to use Exponential distribution for random failures and other lifetime distributions for failure due to aging. Mixture of Exponential distribution and a Gamma distribution with shape parameter 2 is not appropriate in some real life situations. So here we examine the mixture of Exponential distribution and a Gamma distribution with shape parameter 3.

The rest of the paper is organized as follows. Section II discussed Exponential-Gamma$(3,\theta)$ distribution. In Section III, the statistical properties are given. Section IV deals with computation of reliability. Section V described the distribution of maximum and minimum. In Section VI, the maximum likelihood method to estimate the unknown parameter is given and two real data sets are analysed. In Section VII, detailed simulation study is given. The comparison of Exponential-Gamma(3) distribution with Exponential and Exponentiated Exponential distribution (EED) for examples from reliability and survival analysis is discussed in Section VIII. Conclusions are given in section IX.

II. Exponential-Gamma $(3,\theta)$ Distribution

A mixture of Exponential (θ) and Gamma $(3,\theta)$ distribution is considered. It is denoted as $EGD(\theta)$. Probability density function (pdf) of mixture of the Exponential $(\theta)$and Gamma $(3,\theta)$ distribution is as follows:

$$f(x,\theta) = pf_1(x,\theta) + (1-p)f_2(x,3,\theta),$$

where $p = \frac{\theta}{1+\theta}$, $f_1 = \theta e^{-\theta x}$ *and* $f_2 = \theta^3 \frac{x^2}{2} e^{-\theta x}$.

$$f(x,\theta) = \frac{\theta^2}{(1+\theta)}\left[1+\frac{\theta}{2}x^2\right]e^{-\theta x}, x>0, \theta>0. \tag{2.1}$$

The corresponding cumulative distribution function (cdf) of $EGD(\theta)$ distribution is

$$F(x;\theta) = 1 - \frac{(\theta(x(\theta x+2)+2)+2)e^{-\theta x}}{2(1+\theta)}, \ x>0, \ \theta>0. \tag{2.2}$$

The Survival function associated with (2.2) is

$$\bar{F}(x,\theta) = 1 - F(x,\theta) = \frac{(\theta(x(\theta x+2)+2)+2)e^{-\theta x}}{2(1+\theta)}, x>0, \theta>0. \tag{2.3}$$

The first derivative of the pdf is

$$f'(x) = \frac{\theta^3 e^{-\theta x}}{1+\theta}\left(x - 1 - \frac{\theta x^2}{2}\right).$$

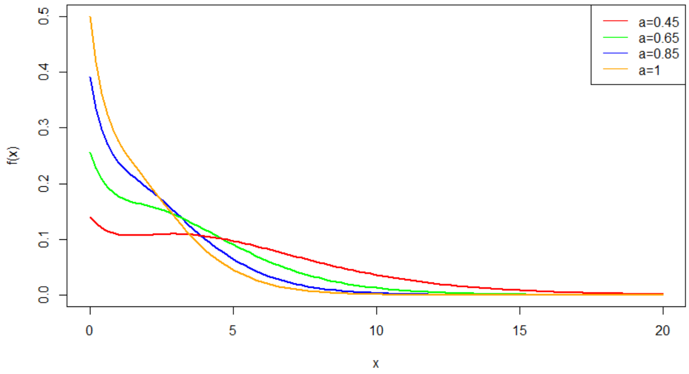
The second derivative of the pdf is

$$f''(x) = \frac{\theta^3 e^{-\theta x}}{1+\theta}\left(1 - 2\theta x + \theta + \frac{\theta^2 x^2}{2}\right).$$

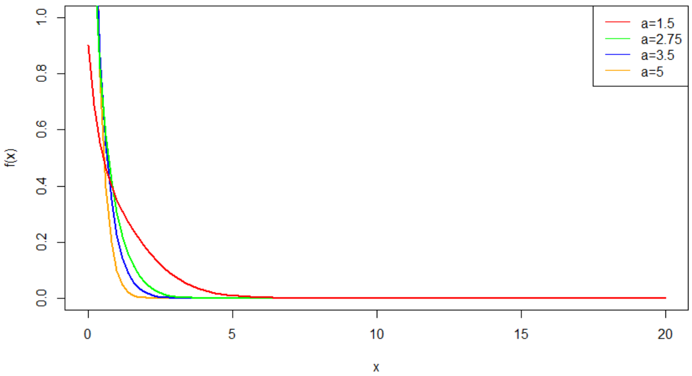
The mode of $f(x)$ is the point $x = x_0$ satisfying $f'(x_0) = 0$. Here $f'(x_0) = 0$ at the

$x_0 = \frac{1 \pm \sqrt{1-\frac{\theta}{2}}}{\theta}$. $f''(x) < 0$ for $0 < x < 1$ and $f''(x) > 0$ for $1 \le x \le 2$.

Shape of the probability density function is given in figure 1 below.



(a)



(b)

Figure 1 (a) & (b): pdf of EGD (θ) for θ =0.45, 0.65, 0.85, 1 and θ = 1.5, 2.75, 3.5, 5.

From the above figures it is obvious that the pdf can be decreasing or unimodal.
The failure rate function of EGD (θ) is given in (2.4) below.

$$h(x)=\frac{f(x,\theta)}{\bar{F}(x,\theta)}=\frac{2(1+\theta)\theta^2\left(1+\theta x^2/2\right)}{\left(\theta(x(\theta x+2)+2)+2\right)},x>0,\,\theta>0 \tag{2.4}$$

The first derivative of failure rate function is

$$h'(x)=2(1+\theta)\theta^2\frac{d}{dx}\left[\frac{1+\theta\frac{x^2}{2}}{\theta(x(\theta x+2)+2)+2}\right]$$

$$=2(1+\theta)\theta^2\frac{\theta x(\theta(x(\theta x+2)+2)+2)-\theta(2\theta x+2)\left(1+\theta\frac{x^2}{2}\right)}{(\theta(x(\theta x+2)+2)+2)^2}$$

$$=\frac{2\theta^2\left(\theta^2x^2+2\theta x-2\theta\right)(1+\theta)}{(\theta(x(\theta x+2)+2)+2)^2}.$$

The second derivative of the failure rate function is given by

$$h''(x)=2(1+\theta)\theta^2\frac{d}{dx}\left[\frac{\theta^2x^2+2\theta x-2\theta}{(\theta(x(\theta x+2)+2)+2)^2}\right]=\frac{4\theta^3(\theta x+1)(-\theta^2x^2+6\theta-2\theta x+2)(1+\theta)}{(\theta(x(\theta x+2)+2)+2)^3}$$

The extremum of h(x) is the point x = x_0 satisfying $h'(x)=0$ and these points correspond to a maximum or a minimum or a point of inflection according as $h''(x)<0$, $h''(x)>0$ and $h''(x)=0$ respectively. Here $h'(x)=0$ at the point $x_0=\frac{-1+\sqrt{1+2\theta}}{\theta}$ and $h''(x)>0$ for $\theta>0$. So h(x) must attain a unique minimum at x = x_0. Initially, plot of h(x) decreases monotonically and then increases giving a bathtub shape.

Figure 2 provide the failure rate functions of EGD(θ) for different parameter values.

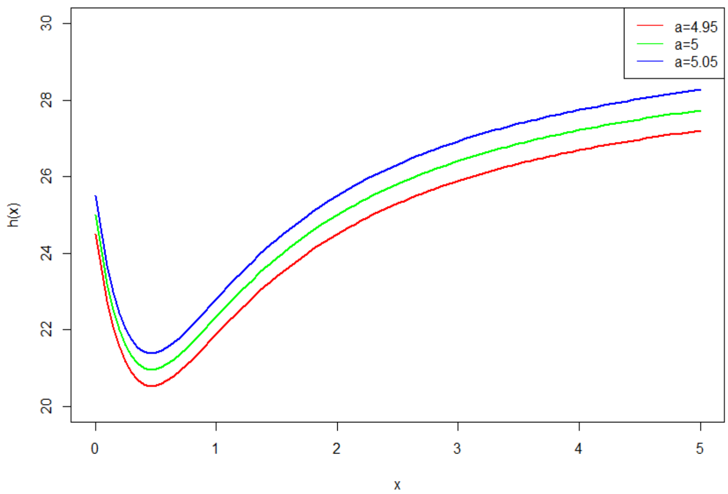


Figure 2: Failure rate function of EGD(θ) for θ = 4.95, 5, 5.15.

III. Statistical Properties

Here, we discuss the statistical measures for the EGD (θ) distribution, such as moments, skewness, kurtosis, moment generating function, characteristic function, quantile function, median, entropy, Lorenz curve and Gini index.

I. Moments

The concept of moment is important in statistical literature. We can measure the central tendency of a population by using moments. Moments also help in measuring the scatteredness, asymmetry and peakedness of a curve for a particular distribution.

The r^{th}raw moment (about origin) of EGD (θ) is

$$\mu_r^{'}=p\frac{r!}{\theta^r}+(1-p)\frac{\Gamma(r+3)}{2\theta^r}=\frac{2\theta r!+\Gamma(r+3)}{2(1+\theta)\theta^r}.$$

Therefore, the mean and variance of EGD(θ) are

$$\mu = \frac{\theta + 3}{\theta(1+\theta)} \text{ and } \sigma^2 = \frac{\theta^2 + 8\theta + 3}{\theta^2(1+\theta)^2}.$$

The skewness and kurtosis can be obtained using these raw moments as

$$\text{Skewness} = \frac{2\theta^3 + 30\theta^2 - 63\theta + 16}{\theta^2 + 8\theta + 3} \text{ and Kurtosis} = \frac{9\theta^4 + 192\theta^3 + 306\theta^2 + 216\theta + 45}{(\theta^2 + 8\theta + 3)^2}$$

II. Moment Generating Function and Characteristic Function

Let X has $EGD(\theta)$ distribution, then the moment generating function of X, $M_X(t) = E[\exp(tX)]$, is

$$M_X(t) = \frac{\theta^2}{1+\theta}\left[-\frac{(t-\theta)^2 + \theta}{(t-\theta)^3}\right]$$

for $t > \theta$. Similarly, the characteristic function of X becomes $\phi(t) = M_X(it)$,

$$\phi(t) = \frac{\theta^2}{1+\theta}\left[-\frac{(it-\theta)^2 + \theta}{(it-\theta)^3}\right]$$

where $i = \sqrt{-1}$.

III. Quantile and Median

Here, we determine the formulas of the quantile and the median of $EGD(\theta)$ distribution. The quantile x_p of the $EGD(\theta)$ is given from

$$F(x_p) = p, 0 < p < 1.$$

We obtain the 100 p^{th} percentile,

$$(\theta(x(\theta x + 2) + 2) + 2)e^{-\theta x} = 2(1-p)(1+\theta) \tag{3.1}$$

Setting $p = 0.5$ in Eq. (3.1), we get the median of $EGD(\theta)$ from

$$(\theta(x(\theta x + 2) + 2) + 2)e^{-\theta x} = 1 + \theta.$$

$x_{0.5}$ is the solution of above monotone increasing function. Using different statistical softwares we can obtain the quantiles or percentiles.

IV. Entropy

An important entropy measure is Rènyi entropy [11]. If X has the $EGD(\theta)$, then Rènyi entropy is defined by

$$\Im_R(\gamma) = \frac{1}{1-\gamma}\log\left\{\int f^{\gamma}(x)dx\right\},$$

where $\gamma > 0$ and $\gamma \neq 1$. Then, we can calculate, for $EGD(\theta)$,

$$\int f^{\gamma}(x)dx = \int_0^{\infty}\left\{\frac{\theta^2}{1+\theta}e^{-\theta x}\left(1+\frac{\theta}{2}x^2\right)\right\}^{\gamma}dx = \left(\frac{\theta^2}{1+\theta}\right)^{\gamma}\int_0^{\infty}\left\{1+\frac{\theta}{2}x^2\right\}^{\gamma}e^{-\gamma\theta x}dx$$

$$= \left(\frac{\theta^2}{1+\theta}\right)^{\gamma}\sum_{k=0}^{\infty}\binom{\gamma}{k}(-1)^k\left(\frac{\theta}{2}\right)^k\int_0^{\infty}x^{2k}e^{-\gamma\theta x}dx = \left(\frac{\theta^2}{1+\theta}\right)^{\gamma}\sum_{k=0}^{\infty}\binom{\gamma}{k}(-1)^k\left(\frac{\theta}{2}\right)^k\frac{\Gamma(2k+1)}{(\gamma\theta)^{2k+1}}.$$

Therefore, Rènyi entropy is given by

$$\Im_R(\gamma) = \frac{1}{1-\gamma}\log\left\{\left(\frac{\theta^2}{1+\theta}\right)^{\gamma}\sum_{k=0}^{\infty}\binom{\gamma}{k}(-1)^k\left(\frac{\theta}{2}\right)^k\frac{\Gamma(2k+1)}{(\gamma\theta)^{2k+1}}\right\}$$

$$= \frac{\gamma}{1-\gamma}\log\left(\frac{\theta}{1+\theta}\right) + \frac{1}{1-\gamma}\log\left\{\sum_{k=0}^{\infty}\binom{\gamma}{k}(-1)^k\left(\frac{\theta}{2}\right)^k\frac{\Gamma(2k+1)}{(\gamma\theta)^{2k+1}}\right\}$$

V. Lorenz Curve and Gini Index

The Lorenz curve and Gini index have applications not only in economics but also in reliability. The Lorenz curve is defined by

$$L(p) = \frac{1}{\mu}\int_0^{q} xf(x)dx$$

or equivalently

$$L(p) = \frac{1}{\mu}\int_0^{p} F^{-1}(x)dx$$

where $\mu = E(X)$ and $q = F^{-1}(p)$. Gini index is defined by

$$G = 1 - 2\int_0^1 L(p)dp$$

If X has $EGD(\theta)$ then

$$L(p) = \frac{1}{\mu}\left[\frac{\theta+3}{\theta(\theta+1)} - \frac{(\theta(q(\theta(q(\theta q+3)+2)+6)+2)+6)e^{-\theta q}}{2\theta(1+\theta)}\right].$$

Gini index is

$$G = 1 - \frac{2}{\mu\theta(1+\theta)}\left[\theta + 3 - \frac{(\theta(q(\theta(q(\theta q+3)+2)+6)+2)+6)e^{-\theta q}}{2}\right], \theta > 0.$$

IV. Reliability

Suppose that X and Y are two independent strength and stress random variables. We derive the reliability $R = P(Y < X)$ when X and Y are independent random variables distributed according to EGD distribution with parameters θ_1 and θ_2, respectively. Then system reliability is

$$R = \int_0^\infty\int_0^x f(x)f(y)dydx = \int_0^\infty\int_0^x \frac{\theta_1^2}{1+\theta_1}(1+\frac{\theta_1}{2}x^2)e^{-\theta_1 x}\frac{\theta_2^2}{1+\theta_2}(1+\frac{\theta_2}{2}y^2)e^{-\theta_2 y}dydx$$

$$= \int_0^\infty \frac{\theta_1^2}{1+\theta_1}(1+\frac{\theta_1}{2}x^2)e^{-\theta_1 x}\left\{\int_0^x \frac{\theta_2^2}{1+\theta_2}(1+\frac{\theta_2}{2}y^2)e^{-\theta_2 y}dy\right\}dx$$

$$= \int_0^\infty \frac{\theta_1^2}{1+\theta_1}(1+\frac{\theta_1}{2}x^2)e^{-\theta_1 x}\left\{\frac{\theta_2}{1+\theta_2}\left[\left(1-e^{-\theta_2 x}\right)\frac{(1+\theta_2)}{\theta_2} - xe^{-\theta_2 x}(1+\theta_2 x)\right]\right\}dx$$

$$= \frac{\theta_1^2}{1+\theta_1}\int_0^\infty\left\{(1+\frac{\theta_1}{2}x^2)e^{-\theta_1 x} - (1+\frac{\theta_1}{2}x^2)e^{-(\theta_1+\theta_2)x} - x\frac{\theta_2}{1+\theta_2}e^{-(\theta_1+\theta_2)x}(1+\theta_2 x)\left(1+\frac{\theta_1}{2}x^2\right)\right\}dx$$

$$= \frac{\theta_1^2}{1+\theta_1}\left\{\frac{1+\theta_1}{\theta_1^2} - \frac{1}{\theta_1+\theta_2} - \frac{\theta_1}{(\theta_1+\theta_2)^3} - \frac{\theta_2}{(1+\theta_2)(\theta_1+\theta_2)^5}\left[(\theta_1+\theta_2)^3 + 2\theta_2(\theta_1+\theta_2)^2 + 3\theta_1(\theta_1+\theta_2) + 12\theta_1\theta_2\right]\right\}.$$

V.Distribution of Maximum and Minimum

Let $X_1, X_2, \ldots, X_n$ be a simple random sample from $EGD(\theta)$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics obtained from this sample. The pdf of $X_{(r)}$ is given by,

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)}[F(x;\theta)]^{r-1}[1-F(x;\theta)]^{n-r} f(x;\theta)$$

where $F(x;\lambda)$, $f(x;\lambda)$ are the cdf and pdf given by (2.1) and (2.2), respectively.

$$f_{r:n}(x) = \frac{1}{B(r,n-r+1)}\left[1-\frac{(\theta(x(\theta x+2)+2)+2)e^{-\theta x}}{2(1+\theta)}\right]^{r-1}\left[\frac{(\theta(x(\theta x+2)+2)+2)e^{-\theta x}}{2(1+\theta)}\right]^{n-r}\frac{\theta^2\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}}{1+\theta}. \tag{5.1}$$

Then the pdf of the smallest and largest order statistics, $X_{(1)}$ and $X_{(n)}$, respectively, are

$$f_1(x) = \frac{1}{B(1,n)}\left[\frac{(\theta(x(\theta x+2)+2)+2)}{2(1+\theta)}\right]^{n-1}\left[\frac{\theta^2\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}}{1+\theta}\right]$$

and

$$f_n(x) = \frac{1}{B(n,1)}\left[1-\frac{(\theta(x(\theta x+2)+2)+2)}{2(1+\theta)}\right]^{n-1}\left[\frac{\theta^2\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}}{1+\theta}\right]$$

The cdf of $X_{(r)}$ is

$$F_{r:n}(x) = \sum_{j=r}^{n}\binom{n}{j}F^j(x)[1-F(x)]^{n-j}$$

$$F_{r:n}(x) = \sum_{j=r}^{n}\binom{n}{j}\left[1-\frac{(\theta(x(\theta x+2)+2)+2)e^{-\theta x}}{2(1+\theta)}\right]^{j}\left[\frac{(\theta(x(\theta x+2)+2)+2)e^{-\theta x}}{2(1+\theta)}\right]^{n-j} \tag{5.2}$$

Then the cdf of the smallest and largest order statistics $X_{(1)}$ and $X_{(n)}$, respectively, are

$$F_1(x) = 1-\left[\frac{(\theta(x(\theta x+2)+2)+2)e^{-\theta x}}{2(1+\theta)}\right]^{n}, \theta > 0$$

and

$$F_n(x) = \left[1-\frac{(\theta(x(\theta x+2)+2)+2)e^{-\theta x}}{2(1+\theta)}\right]^{n}, \theta > 0.$$

These distributions can be used in reliability operations.

VI. Parametric Estimation

In this section, point estimation of the unknown parameter of the $EGD(\theta)$ is described by using the method of maximum likelihood for a complete sample data, as given below.

The likelihood function of $EGD(\theta)$ distribution is

$$L = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n}\frac{\theta^2\left(1+\frac{\theta}{2}x_i^2\right)e^{-\theta x_i}}{1+\theta}.$$

The log-likelihood function is,

$$\log L(x_i,\theta) = 2n\log\theta - n\log(1+\theta) + \sum_{i=1}^{n}\left[\log\left(1+\frac{\theta}{2}x_i^2\right) - \theta x_i\right].$$

The first partial derivatives of the log-likelihood function with respect to θ is

$$\frac{\partial L}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{1+\theta} + \sum_{i=1}^{n}\left(\frac{x_i^2}{2\left(1+\frac{\theta}{2}x_i^2\right)} - x_i\right).$$

Setting the left side of the above equation to zero, we get the likelihood equation as a system of nonlinear equation in θ. Solving this system in θ gives the MLE of θ. It is easy to obtain numerically by using statistical software package like *nlm* package in R programming with arbitrary initial values.

The Fisher information about θ, I (θ), is

$$I(\theta) = E\left\{-\frac{\partial^2}{\partial\theta^2}\ln f(X;\theta)\right\} = E\left\{\frac{2}{\theta^2} - \frac{1}{(1+\theta)^2} + \frac{x^4}{4}\frac{1}{\left(1+\frac{\theta}{2}x^2\right)^2}\right\}$$

$$= \frac{2}{\theta^2} - \frac{1}{(1+\theta)^2} + E\left\{\frac{x^4}{4}\frac{1}{\left(1+\frac{\theta}{2}x^2\right)^2}\right\}.$$

Then the asymptotic 100(1-α) % confidence interval for θ is given by

$$\hat{\theta} \mp z_{\alpha/2}\frac{I^{-1/2}(\hat{\theta})}{\sqrt{n}}$$

VII. Simulation

A simulation study is conducted to illustrate the performance of the accuracy of the estimation method. The following scheme is used:

(i) Specify the value of the parameter θ.

(ii) Specify the sample size n.

(iii) Generate a random sample with size n from $EGD(\theta)$.

(iv) Using the estimation method used in this paper, calculate the point estimate of the parameter θ.

(v) Repeat steps 3-4, N times.

(vi) Calculate the bias and the mean squared error (MSE).

The simulation study is performed at different sample sizes and different parameter values, θ= 1, 1.5, 1.85 and bias and MSEs for the parameter θ is given in table 1. MSE decreases as sample size increases.

Table.1. Simulation Results

Θ	N	Bias	MSE
1	50	-0.000854222	3.648476e^{-05}
	100	0.00039463	1.557328e^{-05}
	500	1.3114e^{-05}	8.59885e^{-08}
	1000	3.6889e^{-05}	1.490841e^{-09}
1.5	50	-0.00072618	2.636687e^{-05}
	100	-0.00058251	3.393179e^{-05}
	500	-3.906e^{-06}	7.628418e^{-09}
	1000	-3.8229e^{-05}	1.461456e^{-06}
1.85	50	0.00174578	0.000152387
	100	0.00092697	8.592734e^{-05}
	500	0.00016791	1.409688e^{-05}
	1000	3.2956e^{-05}	1.086098e^{-06}

VIII. Data Analysis

Applications of the $EGD(\theta)$ distribution is illustrated in two examples.

Data set 1:- We provide a data analysis to see how the new model works. The data set is taken from Klein and Berger [9]. It shows the survival data on the death times of 26 Psychiatric inpatients admitted to the University of Iowa hospitals during the years 1935-1948.

Table 2: The survival data on the death times of Psychiatric inpatients

1	1	2	22	30	28	32	11	14	36	31	33	33
37	35	25	31	22	26	24	35	34	30	35	40	39

We have used different distributions namely, *ED, EED* and $EGD(\theta)$ to analyse the data. The estimate(s) of the unknown parameter(s), corresponding Kolmogorov-Smirnov (K-S) test statistic and Log L values for three different models are given in table 3.

Table 3: The estimates, K-S test statistic and log-likelihood for the dataset 1

Model	Estimates	K-S	LogL
ED	$\hat{\theta} = 0.03784579$	0.3728	-111.1302
EED	$\hat{a} = 1.79724674, \hat{b} = 0.05254319$	0.3146	-108.9871
$EGD(\theta)$	$\hat{\theta} = 0.1050099$	0.2613	-104.5856

We present the p-values, corresponding Akaikes Information Criterion (AIC) (see [1]) and Bayesian Information Criterion (BIC) in the following table 4.

Table 4: The p-value, AIC and BIC of the models based on the dataset 1

Model	p- value	AIC	BIC
ED	0.001455	224.2604	225.5185
EED	0.01162	221.9741	224.4903
$EGD(\theta)$	0.0574	211.1713	212.4294

The table 3 shows the parameter MLEs and log likelihood values of the fitted distributions and table 4 show the values of AIC, BIC and the p-value. The values in tables 3 and 4, indicate that the $EGD(\theta)$ distribution is a strong competitor to other distribution used here for fitting the dataset.

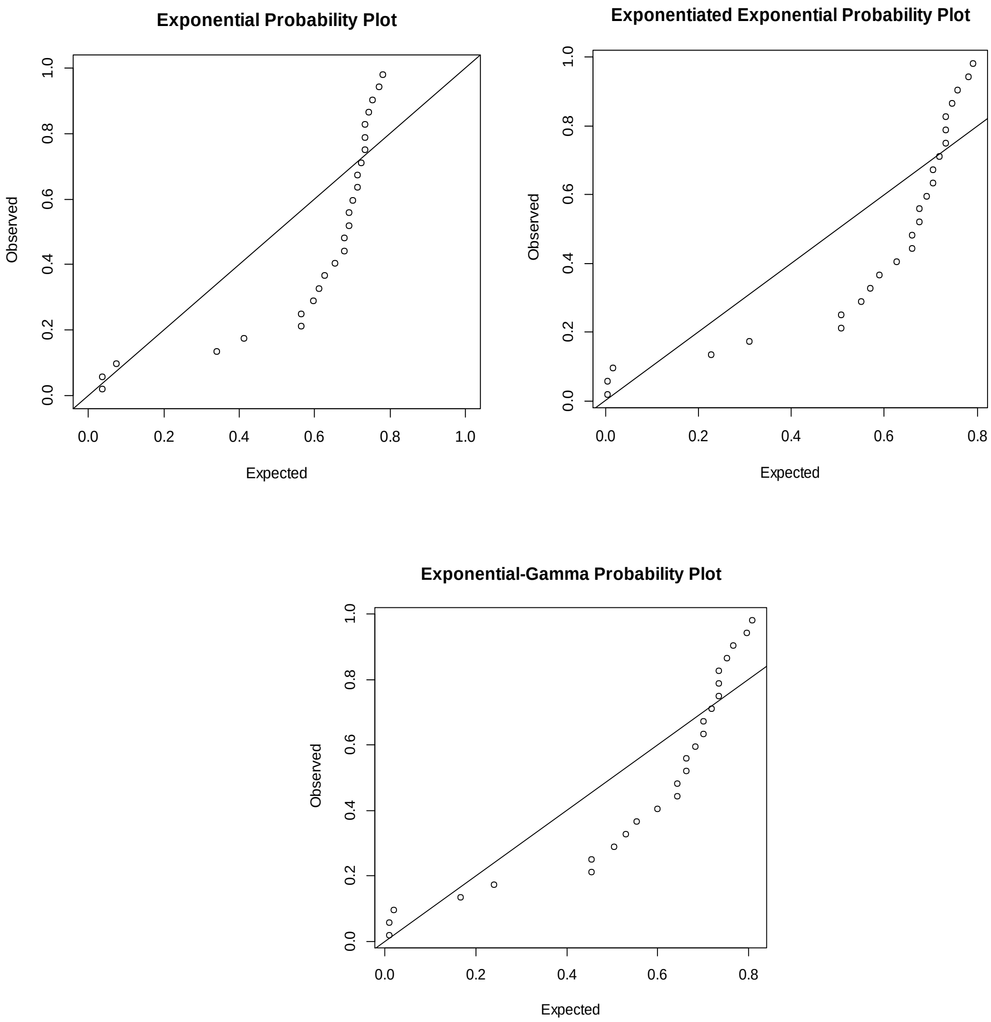


Figure 3: P-P plots for fitted ED, EED and EGD

P-P plot for *ED, EED* and $EGD(\theta)$ are given in Fig.3 which shows that $EGD(\theta)$ model is more plausible than *ED* and *EED* models.

Data set 2:- Chen [6] presented a type-II censoring data of samples, in which there was complete unit failures: 0.29, 1.44, 8.38, 8.66, 10.20, 11.04, 13.44, 14.37, 17.05, 17.13, and 18.35. The estimate(s) of the unknown parameter(s), corresponding Kolmogorov-Smirnov (K-S) test statistic and Log L values for three different models are given in table 5.

Table 5: The estimates, K-S test statistic and log-likelihood for the dataset 2

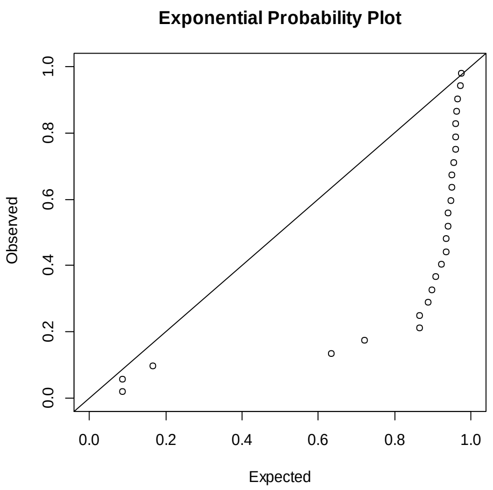
Model	Estimates	K-S	LogL
ED	$\hat{\theta}=0.09139958$	0.3533	-37.3176
EED	$\hat{a}=1.3514168, \hat{b}=0.1090155$	0.3183	-37.04664
EGD	$\hat{\theta}=0.2375122$	0.243	-35.25229

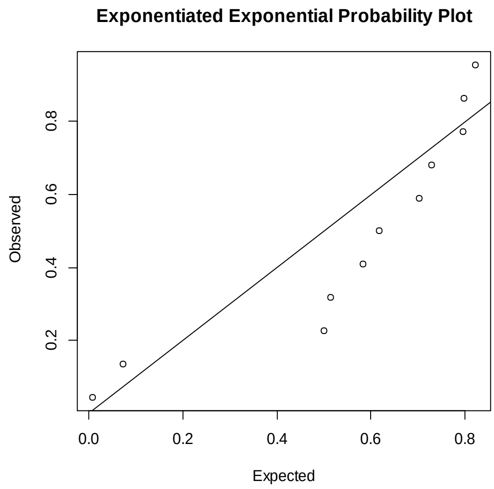
We present the p-values, corresponding Akaikes Information Criterion (AIC) and Bayesian Information Criterion (BIC) for the dataset 2 in the following table 6.

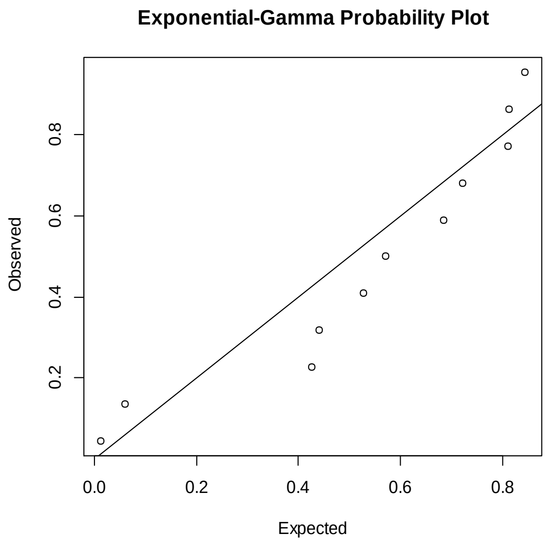
Table 6: The p-value, AIC and BIC of the models based on the dataset 2

Model	P value	AIC	BIC
ED	0.09856	76.6352	77.03309
EED	0.1722	78.09328	78.88907
EGD	0.4625	72.50459	72.90248

The table 5 shows the parameter MLEs and log likelihood values of the fitted distributions and table 6 show the values of AIC, BIC and the p-value. The values in tables 5 and 6, indicate that the $EGD(\theta)$ distribution is a strong competitor to other distribution used here for fitting the dataset.







IX. Conclusion

A bathtub shaped failure rate model, Exponential-Gamma$(3,\theta)$ distribution is considered and its properties are studied. Moments, skewness, kurtosis, moment generating function, characteristic function, etc are derived. Renyi entropy, Lorenz curve and Gini index are obtained. Reliability of stress-strength model is derived. Distribution of maximum and minimum are obtained. We have obtained maximum likelihood estimators. A simulation study is conducted to illustrate the performance of the accuracy of the estimation method used. Applications of $EGD(\theta)$ to real data show that Exponential-Gamma$(3,\theta)$ distribution is effective in providing better fits than the Exponential and Exponentiated Exponential distribution.

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