Identification of Failure Rate Behaviour of Increasing Convex (Concave) Transformations

[1]Deepthi K. S &[2]Chacko V. M

•

Department of Statistics,
St. Thomas' College (Autonomous), Thrissur, Kerala - 680 001 (India)
[1]dipthiks@gmail.com and [2]chackovm@gmail.com

Abstract

In this paper, increasing convex (concave) Total Time on Test (TTT) transform of a lifetime random variable is considered. In order to identify the failure rate model of functions of random variables, the TTT of transformed data can be used. Some properties of the transforms are derived. Some examples are given.

Keywords: Ageing patterns, Total Time on Test Transform, Increasing convex (concave) Total Time on Test transform

I. Introduction

The concept of the Total Time on Test (TTT) plots was first defined by Barlow and Campo (1975). The plots provide information about the identification of failure rate model. Incomplete type 2 censored data can be analysed using TTT transform. Aarset (1985) derived the exact distribution of TTT under the null hypothesis of exponentiality. Gupta and Michalek (1985) developed an explicit method to determine the reliability function by the TTT transform. Vera and Lynch (2005) introduced higher-order TTT transforms by applying definition of TTT recursively to the transformed distributions. Nair et. al (2008) studied the properties of TTT transform of order n and examined their applications in reliability analysis. Nair and Sankaran (2013) listed some known characterizations of common aging notions in terms of the total time on test transform (TTT) function. Franco-Pereira and Shaked (2013) derived two characterizations of the decreasing percentile residual life of order (DPRL(α)) aging notion in terms of the TTT function, and in terms of the observed TTT when X is observed. TTT statistic provide the central value of type 2 data. In order to get the dispersion values, we need the distributions of transformed variables.

The problem of identification of failure rate behavior of increasing convex (concave) transformation of random variable based on distributional properties of the variable is an unexplored one.

In this paper, we consider increasing convex (concave) Total Time on Test (ICXTTT (ICVTTT)) transform of a lifetime random variable. In section II, we provided TTT transform of increasing convex (concave) transformation of the random variable. Some general results about the ageing patterns are given in section III. In section IV, illustrative example is given.

II. Increasing Convex (Concave) TTT transform

In this section, we define ICXTTT (ICVTTT) transform. Given a sample of size n from the non-negative random variable (r.v.) X with distribution *F*, let $X_1 \le X_2 \le \ldots \le X_k \le \ldots \le X_n$ be the order statistics corresponding to the sample. Total time test to the r^{th} failure from distribution F is,

$$T(X_{(r)}) = nX_{(1)} + (n-1)(X_{(2)} - X_{(1)}) + \ldots + (n-r+1)(X_{(r)} - X_{(r-1)})$$
$$= \sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}.$$

Let $H_n^{-1}\left(\frac{r}{n}\right) = \frac{1}{n} T(X_{(r)})$

$$H_n^{-1}\left(\tfrac{r}{n}\right) = \int_0^{F_n^{-1}\left(\frac{r}{n}\right)} (1 - F_n(u))du.$$

The empirical distribution function defined in terms of the order statistics is

$$F_n(u) = \begin{cases} 0, & u < X_{(i)} \\ \frac{i}{n}, & X_{(i)} \le u < X_{(i+1)} \\ 1 & X_{(n)} > u \end{cases}$$

If there exist an inverse function $F_n^{-1}(x) = \inf\{x : F_n(x) \ge u\}$. The fact that $F_n(u) \xrightarrow{a.s} F(u)$ almost surely (a.s.) implies, by Glivenko Cantelli Theorem,

$$\lim_{\substack{\frac{r}{n} \to t \\ n \to \infty}} \int_0^{F_0^{-1}\left(\frac{r}{n}\right)} (1 - F_n(u))du = \int_0^{F^{-1}(t)} (1 - F(u))du$$

uniformly in $t \in [0,1]$. Barlow and Campo (1975) defined TTT transform of F as

$$H_F^{-1}(t) = \int_0^{F^{-1}(t)} (1 - F(u))du \tag{1}$$

Let $g(x)$ be an increasing convex (concave) function. Let $G(x)$ be the distribution function of $g(x)$. Total observed values of transformed variables $g(X)$ under type 2 censored scheme is

$$Tg(X_{(r)}) = ng(X_{(1)}) + (n-1)(g(X_{(2)}) - g(X_{(1)})) + \ldots + (n-r+1)(g(X_{(r)}) - g(X_{(r-1)}))$$
$$= \sum_{i=1}^{r} g(X_{(i)}) + (n-r)g(X_{(r)}).$$

For $g(x)$, define

$$(H_n^{-1})g\left(\tfrac{r}{n}\right) = \int_0^{H_n^{-1}\left(\frac{r}{n}\right)} (1 - H_n(w))dw = \int_0^{g\left(F_n^{-1}\left(\frac{r}{n}\right)\right)} (1 - F_n(u))du$$

where

$$H_n(u) = \begin{cases} 0, & g(u) < g(X_{(i)}) \\ \frac{i}{n}, & g(X_{(i)}) \le g(u) < g(X_{(i+1)}) \\ 1 & g(X_{(n)}) > g(u) \end{cases}$$

$H_n^{-1}(x) = \inf\{x : H_n(x) \ge g(u)\}$ and the fact that $F_n(u) \xrightarrow{a.s} F(u)$ implies, $g(F_n(u)) \xrightarrow{a.s} g(F(u))$, then by Glivenko Cantelli Theorem,

$$\lim_{\substack{\frac{r}{n}\to t \\ n\to\infty}} \int_0^{g\left(F^{-1}\left(\frac{r}{n}\right)\right)} \left(1 - F_n(u)\right)du = \int_0^{g\left(F^{-1}(t)\right)} \left(1 - F(u)\right)du$$

We define TTT of $g(x)$ as

$$\left(H_F^{-1}\right)g(t) = \int_0^{g\left(F^{-1}(t)\right)} \left(1 - F(u)\right)du \quad t \in [0,1]. \tag{2}$$

Then,

$$\frac{d}{dt}\left(H_F^{-1}\right)g(t) = \frac{d}{dt}\int_0^{g\left(F^{-1}(t)\right)} \left(1 - F(u)\right)du$$

$$= \left[1 - \int_0^{g\left(F^{-1}(t)\right)} f(u)du\right] g'\left(F^{-1}(t)\right)\frac{d}{dt}F^{-1}(t)$$

$$= \left[1 - \int_0^{g\left(F^{-1}(t)\right)} f(u)du\right] g'\left(F^{-1}(t)\right)\frac{\frac{d}{dt}H_F^{-1}(t)}{1-t}$$

$$\frac{d}{dt}\left(H_F^{-1}\right)g(t)\Big|_{t=F(x)} = \left[1 - \int_0^{g(x)} f(u)du\right]\frac{g'(x)}{\overline{F}(x)r(x)}.$$

That is,

$$\frac{d}{dt}\left(H_F^{-1}\right)g(t)\Big|_{t=F(x)} = \frac{\overline{F}(g(x))}{\overline{F}(x)} \cdot \frac{g'(x)}{r(x)} \tag{3}$$

Note that $H_F g(\cdot)$ (the inverse of $H_F^{-1} g(\cdot)$) is a distribution with support on $[0, \mu]$,

$$\left(H_F^{-1}\right)g(1) = \int_0^{g\left(F^{-1}(1)\right)} \left(1 - F(u)\right)du = \mu.$$

It is easy to verify that the scaled TTT transform $\frac{\left(H_F^{-1}\right)g(t)}{\left(H_F^{-1}\right)g(1)}$ is continuous increasing function on $[0,1]$. Let $g(x) = x^2$, then

$$\left(H_F^{-1}\right)^2(t) = \int_0^{\left(F^{-1}(t)\right)^2} \left(1 - F(u)\right)du \qquad t \in [0,1] \tag{4}$$

$$\frac{d}{dt}\left(H_F^{-1}\right)^2(t) = \frac{d}{dt}\int_0^{\left(F^{-1}(t)\right)^2} \left(1 - F(u)\right)du$$

$$= \left[1 - \int_0^{\left(F^{-1}(t)\right)^2} f(u)du\right] 2F^{-1}(t)\frac{\frac{d}{dt}H_F^{-1}(t)}{1-t}$$

$$\frac{d}{dt}\left(H_F^{-1}\right)^2(t)\Big|_{t=F(x)} = \left[1 - \int_0^{x^2} f(u)du\right]\frac{2x}{\overline{F}(x)r(x)}.$$

That is,

$$\frac{d}{dt}\left(H_F^{-1}\right)^2(t)\Big|_{t=F(x)} = \frac{\overline{F}(x^2)}{\overline{F}(x)} \cdot \frac{2x}{r(x)} \tag{5}$$

Clearly $\frac{\overline{F}(x^2)}{\overline{F}(x)}$ and x are increasing functions. So that, if $r(x)$ is decreasing, then $\frac{d}{dt}\left(H_F^{-1}\right)^2(t)\Big|_{t=F(x)}$ is increasing. On the other hand, if $r(x)$ is increasing, then $\frac{d}{dt}\left(H_F^{-1}\right)^2(t)\Big|_{t=F(x)}$ is decreasing if rate of increase of $x\frac{\overline{F}(x^2)}{\overline{F}(x)}$ is smaller than the rate of increase of $r(x)$. Note that

$\left(H_F^2\right)$ (the inverse of $\left(H_F^{-1}\right)^2$) is a distribution with support on $[0, \mu]$.

$$\left(H_F^{-1}\right)^2(1) = \int_0^{\left(F^{-1}(1)\right)^2} (1 - F(u))du = \mu.$$

It is easy to verify that the scaled TTT transform $\frac{\left(H_F^{-1}\right)^2(t)}{\left(H_F^{-1}\right)^2(1)}$ is continuous increasing function on $[0,1]$.

Example:- Let $F(x) = 1 - e^{-\frac{x}{\theta}}, x > 0, \theta > 0$ be the distribution function of Exponential distribution with mean θ. Then

$$\left(H_F^{-1}\right)^2(t) = \int_0^{\left(F^{-1}(t)\right)^2} (1 - F(x))dx$$

$$= \int_0^{\left(F^{-1}(t)\right)^2} e^{-\frac{x}{\theta}}dx$$

$$= \int_0^{\left(F^{-1}(t)\right)^2} \theta dF(x)$$

$$\left(H_F^{-1}\right)^2(t)\Big|_{t=F(x)} = \theta F\left(\left(F^{-1}(t)\right)^2\right)$$

$\left(H_F^{-1}\right)^2(1)$ is the mean of exponential distribution. Therefore, $\frac{\left(H_F^{-1}\right)^2(t)}{\left(H_F^{-1}\right)^2(1)} = F\left(\left(F^{-1}(t)\right)^2\right)$.

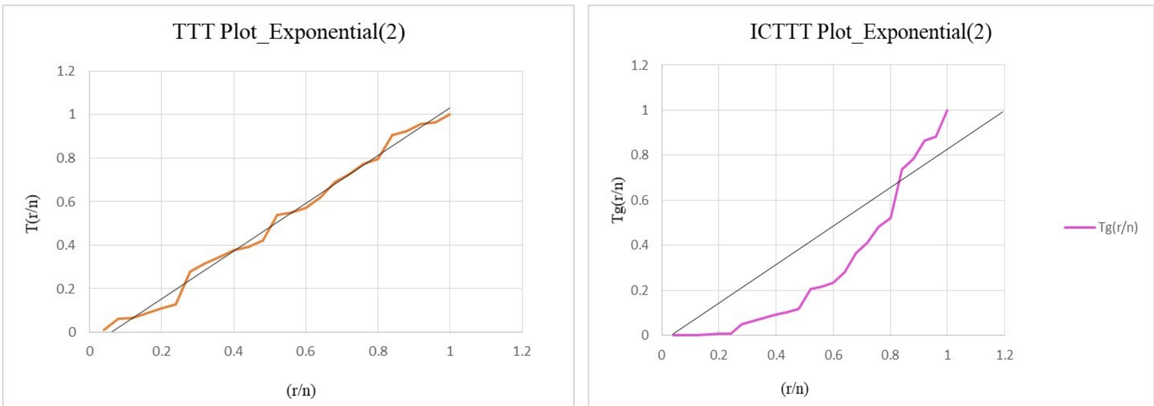


Figure 1: *TTT plot (top) and ICXTTT plot (bottom) for the Exponential Simulated data with parameter* $\theta = 2$.

In this Figure 1, it shows that the TTT-plot of Exponential data set indicates constant failure rate, but ICXTTT-plot for Exponential data set indicates that transformed data follows the bathtub shape failure rate pattern. It is clear that, square of exponential random variable follows some decreasing failure rate model.

III. Ageing Properties

We prove some general results about the ageing patterns of function $g(X)$ using $\frac{\left(H_F^{-1}\right)g(t)}{\left(H_F^{-1}\right)g(1)}$, which is based on the failure rate function $r(x)$ of X.

Proposition 1. G is IFR $\Rightarrow \frac{(H_F^{-1})g(t)}{(H_F^{-1})g(1)}$ is concave in $t \in [0,1]$ if rate of increase of $g'(x)\frac{\overline{F}(g(x))}{\overline{F}(x)}$ is smaller than the rate of increase of $r(x)$. G is DFR $\Rightarrow \frac{(H_F^{-1})g(t)}{(H_F^{-1})g(1)}$ is convex in $t \in [0,1]$.

Proof. G is IFR implies $\frac{d}{dt}(H_F^{-1})g(t)$ is decreasing in $t \in [0,1]$, if rate of increase of $g'(x)\frac{\overline{F}(g(x))}{\overline{F}(x)}$ is smaller than the rate of increase of $r(x)$. $\Rightarrow \frac{(H_F^{-1})g(t)}{(H_F^{-1})g(1)}$ is concave in $t \in [0,1]$. Similarly, G is DFR implies $\frac{d}{dt}(H_F^{-1})g(t)$ is increasing in $t \in [0,1]$, $r(x)$ is decreasing in $x \geq 0$.

$\Rightarrow \frac{(H_F^{-1})g(t)}{(H_F^{-1})g(1)}$ is convex in $t \in [0,1]$, if $r(x)$ is decreasing in $x \geq 0$.

Proposition 2. Let X has distribution F and $Y = g(X)$ has distribution $G(y)$. G is IFRA (DFRA) $\Rightarrow \frac{(H_F^{-1})g(t)}{F(g(F^{-1}(t)))(H_F^{-1})g(1)}$ is decreasing (increasing) in $t \in [0,1]$.

Proof. Let $Y = g(X)$ and $G(y)$ be the distribution function of Y. G is IFRA $\Rightarrow \frac{1}{y}\int_0^y r(u)du$ is increasing in $y \geq 0$.

Let $T(y) = \int_0^y \overline{G}(u)du$. $\frac{T(y)}{y}$ is decreasing in $y \geq 0$, since it is an average of the decreasing function $\overline{G}(y)$.

Then, $\frac{\int_0^y r(u)dT(u)}{T(y)}$ is increasing in $y \geq 0$.

Hence $\frac{G(y)}{\int_0^y \overline{G}(u)du}$ is increasing in $y \geq 0$ and $\frac{\int_0^y \overline{G}(u)du}{G(y)}$ is decreasing in, $y \geq 0$.

Then,

$$\frac{\int_0^y \overline{G}(u)du}{G(y)} = \frac{\int_0^{g(x)} \overline{F}(w)dw}{F(g(x))} \text{ is decreasing in } x \geq 0,$$

since $\overline{G}(u) = P(g(X) > u) = P(X > w) = \overline{F}(w)$ or some w corresponding to u.

$$\frac{\int_0^{g(x)} \overline{F}(w)dw}{F(g(x))} \text{ is decreasing in } x \geq 0.$$

Now make the change of variables $t = F(x)$ and $x = F^{-1}(t)$ and finally we have

$$\frac{\int_0^{g\left(F^{-1}(t)\right)}\overline{F}(w)dw}{F\left(g\left(F^{-1}(t)\right)\right)}=\frac{\left(H_F^{-1}\right)g(t)}{F\left(g\left(F^{-1}(t)\right)\right)}\text{ is decreasing in } t\in[0,1].$$

$$\Rightarrow\frac{\left(H_F^{-1}\right)g(t)}{F\left(g\left(F^{-1}(t)\right)\right)\left(H_F^{-1}\right)g(1)}\text{ is decreasing in } t\in[0,1].$$

Similarly, for G is DFRA,

$$\frac{\left(H_F^{-1}\right)g(t)}{F\left(g\left(F^{-1}(t)\right)\right)}\text{ is increasing in } t\in[0,1].$$

$$\Rightarrow\frac{\left(H_F^{-1}\right)g(t)}{F\left(g\left(F^{-1}(t)\right)\right)\left(H_F^{-1}\right)g(1)}\text{ is increasing in } t\in[0,1].$$

IV. Increasing Convex (Concave) TTT transform order

Let X and Y be two non-negative random variables with distributions F and H respectively. Clearly $X\leq_{ttt}Y$ if and only if

$$\int_0^{F^{-1}(t)}(1-F(u))du\leq\int_0^{H^{-1}(t)}(1-H(u))du \quad t\in[0,1]. \tag{6}$$

A sufficient condition for the order $\leq_{ttt}$ is the usual stochastic order:

$$X\leq_{st}Y\Rightarrow X\leq_{ttt}Y$$

where $X\leq_{st}Y$ means that $\overline{F}(x)\leq\overline{H}(x),\forall x\in R$ (see, Shaked and Shanthikumar (2007)).

Let X and Y be two random variables such that $Tg\left(X_{(n)}\right)\leq Tg\left(Y_{(n)}\right)$ for all convex functions $g:R\to R$ and all samples of size n. Then X is smaller than Y in some stochastic sense, since $\frac{1}{n}Tg\left(X_{(n)}\right)$ is average of total observed convex (concave) transformed time of a test.

Let X and Y be two non-negative random variables with absolutely continuous distribution functions F and H respectively. If

$$\left(H_F^{-1}\right)g(t)\leq\left(H_H^{-1}\right)g(t),\forall t\in[0,1]$$

where g is an increasing convex function, then X is smaller than Y in increasing convex TTT order (denoted as $X\leq_{icxttt}Y$).

Now we prove the relationship of ICXTTT (ICVTTT) transform orders to stochastic orders.

Theorem 1. Let X and Y be two non-negative random variables having absolutely continuous distribution functions F and G respectively. Then $X\leq_{st}Y\Rightarrow X\leq_{icxttt}Y$.

Proof. Let g be the convex (concave) function $g:R\to R$. Since,

$X\leq_{st}Y,\quad g\left(F^{-1}(t)\right)\leq g\left(G^{-1}(t)\right)$ for all $t\in[0,1]$.

Hence,

$$\int_0^{g\left(F^{-1}(t)\right)}(1-F(u))du\leq\int_0^{g\left(G^{-1}(t)\right)}(1-G(u))du \text{ for all } t\in[0,1]$$

Then, $X\leq_{st}Y\Rightarrow X\leq_{icxttt}Y$.

V. Examples

Usually the TTT plot is drawn by plotting $T\left(\frac{r}{n}\right)=\frac{\sum_{i=1}^{r}X_{(i)}+(n-r)X_{(r)}}{\sum_{i=1}^{r}X_{(i)}}$ against $\frac{r}{n}$, where $i=1,2,\ldots,r$ and $r=1,2,\ldots,n.$ A TTT curve may be concave (convex) if corresponding distribution is IFR (DFR) distribution. A concave (convex) and then convex (concave) shape for TTT curve occurs, if the distribution is a bathtub (upside down bathtub) failure rate model. Finally, a TTT curve is straight line if the distribution is exponential.

Then the ICXTTT plot is drawn by plotting $Tg\left(\frac{r}{n}\right)=\frac{\sum_{i=1}^{r}g\left(X_{(i)}\right)+(n-r)g\left(X_{(r)}\right)}{\sum_{i=1}^{r}g\left(X_{(i)}\right)}$ against $\frac{r}{n}$, where $i=1,2,\ldots,r$ and $r=1,2,\ldots,n.$

The scaled TTT-transform, which is independent of scale, is defined for values of t with $0\leq t\leq 1$ and hence the transformed values are in $[0,1]$. Figure 2 and 3 shows scaled TTT-transforms and scaled ICXTTT-transforms of Aarset bathtub shaped failure rate data and Weibul simulated data.

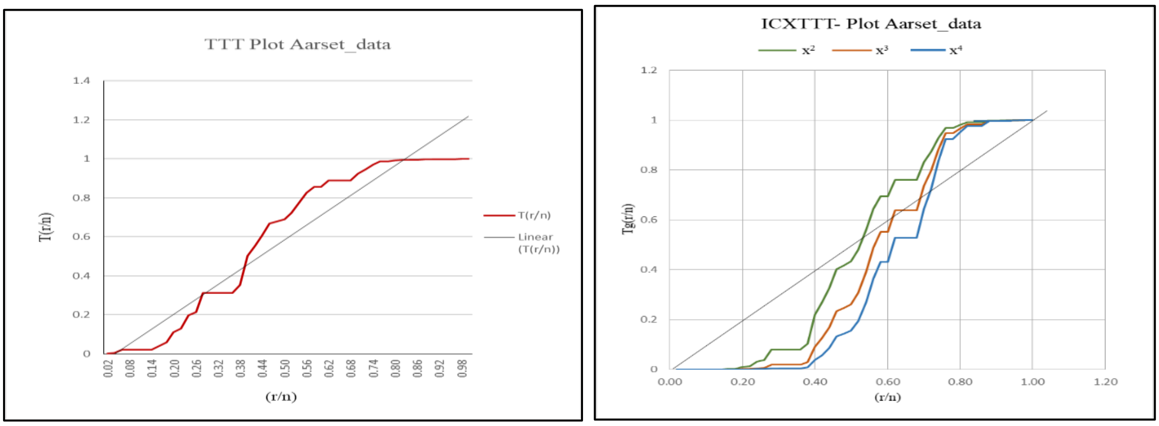
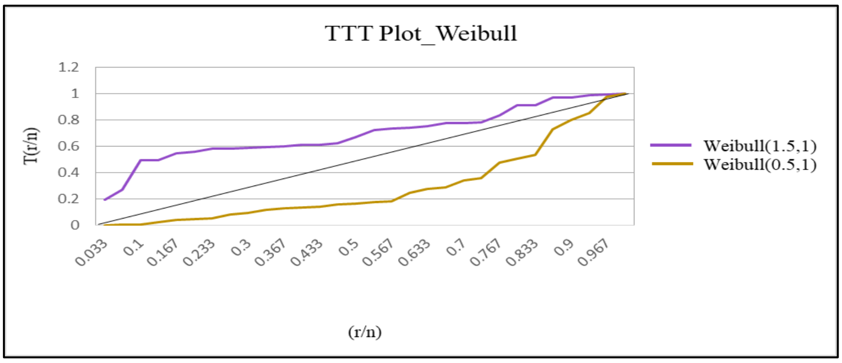


Figure 2. *TTT plot (top) and ICXTTT plot (bottom) for the Aarset data.*

In Figure 2, it shows that the data are known to have a bathtub-shaped failure rate as depicted in TTT plot and ICXTTT-plot.



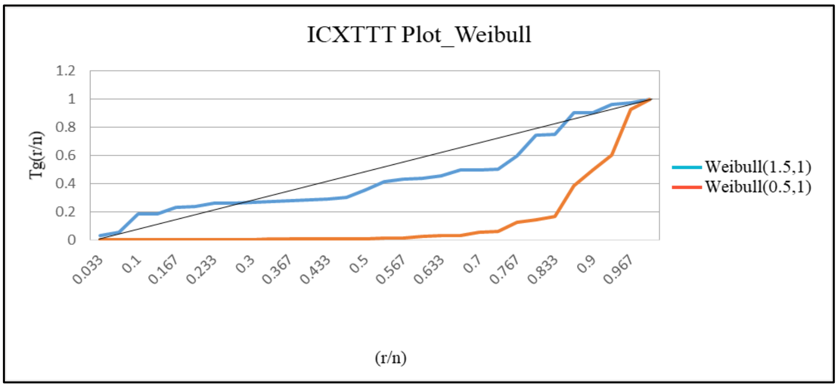


Figure 3. *TTT plot (top) and ICXTTT plot (bottom) for the Weibull Simulated data with parameter* $\alpha = 1.5, 0.5$ *and* $\lambda = 1$ *respectively.*

In Figure 3, it shows that the TTT-plot of $W(\alpha = 1.5, \lambda = 1)$ data set indicate IFR distribution, since it is concave, but ICXTTT-plot for $W(\alpha = 1.5, \lambda = 1)$ data set indicate an inverse bathtub shaped failure rate pattern for the failure rate. The TTT-plot of $W(\alpha = 0.5, \lambda = 1)$ shows $F(t)$ has DFR, since the plot is convex (see Barlow and Campo, 1975) and ICXTTT-plot based on a $W(\alpha = 0.5, \lambda = 1)$ is convex, which indicate a DFR distribution for the transformed data.

VI. Conclusion

This paper considered increasing convex (concave) TTT transform. Identification of the failure rate model of functions of random variables is discussed. Some properties of the transforms are derived with examples.

References

[1] Aarset, M. V. (1985). How to Identify a Bathtub Hazard Rate, *IEEE Transactions on Reliability*, 36(1): 106-108.

[2] Barlow, R. E. and Campo, R. A. (1975). Total time on test processes and applications to failure data analysis, *Reliability and Fault Tree Analysis* (Eds: Barlow, R. E., Fussel, R., and Singpurwalla, N. D.), SIAM, Philadelphia, 451–481.

[3] Chacko, V. M. (2010). On Total Time on Test Transform Order, *Reliability Theory and Applications*, 19(4): 50-55.

[4] Franco-Pereira, A. M. and Shaked, M. (2013). The total time on test transform and the decreasing percentile residual life aging notion, *Statistical Methodology*, http://dx.doi.org/10.1016/j.stamet.2013.09.003.

[5] Gupta, R. C. and Michalek, J. E. (1985). Determination of Reliability Functions by the TTT Transform, *IEEE Transactions on Reliability*, 34(2): 175-176.

[6] Nair, N. U., Sankaran, P. G. and Vinesh Kumar, B. (2008). TTT transforms of order n and their implications in reliability analysis, *Journal of Applied Probability*, 46: 1127-1139.

[7] Nair, N. U. and Sankaran, P. G. (2013). Some new applications of the total time on test transforms. *Statistical Methodology*, 10: 93-102.

[8] Shaked, M. and Shanthikumar, J. G. Stochastic Orders, Springer, New York, 2007.

[9] Vera, F. and Lynch, J. (2005). K-mart stochastic modeling using iterated total time on test transforms. *Modern Statistical and Mathematical Methods in Reliability*, World Scientific, Singapore, 395-409.