Some Contributions to Reliability Theory and Survival Analysis

Submitted to

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for the award of the Degree of

Doctor of Philosophy
in
Statistics

By

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Under the guidance of

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DECLARATION

I, **Beenu Thomas**, hereby declare that the thesis entitled, ***"Some Contributions to Reliability Theory and Survival Analysis"***, submitted to the **University of Calicut** in partial fulfillment of the requirements for the award of the Degree of ***Doctor of Philosophy in Statistics*** is a bonafide research work done by me under the supervision and guidance of **Dr. V M Chacko**, Associate Professor, Department of Statistics, St. Thomas College (Autonomous), Thrissur, Kerala.

I further declare that this thesis has not previously formed the basis of any degree, diploma or any other similar title.

Thrissur **Beenu Thomas**

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CERTIFICATE

This is to certify that the thesis titled ***"Some Contributions to Reliability Theory and Survival Analysis"*** submitted by **Beenu Thomas** to the **University of Calicut** in partial fulfillment of the requirements for the award of the Degree of ***Doctor of Philosophy in Statistics*** is a record of original research work carried out by her under my supervision. The content of this thesis, in full or in parts, has not been submitted by any other candidate to any other University for the award of any degree or diploma.

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Dr. V M Chacko

Research Supervisor

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I hereby certify that, this is the revised version of the thesis entitled ***"Some Contributions to Reliability Theory and Survival Analysis"*** submitted by **Beenu Thomas**, under my guidance, after incorporating the necessary corrections/suggestions made by the adjudicators. I also certify that the contents in the thesis and the soft copy are one and the same.

Thrissur | **Dr. V M Chacko**
24 November, 2023 | Research Supervisor

ABSTRACT

Reliability and survival analysis are widely used in systems engineering and clinical trial experiments. Innovations in reliability methods enhance the safety and reliability of complex technological systems, like engineering systems and offshore pipelines. Survival analysis is generally defined as a set of methods for analyzing data where the outcome variable is the time until an event of interest occurs. It can be a death, illness, failure, or completion of a mission. The time to event or survival time can be measured in days, weeks, years, etc.

The notion of ageing plays a significant role in reliability theory. Ageing has a direct impact on the failure rate function behavior. They can be used in maintenance planning, replacement planning, resource allocation, etc. The increasing failure rate (IFR), decreasing failure rate (DFR), and bathtub failure rate (BFR) distributions are widely used in reliability engineering.

Birnbaum and Saunders (1969) proposed a failure time distribution for fatigue failure caused by cyclic loading. It was also assumed that the failure was due to the development and growth of a dominant crack. Univariate Birnbaum-Saunders (BS) distribution has been used to analyze positively skewed lifetime data. It has received a lot of attention in the last few years. One of the most widely used approaches to reliability estimation is the well-known stress-strength (SS) model. Several physics and engineering applications use this model, including strength failure and the collapse of systems.

The step-stress model is a widely accepted accelerated life testing model. This accelerated testing reduces the time to failure. The data collected from such an accelerated test may then be extrapolated to estimate the underlying distribution of failure times under normal conditions. The step-stress experiment is a special case of accelerated testing that allows for different conditions at various intermediate stages of the experiment.

The thesis entitled *Some Contributions to Reliability Theory and Survival Analysis* has been arranged into 7 chapters. Chapter 1 introduces the basic concepts and definitions to the reader. Also, an extensive review of related literature has

been presented. The summaries of the investigation of study are stated below.

A New Generalization to the DUS Transformation and its Applications

Kumar et al. (2015) proposed a method called DUS transformation to obtain a new parsimonious class of distributions that does not require additional parameters but is found to be a better fit than the baseline distributions. Many of the engineering systems are parallel in nature. If a parallel system has n components and each of the components is well fitted as DUS transformation of any baseline lifetime distribution, then we have to use power generalization. A new transformation called the power generalized DUS transformation (PGDUS) is introduced and proposed new distributions with exponential, Weibull, and Lomax distributions as baseline distributions. Several mathematical properties are examined, including moments, moment generating functions, characteristic functions, quantile functions, order statistics, etc. The maximum likelihood approach to parameter estimation is discussed. Based on several real data sets, the proposed distributions are compared with some of the other failure rate lifetime distributions.

Exponential-Gamma $(\mathbf{3}, \theta)$ Distribution: A Bathtub Shaped Failure Rate Model

Mixture distributions are useful when dealing with lifetime data analysis. A BFR distribution called the exponential-gamma $(3, \theta)$ distribution is examined in detail. An investigation is conducted on the shapes of the probability density function (pdf) and the failure rate. Various properties are discussed, including moments, moment generating functions, characteristic functions, quantile functions, and entropy. Distributions for the minimum and maximum are obtained. In order to estimate the parameters of the distribution, the maximum likelihood method is used. Through the use of a simulation study, biases and mean squared errors are analyzed for maximum likelihood estimators (MLEs). A comparison between the proposed lifetime distribution and other lifetime distributions is conducted using real-world data sets.

Generalized ν-Birnbaum Saunders Distribution

Birnbaum-Saunders (BS) distribution is widely used in reliability literature. In order to analyse data and incorporate flexibility in the form of the distribution, we need to consider the distribution with shape parameters. The generalization of the BS distribution, called the ν-Birnbaum Saunders distribution, is discussed. A number of intriguing and relevant characteristics are investigated in depth. The maximum likelihood principle is employed to estimate the parameters of the univariate ν-BS distribution. To obtain interval estimates, we use asymptotic confidence intervals. Both estimation methodologies have been thoroughly explored in an extensive simulation study. Based on these estimators, the probability coverage of confidence intervals has been evaluated. Real-life applications are provided with three different datasets and compared with the univariate BS distribution.

Inference for $\mathbf{R = P[Y < X]}$ based on the Exponential-Gamma $(\mathbf{3}, \lambda)$ Distribution

When a manufacturer has knowledge of the mechanical reliability of the design through the stress-strength model before production, they can significantly reduce their production costs. A system's longevity is determined by its inherent strength and external stress. A discussion of the stress-strength reliability of the exponential-gamma $(3, \theta)$ distribution is presented. An assessment of the reliability estimation of the single-component model is provided. A simulation study is used to demonstrate how well the MLEs perform. A data application is presented using real data sets to demonstrate how the distribution performs in real-life situations.

A Simple Step-Stress Analysis of Type II Gumbel Distribution

Step-stress reliability analysis is useful in industrial engineering. A simple step-stress accelerated life-testing analysis is provided, incorporating Type-II censoring. Here, a flexible failure rate-based approach to Type II Gumbel distribution for SSALT analysis is considered. The baseline distribution of experimental units at each stress level follows the Type II Gumbel distribution. The MLEs for the model parameters are derived.

Lastly, Chapter 7 presents the concluding remarks of the thesis and proposals for future work.

I dedicate this work

to my parents who paved the way for me during their lifetime

to my beloved husband Joel for his unconditional love, care, and support.

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I have been fortunate to have Dr. V. M. Chacko, Associate Professor, Department of Statistics, St. Thomas College (Autonomous), Thrissur as my guide and supervisor. His flexibility in scheduling, gentle encouragement, and relaxed demeanor made for a positive working relationship and was the impetus for me to finish. He has been a strong and supportive advisor to me throughout my research work.

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Beenu Thomas

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CHAPTER 1

Introduction and Review of Literature

1.1 Introduction

A large number of statistical applications have been seen in reliability theory and survival analysis using various lifetime distributions. However, existing distributions do have limitations when applied to certain data due to their inability to provide a proper fit. To obtain precise probability results, it is necessary to use more appropriate statistical lifetime distributions. Through the study of new statistical distributions for existing data, a fresh perspective can be provided on reliability theory and survival analysis by introducing more flexible models and an in-depth analysis of their characteristics and properties. If a better distribution is available than existing distributions, a researcher would have selected the better model. Fitness for the given data is one of the criteria for selecting the better model. If a distribution is more fitting to the given data, researchers have to leave the existing distribution and use a new model.

In reliability and survival analysis, the general structure of the system could be series or parallel. Therefore, it is essential to investigate the distributional properties of series or parallel systems when components are distributed according to any lifetime distribution available in the literature. The DUS transformation of a lifetime distribution was introduced by Kumar et al. (2015). The name 'DUS' is given according to the first letter of the author's name: Kumar, D., Singh, U., and Singh, S. K.; see Kumar et al. (2015). A DUS transformation provides a new distribution, using some baseline lifetime distribution, without adding new parameters. As far as inference is concerned, the DUS transformation is a good approach since it does not add any new parameters. Increasing the number of parameters leads to complexity in estimation and a lack of consistency. Suppose that the lifetimes of the components are distributed as a DUS-transformed distribution. What would be the distribution of a parallel system consisting of such components? This question has to be answered through research. Several statistical distributions are available, like exponential, Weibull, Lomax, Gamma, etc., for reliability and survival analysis. Also, a large number of transformations are available in the literature, like the DUS transformation, the Kavya-Manoharan (KM) transformation, etc. They are used in reliability analysis. Their performance in engineering systems like series systems and parallel systems is to be investigated in detail.

Mixture distributions are useful when dealing with lifetime data analysis. When a new component switches on for the first time, it may fail at the same moment, or it may fail while working due to overvoltage, jerking, or any such shocks, or it may fail due to degradation. So failure due to random shocks or random failure can be modelled using an exponential distribution. Since there is a chance of failure due to the degradation of components, such failure time may be distributed as any other lifetime distribution if it is fitted to the data. When both situations happen to a group of components installed for a mission, the researcher should use a mixture distribution. Investigation of certain mixtures is essential to knowing how they behave in reliability and survival analysis. Moreover, the estimation of parameters also has to be studied in detail.

Stress-strength reliability analysis using statistical distributions is very important in the fields of engineering, quality control, and medicine. According to stress-strength reliability, a network's ability to withstand stress is the measure of its reliability or safety. It is quite desirable to estimate stress-strength reliability (R) using more appropriate models, especially mixture models. Investigation of the statistical properties of reliability estimators is also imperative when dealing with inferential procedures. This stress-strength parameter, R, measures the performance of systems in the context of mechanical reliability. Apart from the inferential information, R provides the probability of system failure if the system fails while the applied stress exceeds its strength. The field of reliability engineering places a great deal of emphasis on accelerated life testing, in particular, step-stress accelerated life testing. The inference of step stress models based on various statistical distributions is useful for accelerated life testing.

Birnbaum-Saunders (BS) distribution is one of the important distributions when dealing with fatigue failure. Several generalizations of the BS distribution are available in the literature. The usefulness of its generalizations has to be explored more. Due to the complexity of the model, the estimation of parameters becomes more complicated. A detailed study on the estimation of parameters and the estimation of the confidence interval is required for the generalized BS models.

In mathematical statistics, reliability theory and survival analysis examine a specific class of time-to-event random variables. Methods for evaluating and forecasting a product's successful operation or performance are discussed in reliability analysis. Due to rapid advances in technology, the development of highly sophisticated products, intense global competition, and rising customer expectations, manufacturers are under increased pressure to produce high-quality, reliable products. Customers expect engineering products to be reliable and safe when they purchase them. For a substantial time period, systems like vehicles, machines, telecommunication devices, power generation systems, and so forth should be capable of performing their intended functions under normal operating conditions.

In technical terms, reliability can be defined as a system's ability to perform its intended mission within a specified time under normal operating conditions. Enhancing the reliability of products is one of the most important aspects of improving the quality of products. Methods of reliability have been developed and applied to enhance the safety and reliability of complex technological systems, such as nuclear power systems, chemical plants, space systems, hazardous waste facilities, and offshore installations. Survival analysis is generally defined as a set of methods for analyzing data where the outcome variable is the time until an event of interest occurs. It can be death, the onset of a disease, failure, or the completion of a mission. The time to event or survival time may be measured in days, hours, weeks, years, etc.

Reliability theory and survival analysis mainly focus on positive random variables, often called lifetimes. The distribution function provides a complete characterization of a lifetime random variable. How effectively a lifetime is understood depends on failure, death, or some other "end event".

By utilizing appropriate statistical distributions for modeling a lifetime, reliability calculations can be simplified and made more accurate. By analyzing the distribution of the lifetime of the system, reliability, maintenance, and replacement measures can be planned accordingly. An analysis of reliability requires the identification of the failure rate model to ascertain which distribution is suitable for the available data. The distributions available in the literature are often insufficient to explain the distributional characteristics and reliability analysis of the data given. Thus, researchers are constantly forced to come up with more suitable distributions for the data that they are provided with.

1.2 Basic Concepts

1.2.1 Reliability or Survival Function

When it comes to probability, an object's reliability is defined as the probability that it will carry out its intended function for a predetermined timeframe while working under conventional environmental conditions. This probability is referred to as the survival probability in survival analysis.

The probability that an object will survive to time t is defined by the reliability function (or survival function) of the lifetime variable, T, indicated by $R(t)$ (or $S(t)$) where,

$$R(t) = S(t) = P[T > t] = \int_t^\infty f(x)dx.$$

Furthermore, $R(t) = 1 - F(t)$, where $F(.)$ is the cumulative density function (CDF). This feature is often used in reliability analysis.

1.2.2 Failure Rate or Hazard Rate Function

The failure rate function, or hazard rate function (HRF), denoted by $h(t)$, can be considered as the probability that an object will fail in the interval $(t, t + \Delta t)$ for small Δt, assuming that it hasn't already failed before t. It is referred to as the ratio of the probability density function (PDF) to the survival function and is given by

$$h(t) = \lim_{\Delta t \to \infty} \frac{P[t < T \leq t + \Delta t | T > t]}{\Delta t} = \frac{f(t)}{R(t)}.$$

Most applications will result in a reduction in the lifetime of the system beyond the specified age, which is a reasonable assumption. In other words, the survival rate of a system decreases with age. When units exhibit this behavior, their life distributions are considered positive ageing distributions.

Ageing is an important concept in understanding which real-life distributions are suitable for reliability data analysis. An ageing concept largely describes how a device ages with time. Though in most cases, ageing has an adverse effect on a product, there are some other cases in which ageing is beneficial. Ageing has a direct impact on the behavior of the HRF. They can be used in maintenance planning, replacement planning, resource allocation, etc. Using the HRF, one can conveniently define ageing.
The lifetime distributions can be classified into the following categories based on the HRF.

- **Constant Failure Rate (CFR)**
  A constant failure rate is observed during the midlife stage because failures mostly occur as a result of external factors or random failures. In most cases, this period is referred to as the "working life" of a system or component because most systems spend the majority of their lifetimes in this stage.

- **Increasing Failure Rate (IFR)**
  The concept of IFR is intuitively based on the deterioration of components. In the context of failure rates, an IFR is one in which $h(x)$ increases monotonically over x or equivalently, when $-logF(x)$ is convex.

- **Decreasing Failure Rate (DFR)**
  A DFR refers to a process where the probability of an event occurring in the future declines with time. When earlier failures are removed or corrected, there is a DFR during the "infant mortality" period. This corresponds to a situation where the HRF is descending. An improvement in DFR occurs as the system ages. In the context of failure rates, an DFR is one in which $h(x)$

decreases monotonically over x or equivalently, when $-logF(x)$ is concave.

- **Bathtub Shaped Failure Rate (BFR)**
  If the HRF of F decreases initially and then remains constant for a period, and then increases over time, then corresponding distributions are called BFR distributions. BFR refers to the early life, useful life, and wear-out phases of a component or system.

  In the case of humans, failures typically occur during the early life period as a result of birth defects. Failures occurring during useful life can be referred to as chance failures. As the unit ages, the more likely it is to fail in the wear-out region.

  Lifetime distributions with BFR are an important class of lifetime distributions since the lifetime of electronic, electromechanical, and mechanical products is often modeled using them. According to survival analysis, human life exhibits this pattern.

- **Upside-Down Bathtub Shaped Failure Rate (UBFR)**
  A UBFR distribution is characterized by a failure rate $h(x)$, which increases initially for $x \in (0, x_0)$, then becomes constant, and finally decreases for $x > x_0$.

The characterization of distributions, whether IFR or DFR, or BFR reduces the selection of the model in reliability analysis. The bathtub shape is characteristic of the failure rate curve of many well-designed products and components including the human body. Monotonic ageing concepts are found to be popular among many reliability engineers. However, in many practical applications, the effect of age is initially beneficial, but after a certain period, its age-adverse indication is positive.

1.2.3 Some Statistical Distributions

The lifetime distributions used in this thesis are given below.

Exponential Distribution

The exponential distribution is a continuous distribution related to the length of time between events. The exponential distribution was the first to be widely used. In addition to its simple representation of CDF, PDF, and HRF, as well as its availability of simple statistical methods for data analysis, it is also an effective method for predicting the lifetime of many types of manufactured items. While the constant hazard rate may be useful to some extent when considering the use of this distribution, caution should be exercised when considering its use. This is because inferential procedures may be sensitive to deviations from the exponential distribution when utilizing it. Furthermore, this distribution has a lack of memory property. It has the PDF

$$f(t|\mu) = \mu e^{-\mu t}, \;\; t > 0, \mu > 0.$$

The corresponding CDF and HRF are given, respectively, by

$$F(t|\mu) = 1 - e^{-\mu t}, \;\; t > 0, \mu > 0,$$

and

$$h(t|\mu) = \mu, \;\; \mu > 0.$$

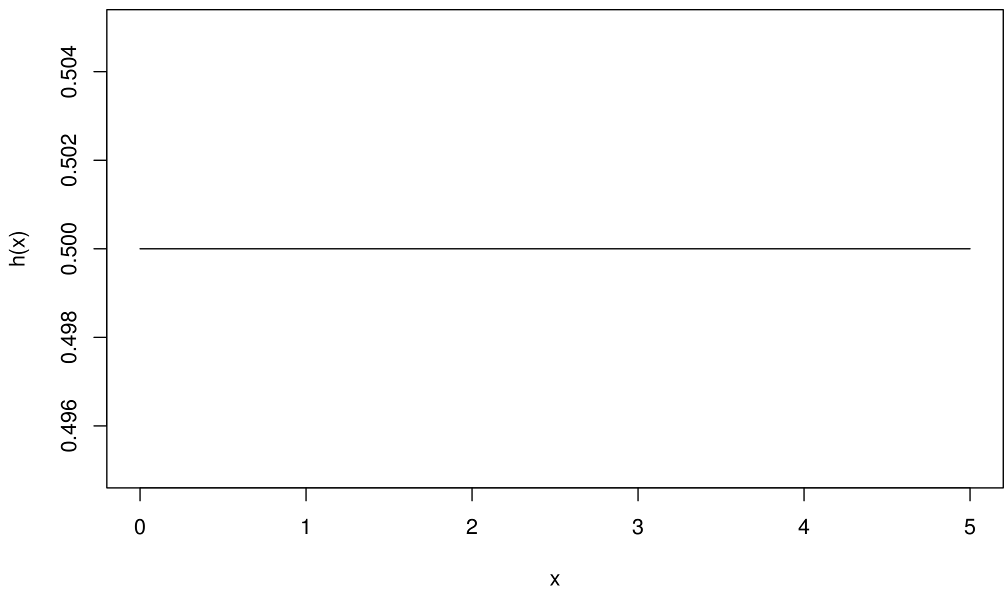


Figure 1.1: Exponential hazard function plot

This distribution has a mean $\frac{1}{\mu}$ and a variance $\frac{1}{\mu^2}$. The standard exponential distribution is defined as the distribution where μ equals 1.

Gamma Distribution

The gamma distribution models the right-skewed data and is one of the most commonly used continuous distributions. As a result of the fame of the exponential distribution, it has become prevalent to use the gamma distribution as a model for the sum of lifetimes with an exponential distribution. In addition to being naturally derived from the convolution of exponential distributions, the gamma life distribution has the disadvantage that it cannot be algebraically treated. It has the PDF

$$f(t|\lambda, \theta) = \frac{\theta^{\lambda}}{\Gamma(\lambda)} t^{\theta-1} e^{-\theta t}, \ t > 0, \lambda > 0, \theta > 0.$$

The CDF is given by

$$F(t|\lambda, \theta) = \frac{\gamma(\lambda, \theta t)}{\Gamma(\lambda)},$$

where $\gamma(\lambda, \theta t)$ is the incomplete gamma function. The mean and variance of gamma

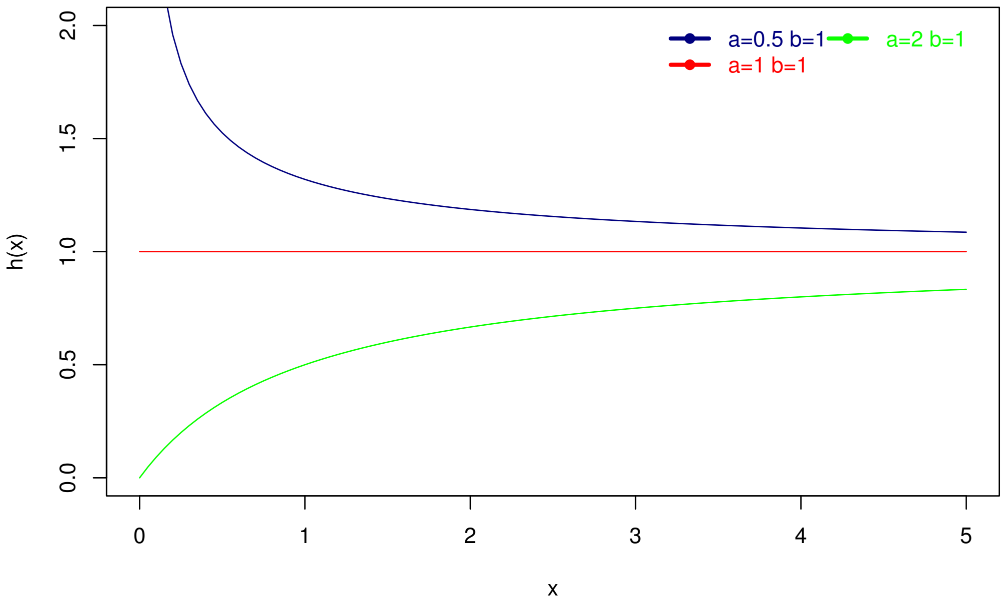


Figure 1.2: Gamma hazard function plot

distribution are $\lambda\theta$ and $\lambda\theta^2$, respectively. The skewness of the gamma distribution only depends on its shape parameter, λ, and it is equal to $2/\sqrt{\lambda}$. In the gamma distribution, HRF is IFR for $\lambda > 1$, DFR for $\lambda < 1$, and CFR for $\lambda = 1$, see Figure 1.2.

Weibull Distribution

Weibull distribution is perhaps the most widely used lifetime distribution. It is convenient to describe different types of hazards using the Weibull distribution, as it is flexible in describing them and mathematically manageable. As a result of its flexibility, the tool is used in a wide range of settings, including quality control, reliability analysis, medical research, and engineering applications. It's PDF is given by

$$f(t|\kappa, \nu) = \kappa\, \nu\, t^{\kappa-1}\, e^{-\nu t^{\kappa}}, t > 0, \kappa > 0, \nu > 0.$$

The corresponding CDF and HRF are given by

$$F(t|\kappa, \nu) = 1 - e^{-\nu t^{\kappa}},\ t > 0,$$

and

$$h(t|\kappa, \nu) = \kappa\, \nu\, t^{\kappa-1},\ t > 0$$

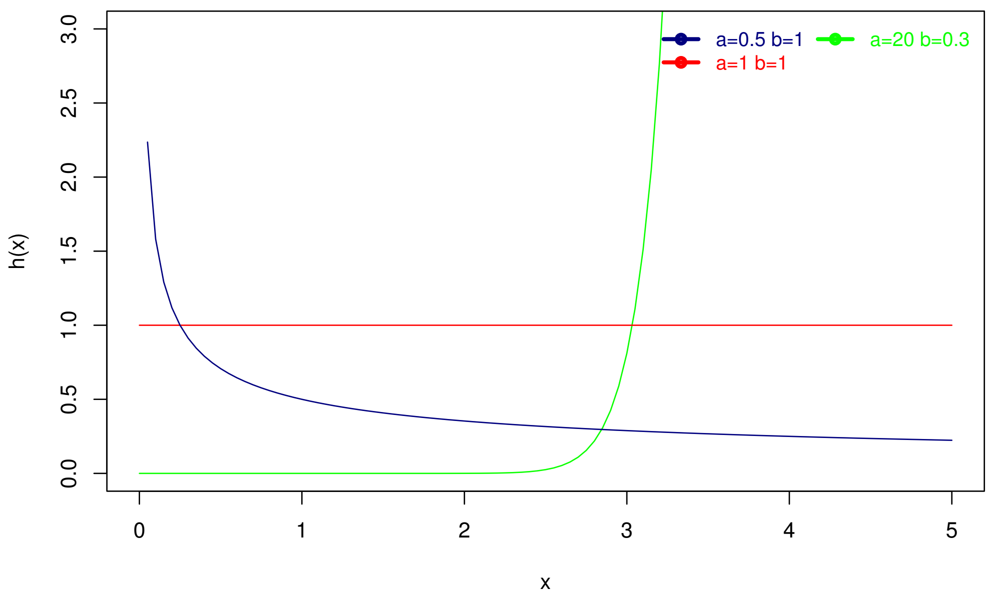


Figure 1.3: Weibull hazard function plot

The mean and variance of this distribution is $\frac{\Gamma(1+1/\kappa)}{\nu^{1/\kappa}}$ and $\frac{1}{\nu^{1/\kappa}}[\Gamma(1+2/\kappa)+\Gamma(1+1/\kappa)^2]$. It may be mentioned here that the Weibull distribution can be positively or negatively skewed depending upon the value of the shape parameter κ. The Weibull distribution demonstrates DFR for $\kappa < 1$, IFR for $\kappa > 1$, and CFR for $\kappa = 1$, see Figure 1.3.

This distribution can be used quite conveniently for censored data as well.

Lomax Distribution

Based on business failure lifetime data, Lomax (1954) developed the Lomax distribution, which has a heavily skewed distribution. Distributions such as this one, a shifted Pareto distribution, are widely used in survival analysis and have many applications in actuarial science, reliability theory, business, network analysis, economics, operations research, medical science, and internet traffic modeling, among others. The PDF of the Lomax distribution has the form

$$f(t|k,\lambda) = \frac{k}{\lambda}\left(1+\frac{t}{k}\right)^{-(\lambda+1)},\ t>0,\ k>0,\ \lambda>0,$$

and the CDF is given by

$$F(t|k,\lambda) = 1-\left(1+\frac{t}{k}\right)^{-\lambda},\ t>0,\ k>0,\ \lambda>0.$$

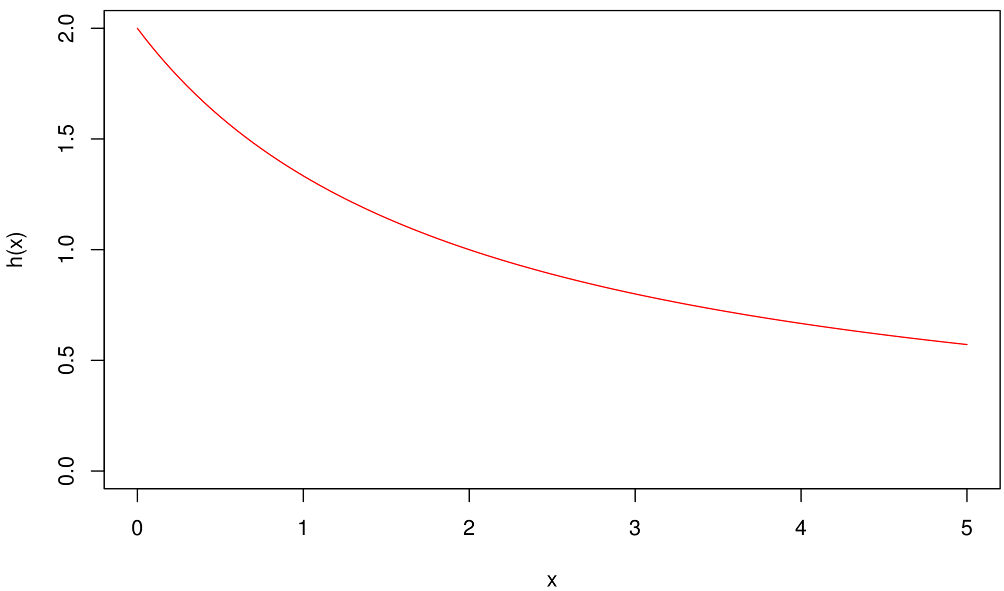


Figure 1.4: Lomax hazard function plot

The mean function is given by

$$E(T) = \frac{k}{\lambda-1}\ for\ \lambda>1,$$

and the variance function is

$$V(T) = \frac{\lambda k^2}{(\lambda - 1)^2(\lambda - 2)}, \; for\; \lambda > 2.$$

The HRF always has a DFR property associated with it, see Figure 1.4.

Birnbaum Saunders Distribution

- **Univariate Birnbaum Saunders Distribution**

  A random variable following the BS distribution is defined through a standard normal random variable. Therefore the PDF and CDF of the BS model can be expressed in terms of the PDF and CDF of the standard normal distribution. The CDF of a two parameter BS random variable T for $\alpha > 0$ and $\beta > 0$ can be written as

$$F_T(t|\alpha, \beta) = \Phi\left[\frac{1}{\alpha}\left(\left(\frac{t}{\beta}\right)^{\frac{1}{2}} - \left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right)\right], \; t > 0, \tag{1.2.1}$$

  where $\Phi(.)$ is the standard normal CDF. The PDF of BS distribution is

$$f_T(t|\alpha, \beta) = \begin{cases} \frac{1}{2\alpha\beta\sqrt{2\pi}}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right] e^{\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta}+\frac{\beta}{t}-2\right)\right]} & \text{if } t > 0 \\ \\ 0 & \text{otherwise} \end{cases} \tag{1.2.2}$$

  Here $\alpha > 0$ and $\beta > 0$ are the shape and scale parameters respectively. The BS distribution has DFR and UBFR for its HRF, see Figure 1.5.

- **Bivariate Birnbaum Saunders Distribution**

  The bivariate Birnbaum-Saunders (BVBS) distribution was introduced by Kundu et al. (2010). The bivariate random vector (T_1, T_2) is said to have a BVBS distribution with parameters $\alpha_1, \beta_1, \alpha_2, \beta_2$, and ρ if the CDF of (T_1, T_2) can be expressed as

$$F(t_1, t_2) = \Phi_2\left[\frac{1}{\alpha_1}\left(\sqrt{\frac{t_1}{\beta_1}} - \sqrt{\frac{\beta_1}{t_1}}\right), \frac{1}{\alpha_2}\left(\sqrt{\frac{t_2}{\beta_2}} - \sqrt{\frac{\beta_2}{t_2}}\right)\right] \tag{1.2.3}$$

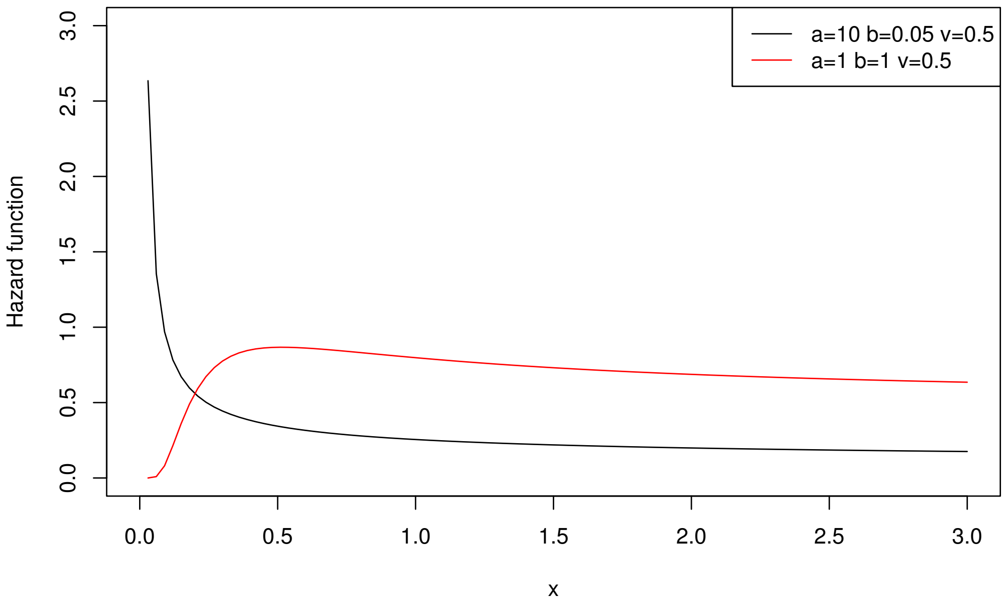


Figure 1.5: Birnbaum Saunders hazard function plot

for $t_1 > 0$, $t_2 > 0$, $\alpha_1 > 0$, $\beta_1 > 0$, α_2, $\beta_2 > 0$, and $-1 < \rho < 1$. Here $\Phi_2(u, v; \rho)$ is the CDF of standard bivariate normal vector (Z_1, Z_2) with correlation coefficient ρ. The PDF corresponding to Eq.(1.2.3) is

$$f(t_1, t_2) = \frac{1}{8\pi\alpha_1\alpha_2\beta_1\beta_2\sqrt{1-\rho^2}}\left[\left(\frac{\beta_1}{t_1}\right)^{\frac{1}{2}} + \left(\frac{\beta_1}{t_1}\right)^{\frac{3}{2}}\right]\left[\left(\frac{\beta_2}{t_2}\right)^{\frac{1}{2}} + \left(\frac{\beta_2}{t_2}\right)^{\frac{3}{2}}\right]$$
$$e^{-\frac{1}{2(1-\rho^2)}\left[\frac{1}{\alpha_1^2}\left(\sqrt{\frac{t_1}{\beta_1}}-\sqrt{\frac{\beta_1}{t_1}}\right)^2+\frac{1}{\alpha_2^2}\left(\sqrt{\frac{t_2}{\beta_2}}-\sqrt{\frac{\beta_2}{t_2}}\right)^2-\frac{2\rho}{\alpha_1\alpha_2}\left(\sqrt{\frac{t_1}{\beta_1}}-\sqrt{\frac{\beta_1}{t_1}}\right)\left(\sqrt{\frac{t_1}{\beta_1}}-\sqrt{\frac{\beta_1}{t_1}}\right)\right]}$$

for $t_1 > 0$, $t_2 > 0$, $\alpha_1 > 0$, $\beta_1 > 0$, α_2, $\beta_2 > 0$, and $-1 < \rho < 1$.

- **Multivariate Birnbaum Saunders Distribution**

Kundu et al. (2013) introduced the multivariate BS distribution. Let $\underline{\alpha}, \underline{\beta} \in \mathbb{R}^p$, where $\underline{\alpha} = (\alpha_1, \cdots, \alpha_p)^T$ and $\underline{\beta} = (\beta_1, \cdots, \beta_p)^T$, with $\alpha_i > 0$, $\beta_i > 0$ for $i = 1, \cdots, p$. Let $\mathbf{\Gamma}$ be a $p \times p$ positive-definite correlation matrix. Then, the random vector $\underline{T} = (T_1, \cdots, T_p)^T$ is said to have a p-variate BS distribution with parameters $(\underline{\alpha}, \underline{\beta}, \mathbf{\Gamma})$ if it has the joint CDF as

$$P(\underline{T} \leq \underline{t}) = P(T_1 \leq t_1, \cdots, T_p \leq t_p) \tag{1.2.4}$$

$$= \Phi_p\left[\frac{1}{\alpha_1}\left(\sqrt{\frac{t_1}{\beta_1}} - \sqrt{\frac{\beta_1}{t_1}}\right), \cdots, \frac{1}{\alpha_p}\left(\sqrt{\frac{t_p}{\beta_p}} - \sqrt{\frac{\beta_p}{t_p}}\right); \mathbf{\Gamma}\right] \tag{1.2.5}$$

for $t_1 > 0, \cdots, t_p > 0$. Here, for $\underline{u} = (u_1, \cdots, u_p)^T$, $\Phi_p(\underline{u}; \mathbf{\Gamma})$ denotes the joint CDF of a standard normal vector $\underline{Z} = (Z_1, \cdots, Z_p)^T$ with correlation matrix $\mathbf{\Gamma}$. The joint PDF of $\underline{T} = (T_1, \cdots, T_p)^T$ can be obtained from the above equation as

$$f_{\underline{T}}(\underline{t}; \underline{\alpha}, \underline{\beta}, \mathbf{\Gamma}) = \phi_p\left(\frac{1}{\alpha_1}\left(\sqrt{\frac{t_1}{\beta_1}} - \sqrt{\frac{\beta_1}{t_1}}\right), \cdots, \frac{1}{\alpha_p}\left(\sqrt{\frac{t_p}{\beta_p}} - \sqrt{\frac{\beta_p}{t_p}}\right); \mathbf{\Gamma}\right)$$
$$\times \prod_{i=1}^{p} \frac{1}{2\alpha_i\beta_i}\left\{\left(\frac{\beta_i}{t_i}\right)^{\frac{1}{2}} + \left(\frac{\beta_i}{t_i}\right)^{\frac{3}{2}}\right\},$$

for $t_1 > 0, \cdots, t_p > 0$; here, for $\underline{u} = (u_1, \cdots, u_p)^T$,

$$\phi_p(u_1, \cdots, u_p; \mathbf{\Gamma}) = \frac{1}{(2\pi)^{\frac{p}{2}}|\mathbf{\Gamma}|^{\frac{1}{2}}} exp\{-\frac{1}{2}\underline{u}^T\mathbf{\Gamma}^{-1}\underline{u}\}$$

is the PDF of the standard normal vector with correlation matrix $\mathbf{\Gamma}$.

Type-II Gumbel Distribution

Type II Gumbel distribution is one of the statistical distributions that are used to model extreme values. The PDF of Type-II Gumbel distribution is

$$f(t|\beta, \theta) = \beta\theta t^{\beta-1} e^{-\theta t^{-\beta}},\ t > 0,\ \beta > 0,\ \theta > 0,$$

and the CDF is given by

$$F(t|\beta, \theta) = e^{-\theta t^{-\beta}},\ t > 0,\ \beta > 0,\ \theta > 0.$$

The Weibull distribution is produced when $\theta = \mu^{-\kappa}$ and $\beta = -\kappa$ are substituted. The mean and variance of Type-II Gumbel distribution are $\theta^{\frac{1}{\beta}}\Gamma(1-\frac{1}{\beta})$ and $\theta^{\frac{2}{\beta}}(\Gamma(1-\frac{1}{\beta}) - \Gamma(1-\frac{1}{\beta})^2)$, respectively. This is always an asymmetric distribution. The HRF plot of this distribution shows both DFR and UBFR, see Figure 1.6.

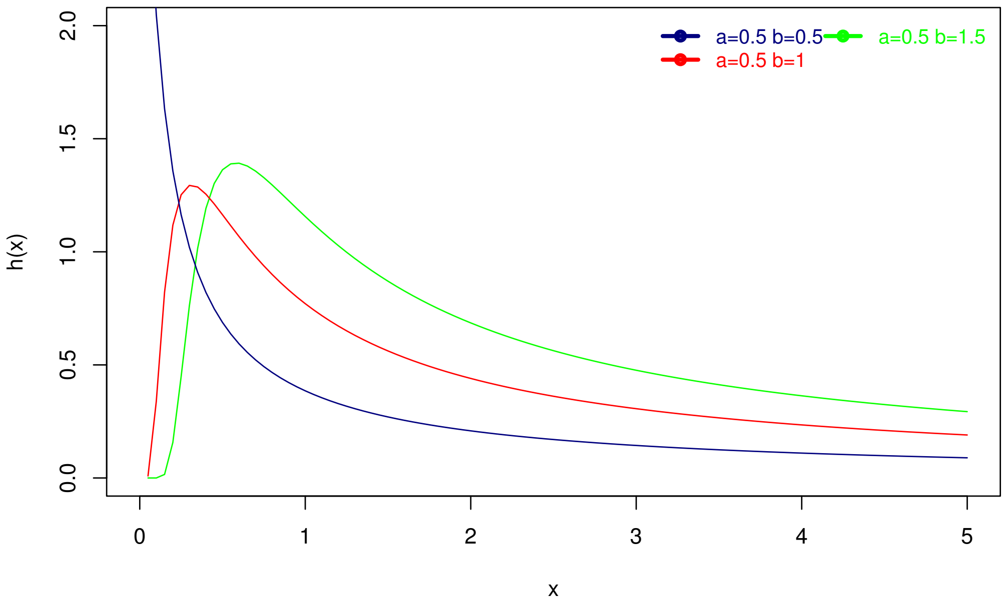


Figure 1.6: Type-II Gumbel hazard function plot

1.2.4 DUS Transformation

DUS (Dinesh-Umesh-Sanjay) transformation is a transformation method used to generate new lifetime distributions proposed by Kumar et al. (2015). In terms of computation and interpretation, this transformation produces a parsimonious result since it does not include any new parameters other than those involved in the baseline distribution.

In the case where $F(x)$ is the CDF of the baseline distribution, then the CDF of the new DUS transformed distribution is as follows:

$$G(x) = \frac{1}{e-1}[e^{F(x)} - 1].$$

Then the PDF becomes

$$g(x) = \frac{1}{e-1}[e^{F(x)} f(x)],$$

where $f(x)$ is the PDF of the baseline distribution.

1.2.5 Censoring

In reliability theory and survival analysis, censoring is a frequently used concept. One scenario involving censored data would be if the study was terminated before all sample items failed. As the investigation has already ended, it is unknown when the remaining parts will fail. As a result, the sample has two sets of observations: one with real failure times, known as complete data (uncensored), and the other with mere constraints on failure times, known as incomplete data (censored data).

Type I Censoring

The most frequently used censoring method in reliability engineering is Type I censoring. The experiment is terminated by Type-I censoring at a predefined time T. For example, in life testing experiment, n items are placed on a test, but lifetimes will be known only for the items that fail by time T. Therefore, according to this scheme, the duration of the experiment is fixed, but the number of failures is random. It has a major advantage that if there are very few failures, statistical analysis will be inefficient.

Type II Censoring

Under the Type II censoring scheme, mth ($m \leq n$) failure terminates the experiment. In a life-testing experiment, n items are tested. Test termination occurs at the time of the mth failure instead of waiting until all n observations have failed. There may be instances in which it takes a long time for the test to fail all n items. A test of this type can be cost-effective and time-saving. There is a fixed number of observed failures in this censoring scheme, but the test duration is random.

1.2.6 Stress-Strength Reliability

A stress-strength (SS) model, which is largely used in reliability engineering but is also utilized in economics, quality control, psychology, and medicine, compares the strength and stresses of a system. Both stresses and strength are viewed as distinct random variables in a SS model.

As a result of technological advancements, a variety of fields have become increasingly concerned with the issue of enhancing network reliability in the modern world. Despite receiving a certain amount of stress, some products can withstand it due to their strength. Appliances, however, tend to malfunction if more stress than strength is given. Assume that Y represents the random stress placed on a certain appliance and X represents the random strength needed to withstand the force. The device will fail if and only if the applied stress ever exceeds its threshold level. Therefore, $R = P(X > Y)$ gives the reliability of a system as a measure.

Component reliability in the SS environment is determined by the PDF $g_1(x)$ of the strength of the unit or system, X, and the PDF $g_2(y)$ of the stress Y. At any given moment, the system will fail if the applied stress exceeds its strength. In the case where X and Y are independently distributed, the SS reliability of the component can be calculated as follows:

$$R = P(X > Y) = \int_0^\infty \left[\int_y^\infty g_1(x)dx \right] g_2(y)dy.$$

1.2.7 Accelerated Life Testing

In an accelerated life test, failure rates are accelerated by subjecting the component to more stress. It follows that the failure time is determined by the stress factor and that higher levels of stress will result in a quicker failure time. At higher temperatures, some components may fail more rapidly; however, at lower temperatures, they may be more likely to last longer. The time required under low-stress conditions may not be sufficient to determine system reliability, which will be tested in conditions of increasing stress, resulting in a relatively short duration of the experiment. With this technology, failures that, in normal circumstances, would take a long time to appear can be seen sooner. In addition, the size of the data can be increased without having to spend a lot of money or time. Reliability testing of this type is known as accelerated life testing (ALT).

Step-Stress ALT

Several types of ALT exist in which stress is applied under accelerated conditions in various ways, including constant stress ALT (CSALT), step-stress ALT (SSALT), and progressive stress ALT (PSALT).

In CSALT, the stress applied to the test product is time-independent. The testing units are subjected to a constant, higher-than-usual level of stress until either all units fail or the test is terminated, resulting in censored test data. In PSALT, the stress applied to a test product continuously increases over time.

Step-stress tests are accelerated life tests in which the amount of stress applied to each unit progressively increases over time. There may be more than one stress change point in this case. According to the simple SSALT model, for example, a random sample of n units is initially placed on the low-stress level x_1 and is allowed to run until the predetermined time T_1 is reached. As soon as time T_1 is up, the stress is changed to x_2 for the remaining unfailed units. Upon changing the stress to x_2, the test continues until all units fail or are censored. Accordingly, these tests are commonly used to estimate the distribution parameters of failure times under normal operating conditions based on observed orders of failure times.

Compared to other test methods, the SSALT model has the primary advantage of reducing overall test duration. SSALT yields faster failures owing to increasing levels of stress. In ALT data analysis, we need to determine the PDF of a test until at design condition from ALT data instead of traditional life test data obtained under normal conditions. To do that, we must have an appropriate life distribution and a life-stress relationship.

1.2.8 Model Selection Criteria

The model selection criterions used in this dissertation are given below.

Kolmogorov-Smirnov Test

To determine whether or not a given sample reflects a population with a specific distribution, Kolmogorov (1933) proposed the Kolmogorov-Smirnov (KS) test. The KS test determines the difference between the estimated CDF of the distribution and the sample's empirical distribution function. In this case, the null hypothesis is H_0: The sample follows the particular distribution, and the alternative hypothesis is H_1: The sample doesn't follow the particular distribution. It is pertinent to note that when comparing more than one distribution, it is more appropriate to choose the distribution with a smaller KS value.

Cramér-Von Mises Test

Based on the sum of squared differences between the empirical distribution function and theoretical distribution function, Cramér-Von Mises (CVM) test statistic can be defined as follows:

$$CVM = \frac{1}{12n} + \sum_{i=1}^{n} \Big[F(x_i, \theta) - \frac{2i-1}{2n}\Big]^2$$

Whenever the value of the CVM test statistic exceeds the critical point, the null hypothesis is rejected.

Akaike's Information Criterion

The Akaike information criterion (AIC) measures the fit of a model to the data it was derived from. The AIC can be used in statistics to determine the most appropriate model for the data by comparing different possible models. It is considered best to select the model with the lowest AIC. In order to calculate the AIC, the following factors must be considered:

- the number of unknown parameters in the model.
- the MLE of the model.

Therefore, the AIC can be defined as

$$AIC = 2v - 2\log L,$$

where v is the number of unknown parameters in the model and L represents the maximized likelihood value.

Bayesian Information Criterion

In statistics, the Bayesian Information Criterion (BIC) is a criterion for selecting between two or more models. It is considered more appropriate to select the model with the lowest BIC. It is defined as

$$BIC = v\log(m) - 2\log(L),$$

where m is the sample size, v is the number of unknown parameters in the model, and L represents the maximized likelihood value.

Corrected Akaike's Information Criterion

For a small sample size, there's a high probability that AIC will choose models containing too many parameters, causing AIC to overfit. In order to mitigate such a risk of overfitting, AICc was introduced: corrected Akaike Information Criterion (AICc) is basically the same as AIC but with a modification for small sample sizes. The AICc is defined as

$$AICc = AIC + \frac{2v(v+1)}{m-v-1},$$

where m is the sample size and v is the number of unknown parameters in the model.

Consistent Akaike's Information Criterion

The consistent Akaike's Information Criterion (CAIC) is one of the information criterion used for selecting different models and is defined as

$$CAIC = -2\log(L) + v[\log(m) + 1],$$

where m is the sample size, v is the number of unknown parameters in the model, and L represents the maximized likelihood value.

1.3 Review of Literature

In the modern world, reliability is well-known and is expanding rapidly. Its objective is to increase the effectiveness of the system by developing new techniques. Therefore it has gained much importance among manufacturers.

1.3.1 Failure Rates

The occurrence of IFR is of interest in a wide variety of real-world systems, Koutras (2011) and Ross et al. (2005). The gamma distribution and the Weibull distribution are the most popular distributions with IFR (both distributions also exhibit DFR and CFR). Exponentiated exponential distribution and weighted exponential distribution have been introduced in place of the gamma and Weibull distributions by Gupta and Kundu (2001, 2009), respectively. Cancho et al. (2011) proposed an IFR lifetime distribution called the Poisson-exponential (PE) distribution. Instead of the Weibull, gamma, exponentiated exponential, weighted exponential, and PE distributions, Bakouch et al. (2014) presented the binomial-exponential 2 (BE2) distribution, a two-parameter lifetime distribution with IFR properties. When the sample size has a zero-truncated binomial distribution, the BE2 distribution is created as a distribution of a random sum of independent exponential random variables.

There have been many instances where the data show DFR function. According to Proschan (1963), air conditioning systems on aircraft follows DFR distribution. Kus (2007) examined earthquakes that have occurred in the North Anatolia fault zone during the last century and concluded that DFR distribution is quite accurate. Adamidis and Loukas (1998) introduced a two-parameter DFR distribution. Alpha power transformed Lindley distribution with DFR and BFR was introduced by Dey et al. (2019) with application to earthquake data.

Among the classes of life distributions that have received considerable attention is the one that exhibits bathtub-shaped failure rates. Detailed accounts of such distributions have been provided by Rajarshi and Rajarshi (1988). F is said to have BFR if its failure rate initially decreases, then remains constant for a duration, and eventually increases over time. The class of lifetime distributions featuring a BFR function is significant since the lifespans of electronic, electromechanical, and mechanical products are typically represented with this feature, as noted in Barlow and Proschan (1975). Moreover, in survival analysis, human lifetimes typically exhibit this pattern.

Kao (1959), Glaser (1980), and Lawless (1982) offer a variety of illustrations of BFR distributions. As a mixture of a group of IFR distributions for competing risk models, Hjorth (1980) portrayed BFR distributions. The BFR distributions were discussed by Lai et al. (2001), while Xie et al. (2002) investigated modified Weibull extension models that have BFR functions helpful for cost analysis and decision-making concerning reliability. Block et al. (2008) examined the continuous mixture of entire families of distributions with BFR functions. The generalized linear failure rate distribution and its characteristics were developed by Sarhan and Kundu (2009).

Mudholkar and Srivastava (1993) and Xie and Lai (1996) proposed modifications to Weibull distributions to make them suitable for BFR data. Additionally, Chen (2000) developed a two-parameter BFR model for analyzing survival data. An additive model for lifetime data with BFR was investigated by Wang (2000) based on the Burr XII distribution. The generalized Rayleigh distribution, also known as the two-parameter Burr Type X distribution, featured an IFR or BFR function and was introduced by Surles and Padgett (2005). A new exponential-type distribution with CFR, IFR, DFR, BFR, and UBFR functions was recently proposed by Lemonte (2013a) and can be utilized to simulate survival data in reliability problems and fatigue life studies. The parameter estimation of a three-parameter Weibull-related model with IFR, DFR, BFR, and UBFRs was studied by Zhang et al. (2013). Weibull extension with BFR function was obtained by Wang et al. (2014) using type-II censored samples.

Due to the ability of some generalized Gamma-type distributions to model different BFR functions, Parsa et al. (2014) studied the differences between the change points of failure rate and mean residual life functions. A novel finite interval lifetime distribution model for fitting the BFR curve was discussed by Wang et al. (2015). Shehla and Ali Khan (2016) used an exponential power model with the BFR function to study reliability analysis. In order to model fuzzy lifetime data, Shafiq and Viertl (2017) provided generalized estimators for the parameters and failure rates of the BFR distributions. A new Lindley Weibull distribution that includes unimodal and BFRs was introduced by Cordeiro et al. (2018). A generalized Weibull uniform distribution that adds DFR or BFR features to the Weibull distribution was proposed by Al-Abbasi et al. (2019). The reliability analysis of gas-turbine engines with BFR distribution was examined by Ahsan et al. (2019). Based on adaptive progressive type-II censored data, Chen and Gui (2020) studied the inferential problem of two parameters of a lifetime distribution using BFR functions. According to Deepthi and Chacko (2020a), an UBFR model can be constructed using DUS Transformation of the Lomax Distribution. Shrahili and Kayid (2022) described a generalized Pareto distribution characterized by a heavy right tail and UBFR. On the basis of a new power function, Sindhu et al. (2023) proposed a bathtub-shaped nonhomogeneous Poisson process software reliability model distribution.

1.3.2 Birnbaum-Saunders Distribution

As a result of the continuous vibration present in commercial aircraft and the problems resulting from it, Birnbaum and Saunders (1968) developed an innovative probabilistic model to describe material specimen lifetimes as a result of fatigue due to cyclical stress and tension caused by exposure to fatigue. Birnbaum and Saunders (1969b); Birnbaum and Saunders (1969a) formulated the fatigue-life distribution that would later bear their names, defining it as a life distribution and establishing an approach for estimating the parameters of this two-parameter distribution.

Unlike most other distributions, the Birnbaum-Saunders distribution is based on cumulative damage that causes fatigue in materials over time. Birnbaum and Saunders (1969a) derived the fatigue-life distribution from a simulation model, which showed that cumulative damage caused by the development and growth of a dominant crack often exceeds a threshold value and results in the failure of the specimen. Some of the assumptions made by Birnbaum and Saunders (1969a) were relaxed by Desmond (1985), strengthening the rationale for using this distribution.

There have been several attempts to extend and generalize the BS distribution. Volodin and Dzhungurova (2000) are credited with extending the BS distribution for the first time. The generalized BS distribution was then introduced by Díaz-García and Leiva (2005); see Leiva et al. (2008) and Sanhueza et al. (2008). A three-parameter extension to the BS distribution was proposed by Owen (2006). Based on skew-elliptical models, Vilca and Leiva (2006) developed a BS distribution. Balakrishnan et al. (2009) used the expectation and maximization algorithm in estimating the parameters of the BS distribution and extended it using a scale-mixture of normal distributions. Based on the slash-elliptic model, Gómez et al. (2009) extended the BS distribution. An extended length-biased version of the BS distribution is provided by Leiva et al. (2009).

A truncated version of the BS distribution was examined by Ahmed et al. (2010). Mixture models based on the BS distribution were presented by Kotz et al. (2010). The epsilon-skew BS distribution has been developed by Vilca et al. (2010) and Castillo et al. (2011). Kundu et al. (2010) introduced the bivariate BS distribution and studied some of its properties and characteristics. The multivariate generalized BS distribution has been introduced, replacing the normal kernel by an elliptically symmetric kernel, by Kundu et al. (2013). The BS mixture distributions were considered in Balakrishnan et al. (2011). The beta-BS distribution has been defined by Cordeiro and Lemonte (2011). A shifted BS distribution was utilized by Leiva et al. (2011) to model wind energy flux.

The Kumaraswamy BS distribution was described by Saulo et al. (2012). Based on a non-homogeneous Poisson process, Fierro et al. (2013) generated the BS distribution. The gamma BS distribution was first introduced by Cordeiro et al. (2013). The Marshall–Olkin–BS distribution was studied by Lemonte (2013b). The exponentiated generalized BS distribution was proposed by Cordeiro and Lemonte (2014), whereas the zero-adjusted BS distribution was introduced by Leiva et al.

(2016). A generalization of BS distribution is done by Chacko et al. (2015). An exhaustive review of BS works can be seen in Balakrishnan and Kundu (2019). Using the skew-Laplace BS distribution, Naderi et al. (2020) demonstrated the modeling of finite mixtures. Benkhelifa (2021) introduced a new extension of the BS distribution based on the Weibull-G family of distributions.

1.3.3 Stress-Strength Reliability

There has been a long history of SS reliability, beginning with the pioneering work of Birnbaum (1956) and Birnbaum and McCarty (1958). Church and Harris (1970) are credited with introducing the term SS reliability. Kotz and Pensky (2003) provided an excellent overview of the various SS models up to 2001. Several authors have published articles on SS models recently. From a reliability perspective, Gupta and Brown (2001) investigated the skew-normal distribution and obtained the strength-stress reliability. See Raqab and Kundu (2005), Kundu and Gupta (2005, 2006), Kundu and Raqab (2009), and Sharma et al. (2015) for further details.

Sometimes the actual SS reliability of the system cannot be evaluated. It is easy to compute R, if the stress and strength are assumed to or fitted to have some well-known statistical distribution. At the same time, if the more fitted probability distributions have more parameters, then the problem becomes complicated. In such situations, one has to estimate the SS reliability, if the values of parameters are not available. SS reliability estimation is very important to investigate the level of strength and level of stress for required reliability. The estimation of SS reliability is more complicated for single-component and multi-component systems. A substantial amount of literature exists regarding the problems associated with the estimation of reliability for single-component SS models. SS reliability analysis using various statistical distributions are available in the literature.

Researchers discuss in detail the estimation of R using various statistical distributions. Using a bivariate Pareto model, Hanagal (1997) derived the maximum likelihood estimator (MLE) of the SS parameter R. A finite mixture of inverse Gaussian distributions was used by Akman et al. (1999) to study reliability estimation. A finite mixture of lognormal components is used by AI-Hussaini and Sultan (2001) to study the estimation of $R = P(Y < X)$. The exponential strength and stress random variables were taken into account by Krishnamoorthy et al. (2007). The estimation of R for the three-parameter generalized exponential distribution was investigated by Raqaab et al. (2008). Lai and Balakrishnan (2009) estimated R in models with correlated stress and strength. A study by Al-Mutairi et al. (2013) examined R estimates based on Lindley distributions. The estimation of reliability $R = P(Y < X)$ where X and Y are independent random variables that follow the Kumaraswamy distribution with varying parameters was discussed by Nadar et al. (2014). Different estimators of the parameter R were produced by Nadar and Kizilaslan (2014) in the context of the Kumaraswamy model with upper record values. Ghitany et al. (2015) discussed the reliability of SS systems based on power Lindley distributions. For a transmuted Rayleigh distribution, Dey et al. (2017) calculated the SS reliability R.

Using progressive first-failure censoring, Krishna et al. (2019) obtain R based on the inverse Weibull distribution. Rao et al. (2019) considered the estimation of stress-strength reliability based on two independent exponential inverse Rayleigh distributions that share a common scale parameter but have different shape parameters. The SS reliability estimation of single and multi-component systems has been studied by Jose et al. (2019) and Xavier and Jose (2021a) based on generalizations of half logistic distributions. On the basis of discrete phase-type distributions, Jose et al. (2022) estimated SS reliability for single and multi-component systems. Deepthi and Chacko (2020b) discussed single-component SS reliability and multi-component SS reliability estimation using the three-parameter generalized Lindley distribution. Alamri et al. (2021) estimated the SS reliability when stress and strength both follow the Rayleigh-half-normal distribution. Assuming that the strength components are distributed independently and identically as power-transformed half-logistic distributions subject to common stress, which is assumed to be independent of either the Weibull distribution or the PHL distribution, Xavier and Jose (2021b) investigated the reliability of the multicomponent stress–strength model. Varghese and Chacko (2022) examined SS reliability using the Akash distribution. Sonker et al. (2023) established stress–strength reliability models for power-Muth distribution.

1.3.4 Accelerated Life Testing

ALT was introduced by Chernoff (1962) and Bessler et al. (1962). A variety of methods can be used for accelerated testing to shorten the life of products or accelerate their degradation. During such tests, it is necessary to obtain data quickly that can be modeled and analyzed. This will enable us to generate the desired information about the product's lifetime and performance under normal conditions. Performing such tests saves a great deal of time and money. There are many ways in which accelerated tests can apply stress loading. Stress loading can be a constant, cyclic, step, or progressive. A discussion of these types of ALT is provided by Nelson (1990).

During CSALT, each unit of the test is monitored until it fails, maintaining constant levels of all stress factors. It has been found that accelerated test models for constant stress are better developed and more reliable for certain materials and products. There are several examples of constant stresses, including temperature, voltage, and current. A comprehensive review of CSALT models can be found in the works of Meeker and Escobar (1998), Escobar and Meeker (2006), Aly and Bleed (2013), and Abdel-Ghaly et al. (2016a). The method proposed by Kim and Bai (2002) for estimating the lifetime distribution for constant stress ALTs uses a mixture of two distributions to describe failure modes. To determine the lifetime of vacuum fluorescent displays, Zhang and Wang (2009) performed four CSALTs with the cathode temperature increase and assumed that the lifetime distribution was lognormal. While Bhattacharyya and Soejoeti (1981) applied the least square method under CSALT to Weibull, exponential, and gamma distributions, Bhattacharyya and Fries (1982) applied it to inverse Gaussian distributions. Using an exponentiated distribution family, Abdel-Ghaly et al. (2016b) examined different estimation methods for CSALT. For CSALT, different estimation methods under the exponentiated power Lindley distribution were discussed by Kumar et al. (2022).

The PSALT procedure involves continuously increasing stress levels on a specimen. In the study of metal fatigue, this test is used to determine the endurance limit of metal. It is likely to be difficult to control the accuracy of the PSALT. It is common for some products to undergo cyclic stress loading when they are in use. As an example, insulation under AC voltage experiences sinusoidal stress. This type of product is subjected to cyclic stress testing by repeatedly subjecting it to the same stress pattern at high levels of stress. Yin and Sheng (1987) examined the MLE of exponential failure times under PSALT. Using the Weibull distribution, Abdel-Hamid and Al-Hussaini (2011) described PSALT under progressive censoring. Using type II progressively censored data from a half-logistic distribution under PSALT, Al-Hussaini et al. (2015) calculated one-sample Bayesian prediction intervals. For the extension of the exponential distribution, Mohie El-Din et al. (2017) investigated both classical and Bayesian inference on progressively type-II censored PSALT. An inference of PSALT was discussed by Kumar Mahto et al. (2020) for the Logistic exponential distribution under progressive type-II censoring.

1.3.5 Step-Stress Accelerated Life Testing

In recent years, SSALT has become one of the most frequently discussed ALTs. This is because the level of stress on each unit increases step by step at predetermined intervals or upon a fixed number of failures. There has been extensive research on SSALTs with exponential lifetime distributions under CEM. Balakrishnan (2009) has written an excellent review article on this topic for the benefit of interested readers. Many authors have examined the optimality of an SSALT in the presence of exponential CEM, such as Miller and Nelson (1983), Bai et al.(1989), Wu et al.(2008), and Kateri et al. (2011) for different censoring methods.

Xiong (1998) discusses the inferences that can be drawn from any sample size considering an exponential lifetime distribution at constant stress and a CEM for the two-step ALT. Weibull CE model properties were examined using SSALT data by Komori (2006). When competing risk factors are independently and exponentially distributed, Balakrishnan and Han (2008) and Han and Balakrishnan (2010) investigated the SSALT under type-II and type-I censoring schemes respectively. The Weibull PH model employed in SSALT was subjected to Bayesian analysis and compared with Weibull CEM by Sha and Pan (2014). Hamada (2015) suggested and explored a generic Bayesian approach to SSALT planning.

It has been proposed by Han and Kundu (2014) that the problem of estimating point and interval estimates may be solved when the distributions of the different risk factors are s-independent Generalized Exponential distributions. El-Din et al. (2016) investigated parametric inference on SSALT for the extension of the exponential distribution with progressive type-II censoring. Chandra et al. (2017) have investigated the optimal quadratic SSALT plan for Weibull distributions with type I censoring. Hakamipour and Rezaei (2017) explored the optimization of simple SSALT using type I censoring for Frechet data. Using a simple SSALT and type II censoring, Basak and Balakrishnan (2018) predicted the censored exponential lifetimes. For a simple SSALT CEM with censored exponential data, Zhu et al. (2020) described exact likelihood-ratio tests. Under the CEM assumption, Samanta et al. (2020) develop a step-stress model with exponential distribution and evaluates the related conclusions based on Type II hybrid stress changing time. Kannan and Kundu (2020) proposed a generalized cumulative risk model for simple SSALT and developed this model on the premise that the underlying population comprised both 'cured' individuals and susceptible individuals. Pal et al. (2021) introduced the failure rate-based simple step-stress model for the Lehmann family of distributions.

Nonparametric methods do not assume the existence of a specific lifetime distribution. By utilizing this distribution-free strategy, we can mitigate the significant error in extrapolating SSALT results when there is a bias in the evaluation of the potential lifetime models or when the models do not provide a good representation of the failure mechanism. For determining the upper confidence bounds of the cumulative failure probability of a product, Hu et al. (2012) suggested a nonparametric PHM. Other research on this subject is found in [Schmoyer (1991) and Pascual and Montepiedra (2003)].

In-depth research has been done on random effects in lifetime trials. When the group impact is statistically significant, León et al. (2007) established a Bayesian approach to conclude ALT data. They compare fixed and random group effect models and demonstrate that the latter offers more precise predictions and estimates. To account for the random group effect in SSALT, Seo and Pan (2017) suggested a generalized linear mixed-effect model. The results from two estimating techniques—adaptive Gaussian quadrature and integrated nested Laplace Approximation—are analyzed. Wang (2020) took into account the Weibull distribution-based data analysis of SSALT data with random group effects.

1.4 Motivation of the present work

Reliability engineering and statistical modeling can benefit from introducing new generalized distributions because they can provide customized solutions for specific challenges, improve model accuracy, foster innovation, handle complex system behaviors, deepen understanding of statistical theory, facilitate interdisciplinary applications, and adapt to emerging data types. Even though existing generalized distributions are valuable, new distributions must be developed to address changing needs and technological advances. The fitness of distributions to the given data is important to draw valid and accurate probability computations. This enables us to accurately model a wide range of real-world situations and fosters cross-disciplinary collaboration. A search for new distributions for modeling lifetime data is essential in this context.

When one considers a parallel system where each of the components has DUS-transformed distributions for its lifetime, we should investigate the distributional properties. Moreover, we have to investigate the distributional properties of the parallel system when components are distributed as DUS transformations of baseline distributions like exponential, Weibull, and Lomax distributions. There are several other distributions that can serve as baseline distributions. After investigating the flexibility of the new distributions in terms of simulation, fitness, estimation, etc. using exponential, Weibull, and Lomax distributions, we can go for other distributions. Generalized exponential distribution (Gupta and Kundu (1999)) was widely accepted by researchers since it was applicable to parallel systems in which components are exponentially distributed. But using an exponential distribution for the lifetime of a component is limited to the case of random failures. But what would be the behavior if we use any other distribution with a non-monotonic failure rate? Nowadays, distributions using the DUS transformation receive high attention since this transformation does not add any more parameters but shows better fitness than the baseline distribution. We have to investigate the power generalization of distribution using the DUS transformation to describe effectiveness, behavior, etc. An attempt towards the power generalization of DUS transformation has to be explored more. A variety of generalizations of BS distributions are available in the reliability literature. A detailed study, especially in the inference part, also has to be explored more.

Mixture distributions are useful when a new component switches on for the first time. They may fail at the same instant of starting operation, or they may fail due to overvoltage, jerking, or any such shocks, or they may fail due to the degradation of the component. Failure due to random shocks is modelled using an exponential distribution. Failure due to degradation can be modelled using any other distribution with a non-monotonic failure rate. In statistical modeling and analysis, for reliability and survival analysis studies, the introduction of a mixture distribution based on exponential and gamma distributions can be seen as extremely useful. In this way, complex data patterns can be captured that cannot be adequately captured by a single distribution. The main purpose of mixture models is to enhance the understanding and description of real-world phenomena by combining several different distributions. A detailed study of mixture distributions has to be carried out to examine failure rate behavior and its inference. The determination of stress-strength reliability has to be addressed in mixture distributions. Estimation of stress-strength reliability is also a research problem when using mixture models. After investigating some mixture distributions and their usefulness in stress-strength analysis, we can investigate the remaining mixtures as per need.

Step-stress accelerated life testing (SSALT) with Type II censoring is a method for assessing the reliability and durability of a product or system while minimizing the amount of testing time and resources required. Using Type II censoring, a product's life is estimated based on how many units fail, and the information gathered is used to determine the product's lifetime. Through SSALT, the units are subjected to progressively higher levels of stress over time or usage, causing the products to age more rapidly. Manufacturers can make informed decisions about product design improvements, warranty policies, and maintenance schedules by designing and analyzing SSALT experiments with Type II censoring to produce robust and reliable products at an affordable cost.

1.5 Objectives of the Study

1. To study the increasing failure rate, decreasing failure rate, bathtub-shaped failure rate, and upside-down bathtub-shaped failure rate distributions and their properties and applications for modeling lifetime data.

2. To study the role of bathtub-shaped failure rate distributions in system engineering and other scientific area and propose new distributions.

3. To study existing step-stress accelerated life testing (SSALT) models, develop an SSALT model, and estimate its model parameters.

4. To study the properties of Birnbaum-Saunders distributions and their generalizations on reliability theory.

5. To study on the stress-strength reliability models and its inferential procedures.

1.6 Outline of the Present Study

A total of seven chapters are included in the thesis. A new generalization of the DUS transformation, the PGDUS transformation, is presented with applications to exponential, Weibull, and Lomax distributions. A new BFR distribution called exponential-gamma $(3, \theta)$ is studied. Further, SS reliability is also calculated for this distribution. Generalization of the BS distribution called ν-BS distribution is then provided. A simple SSALT analysis of Type-II Gumbel distribution under Type-II censoring is given.

The chapters of the thesis are arranged in the following manner. *Chapter 1* provides an overview of the basic concepts and definitions used throughout this thesis. Also, an extensive literature review is given. A comprehensive review study of the IFR, DFR, BFR, UBFR, BS distribution, stress-strength reliability model distributions, and SSALT analysis was conducted to achieve the results of this research work.

Chapter 2 introduces a new transformation called the power generalized DUS transformation and proposes new distributions with exponential, Weibull, and Lomax distributions as baseline distributions. Several mathematical properties are examined, including moments, MGFs, CFs, quantile functions, order statistics, etc. The maximum likelihood approach to parameter estimation is discussed. Based on several real data sets, the proposed distributions are compared with some of the other failure rate lifetime distributions. It has been found that the new distributions fit the data better than the well-known distributions.

Chapter 3 examines in detail a new BFR distribution called the exponential-gamma $(3, \theta)$ distribution. An investigation is conducted into the shapes of the PDF and the failure rate. Various properties are discussed, including moments, MGF, CF, the quantile function, and entropy. Distributions for the minimum and maximum are discovered. In order to estimate the parameters of the distribution, the maximum likelihood method is utilized. Through the use of a simulation study, biases and mean squared errors are analyzed for maximum likelihood estimators (MLEs). A comparison between the proposed lifetime distribution and other lifetime distributions is conducted based on real data sets.

In *Chapter 4* the generalization of the BS distribution, called the ν-Birnbaum Saunders distribution, is discussed. A number of intriguing and relevant characteristics are investigated in depth. The maximum likelihood principle is employed to estimate the parameters of the univariate ν-BS distribution. To obtain interval estimates, we use asymptotic confidence intervals. Both estimation methodologies have been thoroughly explored in an extensive simulation study. Based on these estimators, the probability coverage of confidence intervals has been evaluated. Real-life applications are provided with three different datasets and compared with the univariate BS distribution.

When a manufacturer has knowledge of the mechanical reliability of the design through the stress-strength model before production, they can significantly reduce their production costs. A system's longevity is determined by its inherent strength and external stresses. A discussion of the stress-strength reliability of the exponential-gamma $(3, \theta)$ distribution is presented in *Chapter 5*. An assessment of the reliability estimation of the single-component model is provided. A simulation study is used to demonstrate how well the MLEs perform. A data application is presented using real data sets to demonstrate how the distribution performs in real-life situations.

In *Chapter 6*, a simple SSALT analysis is provided incorporating Type-II censoring. Here, a flexible failure rate-based approach to Type II Gumbel distribution for SSALT analysis is considered. The baseline distribution of experimental units at each stress level follows the Type II Gumbel distribution. The MLE for the model parameters is derived.

In *Chapter 7*, a conclusion of the thesis is presented, as well as recommendations for future research. A list of references is included at the end of the thesis.

CHAPTER 2

A New Generalization to the DUS Transformation and its Applications

2.1 Introduction

Modeling and analysis of lifetime distributions have been extensively used in many fields of science, like engineering, medicine, survival analysis, and biostatistics. Fitting appropriate distributions is essential for proper data analysis. A search for distributions with a better fit is quite essential for data analysis in statistics and reliability engineering. With application to survival data analysis, Kumar et al. (2015) proposed a method called DUS transformation, which has received attention from many engineers and researchers in recent years. In terms of computation and interpretation, this transformation produces a parsimonious result since it does not include any new parameters other than those involved in the baseline distribution.

In the case where $F(x)$ is the CDF of the baseline distribution, the CDF of the DUS transformed distribution is as follows:

$$G(x) = \frac{1}{e-1}[e^{F(x)} - 1].$$

Maurya et al. (2017a) introduced the DUS transformation of the Lindley distribution. Tripathi et al. (2019) studied the DUS transformation of an exponential distribution and its inference based on the upper record values. Recent studies using the DUS transformation can be seen in the works of Deepthi and Chacko (2020a), Kavya and Manoharan (2020), Anakha and Chacko (2021), and Gauthami and Chacko (2021).

In this chapter, a new class of distribution is introduced using an exponentiated generalization of the DUS transformation, called the power generalized DUS (PGDUS) transformation. When we consider a parallel system, we have to apply power transformations to the distribution of components to get the system's distribution. Generalized exponential distribution was introduced by Gupta and Kundu (1999) which is the distribution of a parallel system when components are distributed exponentially. But when a researcher assumes an exponential distribution for its lifetime, only jerking, overvoltage, or any such random shocks are the cause of failure. It is a limitation. Why don't we go for any other lifetime distribution if the cause of failure is degradation? DUS transformation proved the advantage of getting an accurate model for the given data using baseline distributions like Weibull, Lomax, etc. Nevertheless, the question remains: how would the parallel system be distributed when components are distributed based on the DUS transformation of some baseline models? If we use exponential, Weibull, and Lomax distributions as baselines, what would be their distributional properties? An attempt to investigate the applicability of the exponentiated generalization of DUS transformation of some baseline models is addressed in this chapter. The use of other distributions as baseline distributions can be addressed by researchers.

This generalization improves the flexibility and accuracy of the model. The new PGDUS transformed distribution can be obtained as follows: Let X be a random variable with a baseline CDF $F(x)$ and the corresponding PDF $f(x)$. Then, the CDF of the PGDUS distribution is defined as:

$$G(x) = \left(\frac{e^{F(x)} - 1}{e - 1}\right)^{\theta}, \theta > 0, x > 0. \tag{2.1.1}$$

and the corresponding PDF is,

$$g(x) = \frac{\theta}{(e-1)^{\theta}}(e^{F(x)} - 1)^{\theta-1} e^{F(x)} f(x), \theta > 0, x > 0. \tag{2.1.2}$$

The associated survival function is,

$$S(x) = 1 - \left(\frac{e^{F(x)} - 1}{e - 1}\right)^{\theta}, \theta > 0, x > 0.$$

The corresponding HRF is,

$$h(x) = \frac{\theta f(x) e^{F(x)} (e^{F(x)} - 1)^{\theta - 1}}{(e-1)^{\theta} - (e^{F(x)} - 1)^{\theta}}, \theta > 0, x > 0. \tag{2.1.3}$$

The primary motivation for this research stems from the significance of Eq. (2.1.1), as it is the distribution of failures in a parallel system with θ independent components. When researchers deal with parallel systems with components distributed as DUS-transformed lifetime distributions, the PGDUS transformation proves to be an inevitable tool. So the investigation of the PGDUS transformation of various lifetime distributions is relevant in the sense of the selection of appropriate lifetime models for parallel systems. In other words, it assists researchers in determining which distribution transformations best characterize the behavior of individual components in a parallel system, which has consequences for developing reliable systems and predicting their overall performance. As a result, this work is motivated by the need to improve our understanding of how different lifetime distributions can be effectively used in modeling and optimizing parallel systems, resulting in improved decision-making and reliability in a variety of engineering and scientific applications.

The remaining sections are arranged as follows. Section 2.2 introduces the PGDUS transformation of the exponential distribution. Section 2.3 presents the PGDUS transformation of the Weibull distribution, and Section 2.4 presents the PGDUS transformation of the Lomax distribution. The summary is given in section 2.5.

2.2 PGDUS Exponential Distribution

Here, the PGDUS transformation to the exponential distribution is considered. Consider the exponential distribution with parameter λ as the baseline distribution. Invoking the PGDUS transformation given in Eq.(2.1.1), the CDF of the PGDUS transformation of an exponential (PGDUSE) distribution is obtained as

$$G(x) = \left(\frac{e^{1-e^{-\lambda x}} - 1}{e-1}\right)^{\theta}, \lambda > 0, \theta > 0, x > 0. \tag{2.2.1}$$

and the corresponding PDF is given by,

$$g(x) = \frac{\theta\lambda e^{1-\lambda x - e^{-\lambda x}}(e^{1-e^{-\lambda x}} - 1)^{\theta-1}}{(e-1)^{\theta}}, \lambda > 0, \theta > 0, x > 0. \tag{2.2.2}$$

Then, the associated HRF is,

$$h(x) = \frac{\theta\lambda e^{1-\lambda x - e^{-\lambda x}}(e^{1-e^{-\lambda x}} - 1)^{\theta-1}}{(e-1)^{\theta} - (e^{1-e^{-\lambda x}} - 1)^{\theta}}, \lambda > 0, \theta > 0, x > 0. \tag{2.2.3}$$

A PGDUSE distribution with parameters λ and θ is denoted by $PGDUSE(\lambda, \theta)$. Figure 2.1 shows that the density function of $PGDUSE(\lambda, \theta)$ distribution is likely to be unimodal. The HRF plot for different parameter values is given in Figure 2.2.

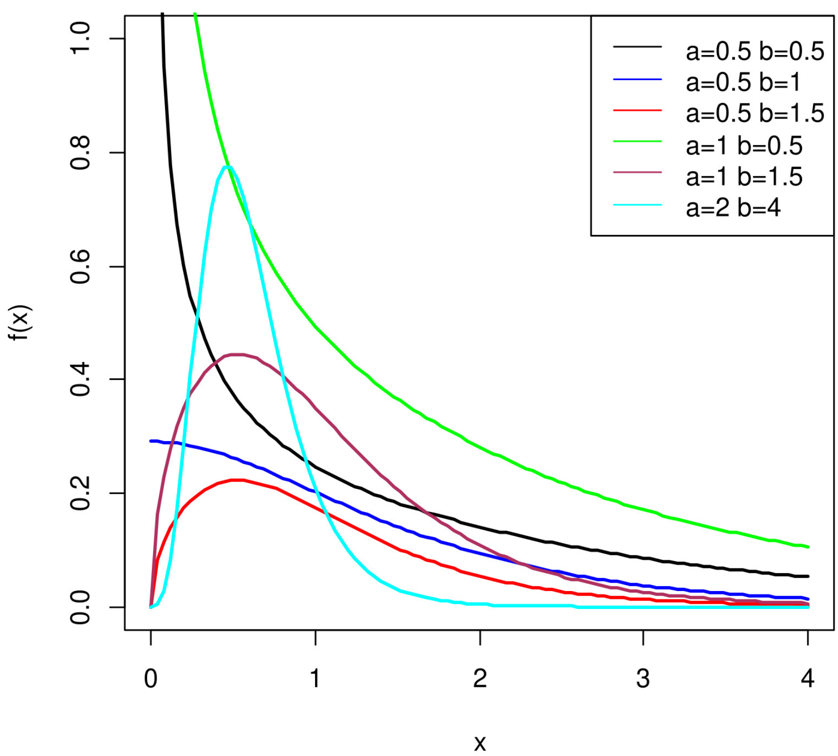


Figure 2.1: Density plot for PGDUSE

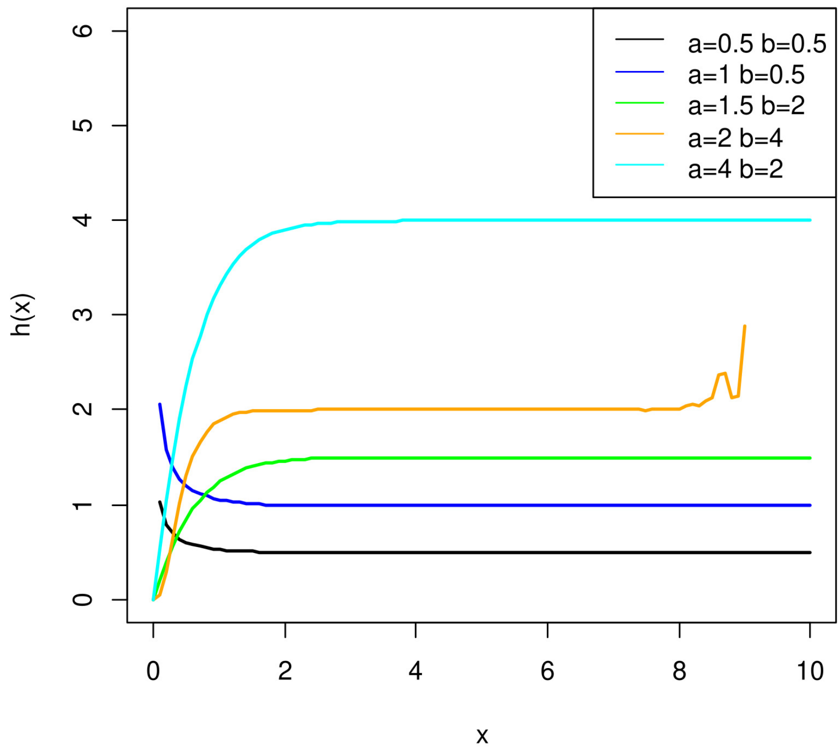


Figure 2.2: Failure rate plot for PGDUSE

2.2.1 Statistical Properties of PGDUSE Distribution

For a distribution, the statistical properties are inevitable. Here, a few statistical properties like moments, moment generating function (MGF), characteristic function (CF), cumulant generating function (CGF), quantile function (QF), order statistics, and entropy of the $PGDUSE(\lambda, \theta)$ distribution are derived.

Moments

The rth raw moment of the $PGDUSE(\lambda, \theta)$ distribution is given by

$$\mu_r' = E(X^r) \quad = \frac{\theta\lambda e}{(e-1)^\theta} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\theta-1}{k} e^{\theta-k-1} (\theta-k)^m \frac{\Gamma(r+1)}{(\lambda+\lambda m)^{r+1}}.$$

By putting r=1, 2, 3... the raw moments can be viewed.

Moment Generating Function

The MGF of $PGDUSE(\lambda, \theta)$ distribution is given by

$$M_X(t) = \frac{\theta\lambda e}{(e-1)^\theta} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\theta-1}{k} e^{\theta-k-1} \frac{(\theta-k)^m}{\lambda+\lambda m-t}.$$

Characteristic Function and Cumulant Generating Function

The characteristic function (CF) is given by

$$\phi_X(t) = \frac{\theta\lambda e}{(e-1)^\theta} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\theta-1}{k} e^{\theta-k-1} \frac{(\theta-k)^m}{\lambda+\lambda m-it},$$

and the cumulant generating function (CGF) is given by

$$K_X(t) = \log\left(\frac{\theta\lambda e}{(e-1)^\theta}\right) + \log\left[\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\theta-1}{k} e^{\theta-k-1} \frac{(\theta-k)^m}{\lambda+\lambda m-it}\right]$$

where $i = \sqrt{-1}$ is the unit imaginary number.

Quantile Function

The qth quantile $Q(q)$ is the solution of the equation $G(Q(q)) = q$. Hence,

$$Q(q) = \frac{-1}{\lambda} \log(1 - \log(q^{\frac{1}{\theta}}(e-1)+1)).$$

The median is obtained by setting $q = 0.5$ in the above equation. Thus,

$$Median = \frac{-1}{\lambda} \log(1 - \log(0.5^{\frac{1}{\theta}}(e-1)+1)).$$

Order Statistic

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics corresponding to the random sample $X_1, X_2, \ldots, X_n$ of size n from the proposed $PGDUSE(\lambda, \theta)$ distribution. The PDF and CDF of rth order statistics of the proposed $PGDUSE(\lambda, \theta)$ distribution are given by

$$g_r(x) = \frac{n!\theta\lambda}{(r-1)!(n-r)!} \frac{e^{1-\lambda x-e^{-\lambda x}}(e^{1-e^{-\lambda x}}-1)^{\theta r-1}}{(e-1)^{2\theta}} \left[1 - \left(\frac{e^{1-e^{-\lambda x}}-1}{e-1}\right)^\theta\right]$$

and

$$G_r(x) = \sum_{i=1}^{n} \binom{n}{i} \left(\frac{e^{1-e^{-\lambda x}} - 1}{e-1} \right)^{\theta i} \left[1 - \left(\frac{e^{1-e^{-\lambda x}} - 1}{e-1} \right)^{\theta} \right]^{n-i}.$$

Then, the PDF and CDF of $X_{(1)}$ and $X_{(n)}$ are obtained by substituting $r = 1$ and $r = n$ respectively in $g_r(x)$ and $G_r(x)$. It is nothing but the distribution of minimum and maximum in series and parallel reliability systems, respectively.

Entropy

Entropy quantifies the measure of information or uncertainty. An important measure of entropy is Rényi entropy (1961). Rényi entropy is defined as

$$\gimel_R(\delta) = \frac{1}{1-\delta} \log \left(\int g^{\delta}(x) dx \right),$$

where $\delta > 0$ and $\delta \neq 1$.

$$\int_0^{\infty} g^{\delta}(x) dx = \frac{\theta^{\delta} \lambda^{\delta} e^{\delta}}{(e-1)^{\theta\delta}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\delta(\theta-1)}{k} (\delta\theta - k)^m e^{\delta\theta-\delta-k} \frac{1}{\lambda(\delta+m)}$$

The Rényi entropy takes the form

$$\gimel_R(\delta) = \tfrac{1}{1-\delta} \log \left[\frac{\theta^{\delta} \lambda^{\delta} e^{\delta}}{(e-1)^{\theta\delta}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\delta(\theta-1)}{k} (\delta\theta - k)^m e^{\delta\theta-\delta-k} \frac{1}{\lambda(\delta+m)} \right]$$

$$= \tfrac{\delta}{1-\delta} \log \left[\frac{\theta\lambda e}{(e-1)^{\theta}} \right] + \tfrac{1}{1-\delta} \log \left[\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \tbinom{\delta(\theta-1)}{k} (\delta\theta - k)^m \frac{e^{\delta\theta-\delta-k}}{\lambda(\delta+m)} \right].$$

2.2.2 Estimation of PGDUSE Distribution

The estimation of parameters by the method of maximum likelihood is discussed. For this, consider a random sample of size n from $PGDUSE(\lambda, \theta)$ distribution. In this case, the likelihood function is given by,

$$L(x) = \prod_{i=1}^{n} g(x) = \prod_{i=1}^{n} \frac{\theta\lambda}{(e-1)^{\theta}} e^{1-\lambda x_i - e^{-\lambda x_i}} (e^{1-e^{-\lambda x_i}} - 1)^{\theta-1}.$$

Then the log-likelihood function becomes,

$$\log L = n\log\theta + n\log\lambda - \theta n\log(e-1) - \lambda\sum_{i=1}^{n} x_i + n - \sum_{i=1}^{n} e^{-\lambda x_i} + (\theta-1)\sum_{i=1}^{n}\log(e^{1-e^{-\lambda x_i}}-1).$$

The maximum likelihood estimators (MLEs) are obtained by maximizing the log-likelihood concerning the unknown parameters λ and θ.

$$\frac{\partial \log L}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i e^{-\lambda x_i} + (\theta-1)\sum_{i=1}^{n}\frac{x_i e^{1-\lambda x_i - e^{-\lambda x_i}}}{e^{1-e^{-\lambda x_i}}-1}.$$

$$\frac{\partial \log L}{\partial\theta} = \frac{n}{\theta} - n\log(e-1) + \sum_{i=1}^{n}\log(e^{1-e^{-\lambda x_i}}-1).$$

These non-linear equations can be numerically solved through statistical software like R using arbitrary initial values. In the case of asymptotic normal MLEs, the confidence interval(CI)s for λ and θ are calculated by computing the observed information matrix given by

$$I = \begin{pmatrix} \frac{\partial^2 \log L}{\partial\lambda^2} & \frac{\partial^2 \log L}{\partial\lambda\partial\theta} \\ \frac{\partial^2 \log L}{\partial\theta\partial\lambda} & \frac{\partial^2 \log L}{\partial\theta^2} \end{pmatrix}$$

where

$$\frac{\partial^2 \log L}{\partial\lambda^2} = -\frac{n}{\lambda^2} - \lambda\sum_{i=1}^{n} x_i e^{-\lambda x_i} - (\theta-1)\lambda\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}((e^{\lambda x_i}-1)e^{1-(\lambda x_i)-e^{-\lambda x_i}}-1)}{(e^{1-e^{-\lambda x_i}}-1)^2},$$

$$\frac{\partial^2 \log L}{\partial\theta\partial\lambda} = \frac{\partial^2 \log L}{\partial\lambda\partial\theta} = \sum_{i=1}^{n} x_i\frac{e^{1-\lambda x_i - e^{-\lambda x_i}}}{(e^{1-e^{-\lambda x_i}}-1)},$$

and

$$\frac{\partial^2 \log L}{\partial\theta^2} = -\frac{n}{\theta^2}.$$

For λ and θ, the $100(1-\gamma)\%$ asymptotic CIs are as follows: $\hat{\lambda} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{11}}$ and $\hat{\theta} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{22}}$, where V_{ij} represents the (i,j)th element in the inverse of the Fisher information matrix I. The computational efficiency of this interval estimation method makes it particularly useful.

2.2.3 Simulation Study

To illustrate the accuracy of the maximum likelihood estimation procedure for $PGDUSE$ distribution, a Monte Carlo simulation study is carried out using the inversion method. Samples of sizes 50, 75, 100, 500, and 1000 for the parameter combinations (0.5, 0.5), (0.5, 1.5), (1, 1.5), and (1.5, 2.5) corresponding to (λ, θ) are generated. The performance of the estimation procedure is studied by calculating the bias and mean square error (MSE) of the MLEs. It can be seen from Table 2.1, 2.2, 2.3, and 2.4 that, as the sample size increases, the bias and MSEs of the estimates decrease.

Table 2.1: Estimate, Biases and MSEs for PGDUSE model at $\lambda = 0.5$ and $\theta = 0.5$

n	Estimated value of Parameters	Bias	MSE
50	$\hat{\lambda}$=0.5248	0.0248	0.0126
	$\hat{\theta}$=0.5223	0.0223	0.0087
75	$\hat{\lambda}$=0.5162	0.0162	0.0086
	$\hat{\theta}$=0.5137	0.0137	0.0055
100	$\hat{\lambda}$=0.5114	0.0114	0.0044
	$\hat{\theta}$=0.5104	0.0104	0.0035
500	$\hat{\lambda}$=0.5101	0.0101	0.0010
	$\hat{\theta}$=0.5066	0.0066	0.0007
1000	$\hat{\lambda}$=0.5019	0.0019	0.0004
	$\hat{\theta}$=0.5042	0.0042	0.0003

2.2.4 Data Analysis

Real data analysis is given to assess how well the proposed distribution works have been performed. The data given in Lawless (1982) that contains the number of million revolutions before the failure of 23 ball bearings put on life test is considered. See Table 2.5.

Further, the proposed distribution has been compared with the generalized DUS exponential (GDUSE) by Maurya et al. (2017b), DUS exponential (DUSE), exponential (ED), and Kavya-Manoharan exponential (KME) by Kavya and Manoharan (2021) distributions. AIC, BIC, the value of KS statistic, p-value,

Table 2.2: Estimate, Biases and MSEs for PGDUSE model at $\lambda = 0.5$ and $\theta = 1.5$

n	Estimated value of Parameters	Bias	MSE
50	$\hat{\lambda}$=0.5192	0.0192	0.0066
	$\hat{\theta}$=1.6242	0.1242	0.1442
75	$\hat{\lambda}$=0.5165	0.0165	0.0042
	$\hat{\theta}$=1.5711	0.0790	0.0758
100	$\hat{\lambda}$=0.5158	0.0158	0.0031
	$\hat{\theta}$=1.5719	0.0719	0.0607
500	$\hat{\lambda}$=0.5025	0.0025	0.0005
	$\hat{\theta}$=1.5122	0.0122	0.0083
1000	$\hat{\lambda}$=0.5009	0.0009	0.0003
	$\hat{\theta}$=1.4709	-0.0291	0.0045

Table 2.3: Estimate, Biases and MSEs for PGDUSE model at $\lambda = 1$ and $\theta = 1.5$

n	Estimated value of Parameters	Bias	MSE
50	$\hat{\lambda}$=1.0236	0.0236	0.0255
	$\hat{\theta}$=1.5655	0.0655	0.1267
75	$\hat{\lambda}$=1.0190	0.0190	0.0161
	$\hat{\theta}$=1.5484	0.0484	0.0793
100	$\hat{\lambda}$=1.0116	0.0116	0.0113
	$\hat{\theta}$=1.5062	0.0062	0.0434
500	$\hat{\lambda}$=1.0091	0.0091	0.0023
	$\hat{\theta}$=1.5178	0.0178	0.0098
1000	$\hat{\lambda}$=0.9889	-0.0111	0.0010
	$\hat{\theta}$=1.4805	-0.0195	0.0039

and log-likelihood value have been used for model selection.

Table 2.6 elucidates that the proposed distribution gives the lowest AIC, BIC, and KS values, the greatest log-likelihood, and the p-value. Thus, it can be concluded that the $PGDUSE(\lambda, \theta)$ distribution provides a better fit for the given data set when compared with other competing distributions. The empirical cumulative density function (ECDF) plot is depicted in Figure 2.3.

Table 2.4: Estimate, Biases and MSEs for PGDUSE model at $\lambda = 1.5$ and $\theta = 2.5$

n	Estimated value of Parameters	Bias	MSE
50	$\hat{\lambda}$=1.5536	0.0536	0.0453
	$\hat{\theta}$=2.7200	0.2200	0.4536
75	$\hat{\lambda}$=1.5363	0.0363	0.0290
	$\hat{\theta}$=2.6169	0.1169	0.2836
100	$\hat{\lambda}$=1.5229	0.0229	0.0210
	$\hat{\theta}$=2.6154	0.1154	0.2005
500	$\hat{\lambda}$=1.5052	0.0052	0.0040
	$\hat{\theta}$=2.5271	0.0271	0.0314
1000	$\hat{\lambda}$=1.4897	-0.0103	0.0020
	$\hat{\theta}$=2.4774	-0.0226	0.0144

Table 2.5: Ball bearings dataset

17.88	28.92	33.00	41.52	42.12	45.60
48.80	51.84	51.96	54.12	55.56	67.80
68.64	68.64	68.88	84.12	93.12	98.64
105.12	105.84	127.92	128.04	173.40	

Table 2.6: MLEs of the parameters, Log-likelihoods, AIC, BIC, KS Statistics and p-values of the fitted models

Model	MLEs	log **L**	AIC	BIC	KS	p-value
PGDUSE	$\hat{\lambda} = 0.0336$ $\hat{\theta} = 3.8066$	-113.0030	230.0060	232.2770	0.1103	0.9425
GDUSE	$\hat{\alpha} = 4.7391$ $\hat{\beta} = 0.0355$	-113.0466	230.0931	232.3641	0.1179	0.9064
DUSE	$\hat{a} = 0.0182$	-127.4622	256.9244	261.1954	0.2774	0.0580
KME	$\hat{\theta} = 0.0095$	-123.1065	248.2129	252.4839	0.3110	0.0234
ED	$\hat{\theta} = 0.0138$	-121.4393	244.8786	246.0141	0.30673	0.0264

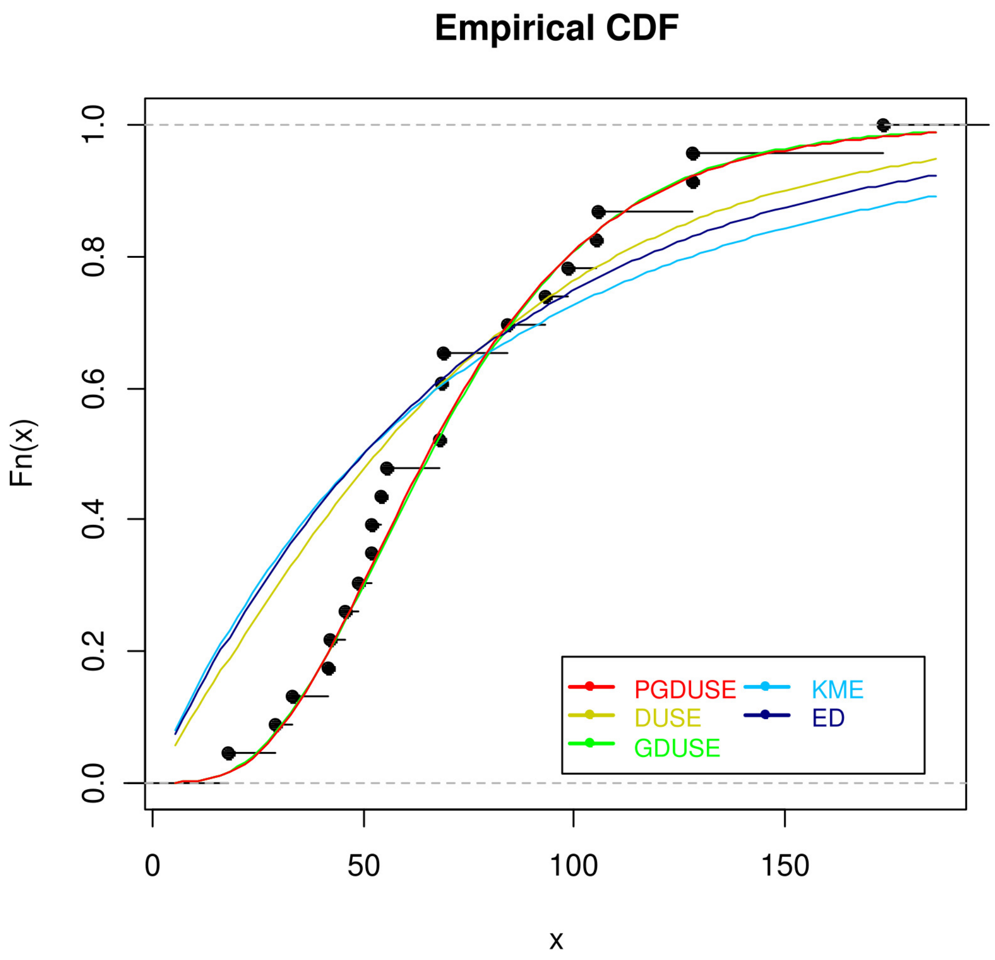


Figure 2.3: The empirical CDFs of the models.

2.3 PGDUS Weibull Distribution

Weibull distribution is used as the baseline distribution for PGDUS transformation and investigated the distributional properties. The CDF of Weibull distribution with parameters α and β is

$$G(x) = 1 - e^{-(x\beta)^{\alpha}}, \alpha, \beta > 0, x > 0. \tag{2.3.1}$$

and corresponding PDF is

$$g(x) = \alpha\beta(x\beta)^{\alpha-1}e^{-(x\beta)^{\alpha}}, \alpha, \beta > 0, x > 0 \tag{2.3.2}$$

Using Eq.(2.3.1) in Eq.(2.1.1), the CDF of PGDUS transformation of Weibull

(PGDUSW) distribution is as follows:

$$F(x) = \left(\frac{e^{1-e^{-(x\beta)^{\alpha}}} - 1}{e-1}\right)^{\theta}, \alpha, \beta > 0, \theta > 0, x > 0. \tag{2.3.3}$$

and the corresponding PDF is

$$f(x) = \frac{\theta\alpha\beta^{\alpha}}{(e-1)^{\theta}} x^{\alpha-1}(e^{1-e^{-(x\beta)^{\alpha}}} - 1)^{\theta-1} e^{1-(x\beta)^{\alpha}-e^{-(x\beta)^{\alpha}}}, \alpha, \beta, \theta > 0, x > 0. \tag{2.3.4}$$

In relation to Eq.(2.3.3) and Eq.(2.3.4), the HRF is,

$$h(x) = \frac{\theta\alpha\beta^{\alpha}x^{\alpha-1}(e^{1-e^{-(x\beta)^{\alpha}}} - 1)^{\theta-1} e^{1-(x\beta)^{\alpha}-e^{-(x\beta)^{\alpha}}}}{(e-1)^{\theta} - (e^{1-e^{-(x\beta)^{\alpha}}} - 1)^{\theta}}, \alpha, \beta, \theta > 0, x > 0. \tag{2.3.5}$$

The distribution with CDF Eq.(2.3.3) and PDF Eq.(2.3.4) is referred to as PGDUSW distribution with parameters α, β and θ and is denoted as $PGDUSW(\alpha, \beta, \theta)$. Figures 2.4 and 2.5 provide the graphical representation of the pdf and HRF respectively for various parameter values.

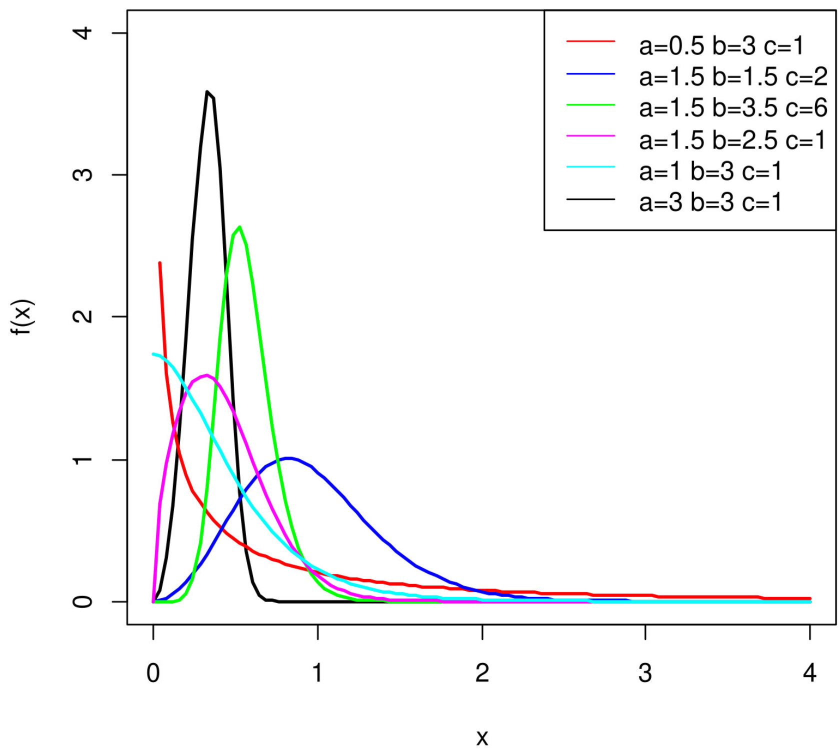


Figure 2.4: Density plot for PGDUSW

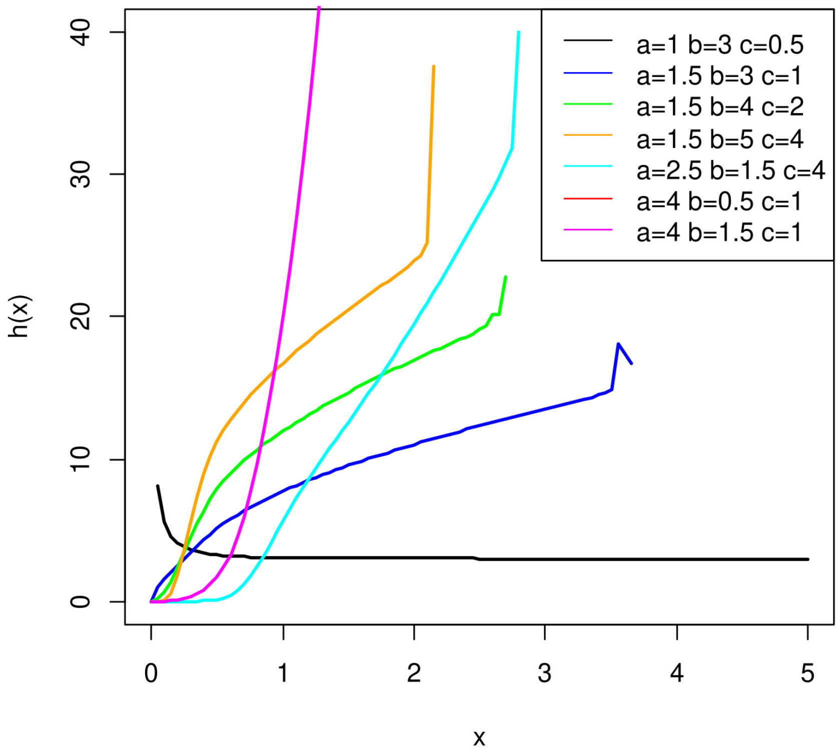


Figure 2.5: Failure rate plot for PGDUSW

2.3.1 Statistical Properties of PGDUSW Distribution

Moments, MGF, CF, CGF, QF, distribution of order statistics, and Rényi entropy of the proposed $PGDUSW(\alpha, \beta, \theta)$ distribution are derived.

Moments

The rth raw moment of the $PGDUSW(\alpha, \beta, \theta)$ distribution is given by

$$\mu_r' = \frac{\theta \beta^{-r} e}{(e-1)^{\theta}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+k}}{m!} e^{\theta-k-1} \binom{\theta-1}{k} (\theta-k)^m \frac{\Gamma(\frac{r}{\alpha}+1)}{(1+m)^{\frac{r}{\alpha}+1}}.$$

Moment Generating Function

The MGF of $PGDUSW(\alpha, \beta, \theta)$ distribution is

$$M_X(t) = \frac{\theta \alpha}{(e-1)^{\theta}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{k+m+n}}{m!n!} \binom{\theta-1}{k} e^{\theta-k} (\theta-k)^m (1+m)^n \beta^{\alpha+\alpha n} \frac{\Gamma(\alpha+\alpha n)}{t^{\alpha+\alpha n}}.$$

Characteristic Function and Cumulant Generating Function

The CF of $PGDUSW(\alpha, \beta, \theta)$ is given by

$$\phi_X(t) = \frac{\theta\alpha}{(e-1)^\theta} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{k+m+n}}{m!n!} \binom{\theta-1}{k} e^{\theta-k}(\theta-k)^m(1+m)^n \beta^{\alpha+\alpha n} \frac{\Gamma(\alpha+\alpha n)}{(it)^{\alpha+\alpha n}},$$

and the CGF of $PGDUSW(\alpha, \beta, \theta)$ is given by

$$\begin{aligned} K_X(t) &= \log \phi_X(t) \\ &= \log\left[\tfrac{\theta\alpha}{(e-1)^\theta} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{k+m+n}}{m!n!} \binom{\theta-1}{k} e^{\theta-k}(\theta-k)^m(1+m)^n \beta^{\alpha+\alpha n} \tfrac{\Gamma(\alpha+\alpha n)}{(it)^{\alpha+\alpha n}}\right] \end{aligned}$$

where $i = \sqrt{-1}$ is the unit imaginary number.

Quantile Function

The pth quantile $Q(p)$ of the $PGDUSW(\alpha, \beta, \theta)$ is the real solution of the following equation

$$((e^{1-e^{-(\beta Q(p))^\alpha}} - 1)/(e-1))^\theta = p$$

where $p \sim Uniform(0, 1)$. Solving the above equation for $Q(p)$, it is obtained that

$$Q(p) = \frac{-1}{\beta^\alpha} \log[1 - \log(e-1)p^{\frac{1}{\theta}} + 1]^{\frac{1}{\alpha}}. \tag{2.3.6}$$

Setting $p = 0.5$ in the Eq.(2.3.6) yields the median. Thus,

$$Median = \frac{-1}{\beta^\alpha} \log[1 - \log(e-1)0.5^{\frac{1}{\theta}} + 1]^{\frac{1}{\alpha}}.$$

Similarly, the quartiles Q_1 and Q_3 are obtained respectively by setting $p = \frac{1}{4}$ and $p = \frac{3}{4}$ in Eq.(2.3.6).

Distribution of Order Statistic

Let $X_1, X_2, \ldots, X_m$ be m independent random variables from the $PGDUSW(\alpha, \beta, \theta)$ distribution with CDF Eq.(2.3.3) and PDF Eq.(2.3.4). Then the PDF of rth order statistics $X_{(r)}$ of the $PGDUSW(\alpha,\beta,\theta)$ distribution is given by

$$f_{X_{(r)}} = \frac{m!}{(r-1)!(m-r)!}\frac{\theta\alpha\beta^{\alpha}x^{\alpha-1}}{(e-1)^{\theta m}}\left(e^{1-e^{-(x\beta)^{\alpha}}}-1\right)^{\theta r-1} e^{1-(x\beta)^{\alpha}-e^{-(x\beta)^{\alpha}}} \\ \left[(e-1)^{\theta}-(e^{1-e^{-(x\beta)^{\alpha}}})^{\theta}\right]^{m-r}, r=1,2,\ldots,m. \tag{2.3.7}$$

Then, the PDF of $X_{(1)}$ and $X_{(m)}$ are obtained by setting $r = 1$ and $r = m$ respectively in Eq.(2.3.7). This can be used in reliability analysis of series and parallel system.

Rényi Entropy

Rényi entropy introduced by Rényi (1961) is defined as

$$\beth_R(\nu) = \frac{1}{1-\nu}\log\left(\int f^{\nu}(x)dx\right)$$

where $\nu > 0$ and $\nu \neq 1$.

$$\int_0^{\infty} f^{\nu}(x)dx = \frac{(\theta\alpha)^{\nu}}{(e-1)^{\theta\nu}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!}\binom{\nu\theta-\nu}{k}(\nu\theta-k)^{m}e^{\nu\theta-k}\frac{\Gamma(\nu-\frac{\nu}{\alpha}+1)}{(\nu+m)^{\nu-\frac{\nu}{\alpha}+1}\beta^{\alpha-\nu}}$$

Then the Rényi entropy of the $PGDUSW(\alpha,\beta,\theta)$ becomes

$$\beth_R(\nu) = \tfrac{1}{1-\nu}\log\left[\frac{(\theta\alpha)^{\nu}}{(e-1)^{\theta\nu}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!}\binom{\nu\theta-\nu}{k}(\nu\theta-k)^{m}e^{\nu\theta-k}\frac{\Gamma(\nu-\frac{\nu}{\alpha}+1)}{(\nu+m)^{\nu-\frac{\nu}{\alpha}+1}\beta^{\alpha-\nu}}\right]$$

2.3.2 Estimation of PGDUSW Distribution

To estimate the unknown parameters of the $PGDUSW(\alpha,\beta,\theta)$, the maximum likelihood estimation method is utilized. For this, a random sample of size n from the $PGDUSW(\alpha,\beta,\theta)$ distribution was chosen. Therefore, the likelihood function is given by,

$$L(x) = \prod_{i=1}^{n} f(x) = \prod_{i=1}^{n}\frac{\theta\alpha\beta^{\alpha}}{(e-1)^{\theta}}x^{\alpha-1}e^{1-(x_i\beta)^{\alpha}-e^{-(x_i\beta)^{\alpha}}}(e^{1-e^{-(x_i\beta)^{\alpha}}}-1)^{\theta-1} \tag{2.3.8}$$

Applying the natural logarithm to Eq.(2.3.8), the log-likelihood function becomes

$$\log L = n\log(\theta) + n\log(\alpha) + \alpha n\log(\beta) - \theta n\log(e-1) + n + \sum_{i=0}^{n}(\alpha-1)\log(x_i)$$

$$-\sum_{i=0}^{n}(x_i\beta)^{\alpha} - \sum_{i=0}^{n} e^{-(x_i\beta)^{\alpha}} + (\theta-1)\sum_{i=0}^{n}\log(e^{1-e^{-(x_i\beta)^{\alpha}}}-1).$$

Computing the first order partial derivatives,

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=0}^{n}(x_i\beta)^{\alpha}\log(x_i\beta) + \sum_{i=0}^{n}\log(x_i) + \sum_{i=0}^{n}(x_i\beta)^{\alpha}e^{-(x_i\beta)^{\alpha}}\log(x_i\beta)$$

$$+n\log(\beta) + \frac{(\theta-1)(x_i\beta)^{\alpha}}{(e^{1-e^{-(x_i\beta)^{\alpha}}}-1)}\log(x_i\beta)e^{1-(x_i\beta)^{\alpha}-e^{-(x_i\beta)^{\alpha}}}, \tag{2.3.9}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n\alpha}{\beta} - \sum_{i=0}^{n}\frac{\alpha(x_i\beta)^{\alpha}}{\beta} + \sum_{i=0}^{n}\frac{\alpha(x_i\beta)^{\alpha}}{\beta}e^{-(x_i\beta)^{\alpha}}$$

$$+(\theta-1)\frac{\alpha}{\beta}\sum_{i=0}^{n}(x_i\beta)^{\alpha}\frac{e^{1-(x_i\beta)^{\alpha}-e^{-(x_i\beta)^{\alpha}}}}{(e^{1-e^{-(x_i\beta)^{\alpha}}}-1)}, \tag{2.3.10}$$

and

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - n\log(e-1) + \sum_{i=0}^{n}\log(e^{1-e^{-(x_i\beta)^{\alpha}}}-1). \tag{2.3.11}$$

Equations (2.3.9), (2.3.10) and (2.3.11) are not in closed form. The solution to these explicit equations can be obtained analytically and can be solved numerically using R software by taking arbitrary initial values. In the case of asymptotic normal MLEs, the confidence interval(CI)s for α, β, and θ are calculated by computing the observed information matrix given by

$$I = \begin{pmatrix} \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \beta} & \frac{\partial^2 \log L}{\partial \alpha \partial \theta} \\ \\ \frac{\partial^2 \log L}{\partial \beta \partial \alpha} & \frac{\partial^2 \log L}{\partial \beta^2} & \frac{\partial^2 \log L}{\partial \beta \partial \theta} \\ \\ \frac{\partial^2 \log L}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L}{\partial \theta \partial \beta} & \frac{\partial^2 \log L}{\partial \theta^2} \end{pmatrix}$$

where

$$\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{n}{\alpha} - \sum_{i=1}^{n} \log^2(x_i\beta)(x_i\beta)^\alpha((x_i\beta)^\alpha - 1)e^{-(x_i\beta)^\alpha}$$

$$+(\theta - 1)\textstyle\sum_{i=1}^{n} \frac{(x_i\beta)^\alpha \log^2(x_i\beta)e^{1-(x_i\beta)^\alpha - e^{-(x_i\beta)^\alpha}}((((x_i\beta)^\alpha - 1)e^{1-e^{-(x_i\beta)^\alpha}} - (x_i\beta)^\alpha e^{-(x_i\beta)^\alpha}) + (1-(x_i\beta)^\alpha))}{(e^{1-e^{-(x_i\beta)^\alpha}} - 1)^2},$$

$$\frac{\partial^2 \log L}{\partial \alpha \partial \beta} = -\sum_{i=1}^{n} (x_i\beta)^\alpha e^{-(x_i\beta)^\alpha}[\alpha((x_i\beta)^\alpha - 1)\ln(x_i\beta) - 1]$$

$$+\frac{n}{\beta} - (\theta - 1)\sum_{i=1}^{n} \frac{(x_i\beta)^\alpha([\alpha[(x_i\beta)^\alpha - 1]e^{(x_i\beta)^\alpha} - \alpha(x_i\beta)^\alpha]e^{1-(x_i\beta)^\alpha - e^{-(x_i\beta)^\alpha}}}{\beta(e^{1-e^{-(x_i\beta)^\alpha}} - 1)^2}$$

$$-\sum_{i=1}^{n} x_i^\alpha \beta^{\alpha-1}(\alpha \ln(x_i\beta) + 1) - (\theta - 1)\sum_{i=1}^{n} \frac{\alpha(1 - (x_i\beta)^\alpha)\ln(x_i\beta) - e^{1-e^{1-(x_i\beta)^\alpha}} + 1)}{\beta(e^{1-e^{-(x_i\beta)^\alpha}} - 1)^2},$$

$$\frac{\partial^2 \log L}{\partial \alpha \partial \theta} = \sum_{i=1}^{n} \frac{(x_i\beta)^\alpha \log(x_i\beta)e^{1-(x_i\beta)^\alpha - e^{-(x_i\beta)^\alpha}}}{(e^{1-e^{-(x_i\beta)^\alpha}} - 1)},$$

$$\frac{\partial^2 \log L}{\partial \beta^2} = \frac{-\alpha n}{\beta^2} - \frac{\alpha(\alpha-1)}{\beta^2}\sum_{i=1}^{n}(x_i\beta)^{\alpha} - \frac{\alpha}{\beta^2}\sum_{i=1}^{n}(x_i\beta)^{\alpha}(\alpha(x_i\beta)^{\alpha}-\alpha+1)e^{-(x_i\beta)^{\alpha}}$$

$$+(\theta-1)\tfrac{\alpha}{\beta^2}\textstyle\sum_{i=1}^{n}(x_i\beta)^{\alpha}\frac{(((\alpha(x_i\beta)^{\alpha}-\alpha+1)e^{(x_i\beta)^{\alpha}}-\alpha(x_i\beta)^{\alpha})e^{1-(x_i\beta)^{\alpha}-e^{-(x_i\beta)^{\alpha}}}(-\alpha(x_i\beta)^{\alpha}+\alpha-1))}{(e^{1-e^{-(x_i\beta)^{\alpha}}}-1)^2}$$

$$\frac{\partial^2 \log L}{\partial \beta \partial \theta} = \frac{\alpha}{\beta}\sum_{i=1}^{n}\frac{(x_i\beta)^{\alpha}e^{1-(x_i\beta)^{\alpha}-e^{-(x_i\beta)^{\alpha}}}}{(e^{1-e^{-(x_i\beta)^{\alpha}}}-1)},$$

and

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{n}{\theta^2}.$$

For α, β, and θ, the $100(1-\gamma)\%$ asymptotic CIs are as follows: $\hat{\alpha} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{11}}$, $\hat{\beta} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{22}}$, and $\hat{\theta} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{33}}$, where V_{ij} represents the (i, j)th element in the inverse of the Fisher information matrix I.

2.3.3 Simulation Study

To illustrate the performance of the maximum likelihood method for $PGDUSW(\alpha, \beta, \theta)$ distribution, the inverse transformation method is used. For different values of α, β and θ, samples of sizes $n = 100, 250, 500, 750$ and 1000 are generated from the proposed model. For 1000 repetitions, the bias and mean square error (MSE) of the estimated parameters are computed. The selected parameter values are $\alpha = 0.5, \beta = 0.5$ and $\theta = 0.5$, $\alpha = 0.5, \beta = 1$ and $\theta = 0.5$ and $\alpha = 1, \beta = 1$ and $\theta = 0.5$. From the Tables 2.7, 2.8 and 2.9, it is noted that bias and MSE decrease for the selected parameter values as sample size increases.

2.3.4 Data Analysis

A real data analysis is carried out to determine the performance of the proposed model. For this, the data on the number of million revolutions before the failure of 23 ball bearings put on test is considered (Lawless (1982)), see Table 2.5.

Table 2.7: Estimate, Biases and MSEs for PGDUSW model at $\alpha = 0.5, \beta = 0.5$ and $\theta = 0.5$

n	Estimated Parameter values	Bias	MSE
100	$\hat{\alpha}$=0.5668	0.0668	0.0473
	$\hat{\beta}$=0.7541	0.2541	1.0617
	$\hat{\theta}$=0.5021	0.0031	0.0413
250	$\hat{\alpha}$=0.5251	0.0251	0.0118
	$\hat{\beta}$=0.5831	0.0831	0.1488
	$\hat{\theta}$=0.5032	0.0022	0.0165
500	$\hat{\alpha}$=0.5297	0.0189	0.0057
	$\hat{\beta}$=0.4929	0.0177	0.0318
	$\hat{\theta}$=0.4922	0.0007	0.0068
750	$\hat{\alpha}$=0.5188	0.0188	0.0034
	$\hat{\beta}$=0.4935	-0.0065	0.0223
	$\hat{\theta}$=0.5026	0.0003	0.0050
1000	$\hat{\alpha}$=0.5165	0.0165	0.0025
	$\hat{\beta}$=0.4795	-0.0205	0.0159
	$\hat{\theta}$=0.4922	-0.0078	0.0035

Different distributions namely, Inverse Weibull (IW) distribution, DUS Exponential (DUSE) distribution by Kumar et al. (2015), and Kavya-Manoharan Weibull (KMW) by Kavya and Manoharan (2021) distribution are used to compare the performance with the proposed $PGDUSW(\alpha, \beta, \theta)$ distribution.

To check the acceptability of the $PGDUSW(\alpha, \beta, \theta)$ distribution for the given data set AIC, Corrected Akaike Information Criterion (AICc), log-likelihood value,

Table 2.8: Estimate, Biases and MSEs for PGDUSW model at $\alpha = 0.5, \beta = 1$ and $\theta = 0.5$

n	**Estimated Parameter values**	**Bias**	**MSE**
100	$\hat{\alpha}$=0.5729	0.0729	0.0460
	$\hat{\beta}$=1.4827	0.4827	3.7354
	$\hat{\theta}$=0.51341	0.0434	0.0485
250	$\hat{\alpha}$=0.5019	0.0019	0.0083
	$\hat{\beta}$=1.2852	0.2852	0.6372
	$\hat{\theta}$=0.5333	0.0393	0.0169
500	$\hat{\alpha}$=0.4943	-0.0057	0.0041
	$\hat{\beta}$=1.2236	0.2236	0.2915
	$\hat{\theta}$=0.5399	0.0339	0.0102
750	$\hat{\alpha}$=0.4886	-0.0109	0.0023
	$\hat{\beta}$=1.1045	0.1814	0.1353
	$\hat{\theta}$=0.5244	0.0244	0.0050
1000	$\hat{\alpha}$=0.4822	-0.0178	0.0022
	$\hat{\beta}$=1.1814	0.1045	0.1195
	$\hat{\theta}$=0.5207	0.0207	0.0042

and KS goodness of fit test statistic with the p-value are used and the computed values are provided in Table 2.10. It is worth noting that in the goodness of fit test, the purpose is to determine whether the sets of data with the distribution function $F(y)$ and the hypothesised distribution $F_{PGDUSW}(y)$ are compatible. This problem can be formulated as $H_0 : F(y) = F_{PGDUSW}(y)$ versus the alternative $H_1 : F(y) \neq F_{PGDUSW}(y)$.

Table 2.9: Estimate, Biases and MSEs for PGDUSW model at $\alpha = 1, \beta = 1$ and $\theta = 0.5$

n	Estimated Parameter values	Bias	MSE
100	$\hat{\alpha}$=1.1273	0.1273	0.1628
	$\hat{\beta}$=1.1460	0.1460	0.8851
	$\hat{\theta}$=0.5223	0.0223	0.0545
250	$\hat{\alpha}$=1.0184	0.0184	0.0449
	$\hat{\beta}$=1.0889	0.0889	0.1068
	$\hat{\theta}$=0.5205	0.0205	0.0177
500	$\hat{\alpha}$=1.0109	0.0109	0.0185
	$\hat{\beta}$=1.0490	0.0490	0.0447
	$\hat{\theta}$=0.5151	0.0151	0.0085
750	$\hat{\alpha}$=1.0056	0.0056	0.0107
	$\hat{\beta}$=1.0381	0.0381	0.0260
	$\hat{\theta}$=0.5095	0.0095	0.0049
1000	$\hat{\alpha}$=0.9851	-0.0149	0.0074
	$\hat{\beta}$=1.0239	0.0239	0.0167
	$\hat{\theta}$=1.0012	0.0012	0.0035

From Table 2.10, it is noted that the $PGDUSW(\alpha, \beta, \theta)$ distribution fits well for the given data set. To facilitate a better understanding of the results, the plot of the ECDF is shown in the Figure 2.6 along with the plot of fitted densities in the Figure 2.7 of the distributions for the ball bearings dataset. Furthermore, our proposed distribution is found to fit better than those of the other distributions.

Table 2.10: Findings for PGDUSW Distribution

Model	MLEs	log **L**	AIC	AICc	KS	p-value
IW	$\hat{\lambda} = 1.8341$ $\hat{\theta} = 0.0206$	-115.7887	235.5774	236.1774	0.1328	0.8118
DUSE	$\hat{a} = 0.0182$	-127.4622	256.9244	257.1149	0.2774	0.0580
KMW	$\hat{\lambda} = 2.3169$ $\hat{\kappa} = 0.0107$	-113.4076	230.8152	231.4152	0.1421	0.7419
PGDUSW	$\hat{\alpha} = 0.9362$ $\hat{\beta} = 0.0383$ $\hat{\theta} = 4.4478$	-113.0114	230.0228	230.6228	0.10921	0.9467

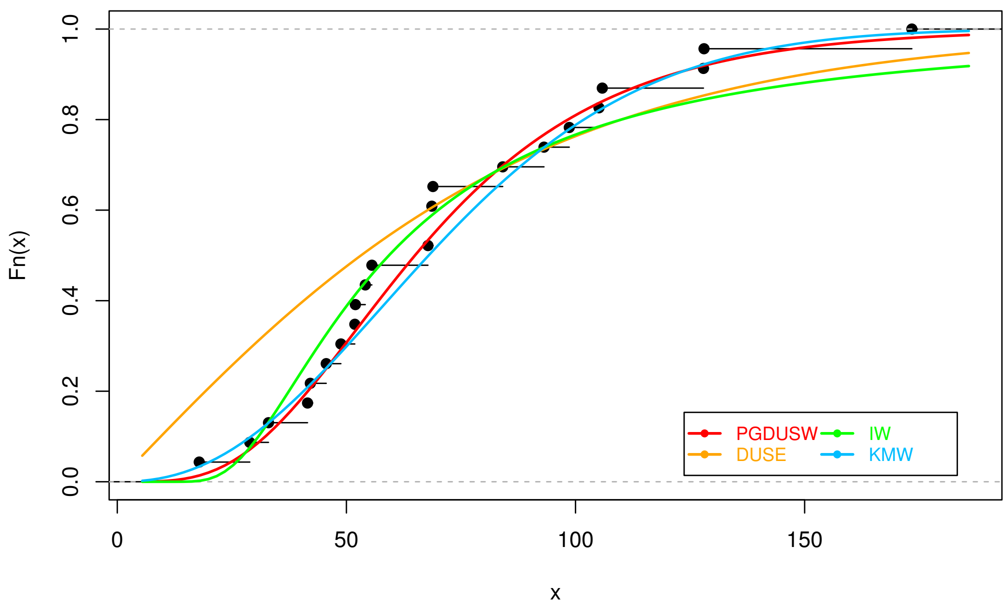


Figure 2.6: ECDF plot for various distributions.

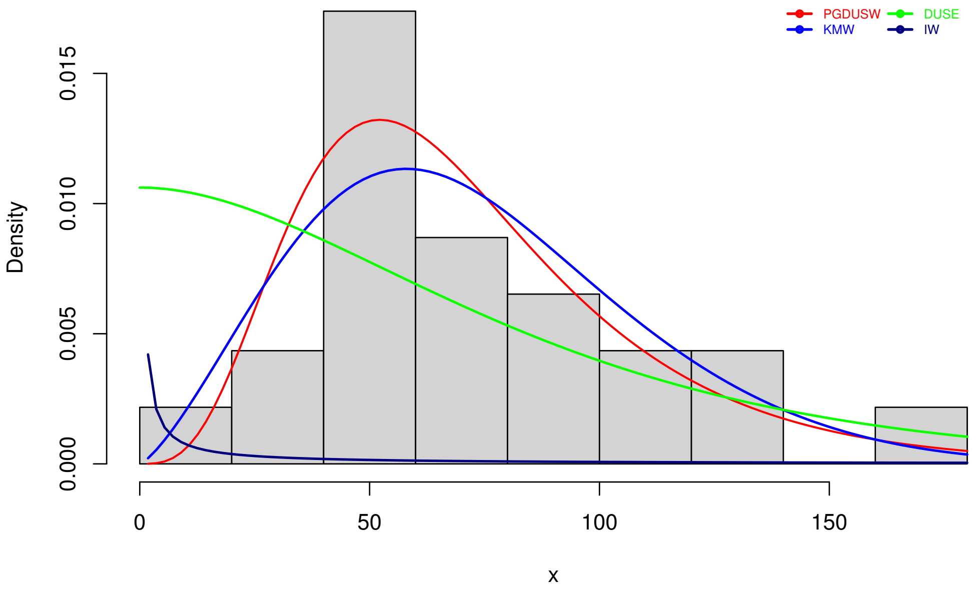


Figure 2.7: Fitted Density plot for various distributions.

2.4 PGDUS Lomax Distribution

Power Generalized DUS Lomax (PGDUSL) Distribution, denoted as $PGDUSL(\alpha, \beta, \theta)$, is obtained using PGDUS transformation with Lomax distribution as baseline distribution. Then the CDF of the $PGDUSL(\alpha, \beta, \theta)$ distribution using Eq.(2.1.1) is given by

$$F(x) = \left(\frac{e^{1-(1+x\beta)^{-\alpha}} - 1}{e - 1} \right)^{\theta}, \alpha, \beta > 0, \theta > 0, x > 0. \tag{2.4.1}$$

Then the PDF is

$$f(x) = \frac{\theta\alpha\beta}{(e-1)^{\theta}} (e^{1-(1+x\beta)^{-\alpha}} - 1)^{\theta-1} e^{1-(1+x\beta)^{-\alpha}} (1 + x\beta)^{-(\alpha+1)}. \tag{2.4.2}$$

The HRF is

$$h(x) = \frac{\theta\alpha\beta(e^{1-(1+x\beta)^{-\alpha}} - 1)^{\theta-1} e^{1-(1+x\beta)^{-\alpha}} (1 + x\beta)^{-(\alpha+1)}}{(e-1)^{\theta} - (e^{1-(1+x\beta)^{-\alpha}} - 1)^{\theta}}$$

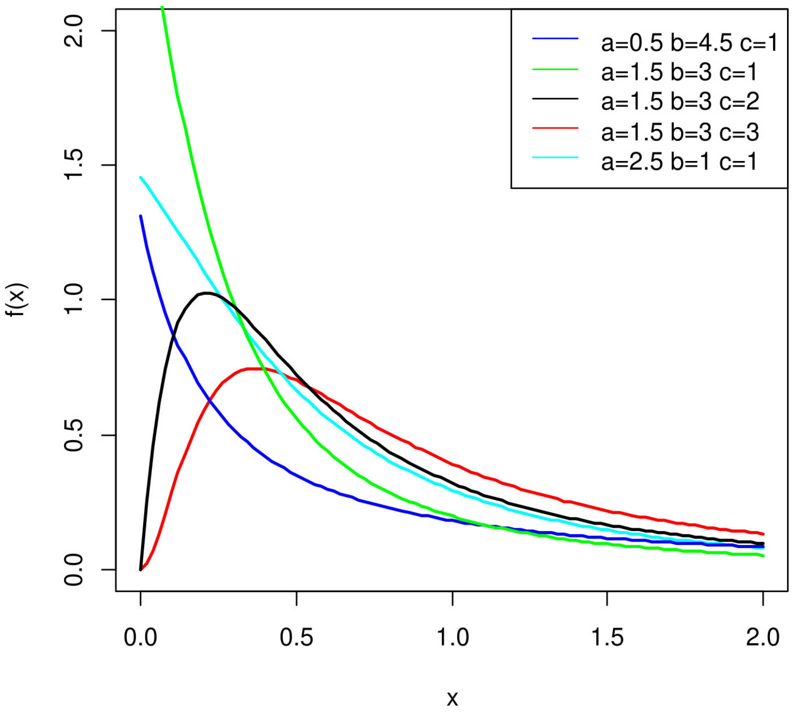


Figure 2.8: PGDUSL distribution density plot for various parameter values.

2.4.1 Properties of PGDUSL Distribution

Here, a few properties of the PGDUSL distribution are explored.

Moments

The r^{th} raw moments of $PGDUSL(\alpha, \beta, \theta)$ is

$$\mu_r' = \frac{\theta\alpha}{(e-1)^{\theta}} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{k+m+n}}{n!} \binom{\alpha+k}{k} \binom{\theta-1}{m} \beta^{k+1}\, e^{\theta-m}\,(\theta-m)^n$$
$$B(r+k+1, \alpha n-r-k-1).$$

Moment Generating Function

The MGF of $PGDUSL(\alpha, \beta, \theta)$ is

$$M_X(t) = \frac{\theta\alpha}{(e-1)^{\theta}} \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{k+m+n}}{n!\; l!\; \beta^{l}} \binom{\alpha-k}{k} \binom{\theta-1}{m} e^{\theta-m}\,(\theta-m)^n\, t^l$$

$$B(k+l+1, \alpha\, n-k-l-1).$$

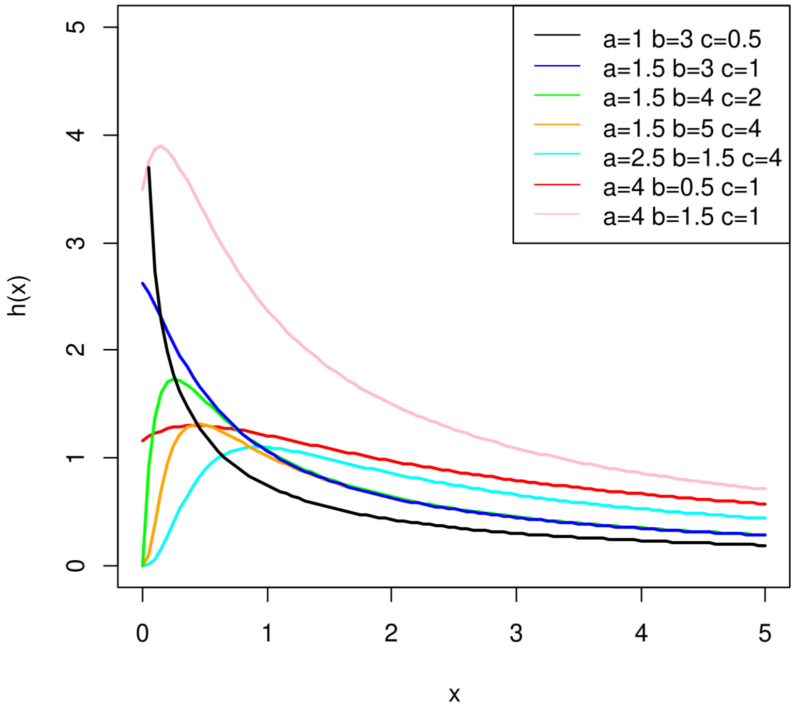


Figure 2.9: PGDUSL distribution HRF plot for various parameter values.

Characteristic Function and Cumulant Generating Function

The CF of the proposed distribution is given by

$$\phi_X(t) = \frac{\theta\alpha}{(e-1)^{\theta}} \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{k+m+n}}{n!\, l!\, \beta^{l}} \binom{\alpha-k}{k}\binom{\theta-1}{m} e^{\theta-m}\,(\theta-m)^{n}\,(it)^{l}$$

$$B(k+l+1, \alpha\, n-k-l-1).$$

The CGF of the proposed distribution is given by

$$K_X(t) = \log\left\{ \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{k+m+n}}{n!\, l!\, \beta^{l}} \binom{\alpha-k}{k}\binom{\theta-1}{m} e^{\theta-m}\,(\theta-m)^{n}\,(it)^{l} \right.$$

$$\left. B(k+l+1, \alpha\, n-k-l-1)\right\} + \log\left(\frac{\theta\alpha}{(e-1)^{\theta}}\right).$$

Quantile Function

The pth quantile $Q(p)$ of the $PGDUSL(\alpha,\beta,\theta)$ is the real solution of the following equation

$$((e^{1-1+(\beta Q(p))^{\alpha}}-1)/(e-1))^{\theta}=p,$$

where $p\sim Uniform(0,1)$. Solving the above equation for $Q(p)$, it can be obtained as

$$Q(p)=\frac{1}{\beta}\Big\{\Big[1-\log\Big[p^{\frac{1}{\theta}}(e-1)+1\Big]\Big]^{\frac{-1}{\alpha}}-1\Big\}.$$

Setting $p=0.5$ in the above equation yields median. Thus,

$$Median=\frac{1}{\beta}\Big\{\Big[1-\log\Big[0.5^{\frac{1}{\theta}}(e-1)+1\Big]\Big]^{\frac{-1}{\alpha}}-1\Big\}.$$

2.4.2 Estimation of PGDUSL Distribution

Method of Maximum likelihood estimation is used to estimate the unknown parameters of $PGDUSL(\alpha,\beta,\theta)$. For this, a random sample of size n from $PGDUSL(\alpha,\beta,\theta)$ distribution was chosen. Then the likelihood function is given by,

$$L(x)=\prod_{i=1}^{n}f(x)=\frac{(\theta\alpha\beta)^{n}}{(e-1)^{\theta n}}\prod_{i=1}^{n}(e^{1-(1+x_i\beta)^{-\alpha}}-1)^{\theta-1}e^{1-(1+x_i\beta)^{-\alpha}}(1+x_i\beta)^{-\alpha+1} \quad (2.4.3)$$

The log-likelihood function becomes

$$\log L=n\log(\theta)+n\log(\alpha)+n\log(\beta)-\theta n\log(e-1)+n-\sum_{i=1}^{n}(1+x_i\beta)^{-\alpha}$$

$$-(\alpha+1)\sum_{i=1}^{n}\log(1+x_i\beta)+(\theta-1)\sum_{i=1}^{n}\log(e^{1-(1+x_i\beta)^{-\alpha}}-1). \quad (2.4.4)$$

Computing the first order partial derivatives of Eq.(2.4.4),

$$\frac{\partial\log L}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log(1+x_i\beta)(1+x_i\beta)^{-\alpha}-\sum_{i=1}^{n}\log(1+x_i\beta)$$
$$+\sum_{i=1}^{n}\frac{(\theta-1)\log(1+x_i\beta)e^{1-(1+x_i\beta)^{-\alpha}}(1+x_i\beta)^{-\alpha}}{(e^{1-(1+x_i\beta)^{-\alpha}}-1)}. \quad (2.4.5)$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \alpha x_i (1 + x_i\beta)^{-(\alpha+1)} - (\alpha + 1) \sum_{i=1}^{n} \frac{x_i}{1 + x_i\beta}$$

$$- \sum_{i=1}^{n} \frac{\alpha x_i (\theta - 1)(1 + x_i\beta)^{-(\alpha+1)}}{(e^{1-(1+x_i\beta)^{-\alpha}} - 1)}, \tag{2.4.6}$$

and

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - n\log(e - 1) + \sum_{i=1}^{n} \log(e^{1-(1+x_i\beta)^{-\alpha}} - 1). \tag{2.4.7}$$

Equations (2.4.5), (2.4.6) and (2.4.7) are not in closed form. The solution of these explicit equations can be obtained analytically and can be solved numerically using R software by taking arbitrary initial values.

In the case of asymptotic normal MLEs, the confidence interval(CI)s for α, β, and θ are calculated by computing the observed information matrix given by

$$I = \begin{pmatrix} \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \beta} & \frac{\partial^2 \log L}{\partial \alpha \partial \theta} \\ \frac{\partial^2 \log L}{\partial \beta \partial \alpha} & \frac{\partial^2 \log L}{\partial \beta^2} & \frac{\partial^2 \log L}{\partial \beta \partial \theta} \\ \frac{\partial^2 \log L}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L}{\partial \theta \partial \beta} & \frac{\partial^2 \log L}{\partial \theta^2} \end{pmatrix}$$

where

$$\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \sum_{i=1}^{n} (1 + \beta x_i)^{-\alpha} \log^2(1 + \beta x_i)$$

$$+ (\theta - 1) \sum_{i=1}^{n} \frac{\log^2(1 + x_i\beta) e^{1-(1+x_i\beta)^{-\alpha}} (1 + x_i\beta)^{-\alpha} [1 - (1 + x_i\beta)^{-\alpha} - e^{1-(1+x_i\beta)^{-\alpha}}]}{(e^{1-(1+x_i\beta)^{-\alpha}} - 1)^2},$$

$$\frac{\partial^2 \log L}{\partial\alpha\partial\beta} = -\sum_{i=1}^{n} x_i A^{-1} - (\theta-1)\sum_{i=1}^{n} A^{-(\alpha+1)} e^{1-A^{-\alpha}} \frac{[x_i(e^{1-A^{-\alpha}} - 1 - \alpha x_i A^{-(\alpha+1)})]}{(e^{1-A^{-\alpha}}-1)^2}$$

$$+ (\theta-1)\sum_{i=1}^{n} A^{-(\alpha+1)} e^{1-A^{-\alpha}} \frac{[\alpha\log(A) e^{1-A^{-\alpha}}(A^{-\alpha}\log(A) - 1 + e^{-(1-A^{-\alpha})} - A^{-\alpha}e^{-(1-A^{-\alpha})})]}{(e^{1-A^{-\alpha}}-1)^2}$$

$$-\sum_{i=1}^{n} x_i A^{-(\alpha+1)}[\alpha\log(A) - 1]$$

where $A = (1 + x_i\beta)$

$$\frac{\partial^2 \log L}{\partial\alpha\partial\theta} = \frac{\partial^2 \log L}{\partial\theta\partial\alpha} = \sum_{i=1}^{n} \frac{\log(1+\beta x_i) e^{1-(1+\beta x_i)^{-\alpha}}(1+\beta x_i)^{-\alpha}}{(e^{1-(1+\beta x_i)^{-\alpha}} - 1)},$$

$$\frac{\partial^2 \log L}{\partial\beta^2} = -\frac{n}{\beta^2} - \sum_{i=1}^{n} \alpha(\alpha+1)x_i^2(1+\beta x_i)^{-(\alpha+2)} + (\alpha+1)\sum_{i=1}^{n} x_i^2(1+\beta x_i)^{-2} +$$

$$\alpha(\theta-1)\textstyle\sum_{i=1}^{n} x_i^2 \frac{(\alpha+1)(1+\beta x_i)^{-(\alpha+2)}(e^{1-(1+\beta x_i)^{-\alpha}}-1) - \alpha(1+\beta x_i)^{-2(\alpha+1)}e^{1-(1+\beta x_i)^{-\alpha}}}{(e^{1-(1+\beta x_i)^{-\alpha}}-1)^2},$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\theta} = -\sum_{i=1}^{n} \frac{\alpha x_i(1+\beta x_i)^{-(\alpha+1)}}{(e^{1-(1+\beta x_i)^{-\alpha}} - 1)},$$

and

$$\frac{\partial^2 \log L}{\partial\theta^2} = -\frac{n}{\theta^2}.$$

For α, β, and θ, the $100(1-\gamma)\%$ asymptotic CIs are as follows: $\hat{\alpha} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{11}}$, $\hat{\beta} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{22}}$, and $\hat{\theta} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{33}}$, where V_{ij} represents the (i,j)th element in the inverse of the Fisher information matrix I.

2.4.3 Simulation Study

In order to demonstrate the performance of the maximum likelihood method for the proposed $PGDUSL(\alpha,\beta,\theta)$ distribution, the inverse transformation method is used.

For different combinations of values of α, β, and θ, samples of sizes n = 250, 500, 750, and 1000 are generated from the $PGDUSL(\alpha, \beta, \theta)$ model. The bias and mean square error (MSE) of the estimated parameters are calculated for 1000 iterations. The selected parameter values are $\alpha = 0.5, \beta = 0.5$ and $\theta = 0.5$, $\alpha = 1, \beta = 1.5$ and $\theta = 0.5$ and $\alpha = 1, \beta = 1.5$ and $\theta = 1$. From Tables 2.11, 2.12, and 2.13, it is observed that bias and MSE decreases for the selected parameter values as the sample size increases.

Table 2.11: Estimate, Biases and MSEs for PGDUSL model at $\alpha = 0.5, \beta = 0.5$ and $\theta = 0.5$

n	Estimated value of Parameters	Bias	MSE
250	$\hat{\alpha}$=0.5100	0.0100	0.0031
	$\hat{\beta}$=0.5520	0.0720	0.0665
	$\hat{\theta}$=0.5218	0.0218	0.0049
500	$\hat{\alpha}$=0.4921	-0.0039	0.0016
	$\hat{\beta}$=0.5926	0.0526	0.0422
	$\hat{\theta}$=0.5197	0.0197	0.0023
750	$\hat{\alpha}$=0.4960	-0.0079	0.0010
	$\hat{\beta}$=0.5313	0.0343	0.0181
	$\hat{\theta}$=0.5088	0.0088	0.0013
1000	$\hat{\alpha}$=0.4889	-0.0111	0.0008
	$\hat{\beta}$=0.5343	0.0313	0.0134
	$\hat{\theta}$=0.5046	0.0046	0.0009

2.4.4 Real Data Application

Real data analysis is used to determine the applicability of the PGDUSL model. The data set shown in Table 2.14 is uncensored. Among 128 patients with bladder cancer in a random sample, it corresponds to the number of months they experienced

Table 2.12: Estimate, Biases and MSEs for PGDUSL model at $\alpha = 1, \beta = 1.5$ and $\theta = 0.5$

n	Estimated value of Parameters	Bias	MSE
250	$\hat{\alpha}$=1.0268	0.0268	0.0314
	$\hat{\beta}$=1.6452	0.1800	0.4484
	$\hat{\theta}$=0.5217	0.0217	0.0037
500	$\hat{\alpha}$=1.0140	0.0140	0.0131
	$\hat{\beta}$=1.6800	0.1452	0.2215
	$\hat{\theta}$=0.5187	0.0187	0.0017
750	$\hat{\alpha}$=0.9838	-0.0070	0.0080
	$\hat{\beta}$=1.6374	0.1374	0.1404
	$\hat{\theta}$=0.5040	0.0050	0.0008
1000	$\hat{\alpha}$=0.9930	-0.0162	0.0059
	$\hat{\beta}$=1.6070	0.1070	0.0906
	$\hat{\theta}$=0.5050	0.0040	0.0006

remission, as reported by Lee and Wang (2003). Different distributions, namely the Lomax distribution (LD) by Lomax (1954), the DUSE distribution by Kumar et al. (2015), and the DUS Lomax (DUSL) distribution by Deepthi and Chacko (2020), are used to compare the performance with the proposed $PGDUSL(\alpha, \beta, \theta)$ distribution.

To check the acceptability of the $PGDUSL(\alpha, \beta, \theta)$ distribution for the given data set AIC, Corrected AIC (AICc), log-likelihood value, KS value and p-value are used and the computed values are provided in Table 2.15. From Table 2.15, it is clear that $PGDUSL(\alpha, \beta, \theta)$ distribution fits well for the given data set. To facilitate a better understanding of the results, the plot of the ECDF is shown in the Figure 2.10 along with fitted density plot in the Figure 2.11 of the distributions

Table 2.13: Estimate, Biases and MSEs for PGDUSL model at $\alpha = 1, \beta = 1.5$ and $\theta = 1$

n	Estimated value of Parameters	Bias	MSE
250	$\hat{\alpha}$=1.0284	0.0284	0.0194
	$\hat{\beta}$=1.69386	0.19386	0.71071
	$\hat{\theta}$=1.05298	0.05297	0.03426
500	$\hat{\alpha}$=1.0179	0.0179	0.0082
	$\hat{\beta}$=1.5999	0.0999	0.1999
	$\hat{\theta}$=1.0472	0.0472	0.0144
750	$\hat{\alpha}$=0.9917	-0.0083	0.0049
	$\hat{\beta}$=1.5596	0.0596	0.1101
	$\hat{\theta}$=1.0145	0.0145	0.0068
1000	$\hat{\alpha}$=0.9836	-0.0164	0.0033
	$\hat{\beta}$=1.5187	0.0187	0.0755
	$\hat{\theta}$=0.9967	-0.0033	0.0051

for the blood cancer patients dataset. Furthermore, our proposed distribution is found to fit better than those of the other distributions.

2.5 Summary

In this chapter, a new class of distribution generalizing the DUS transformation, called the PGDUS transformation, is introduced. A new lifetime distribution called the PGDUSE distribution with exponential as the baseline distribution is proposed. The generalized form provides greater flexibility in modeling real datasets. When a parallel system is considered, if the components are distributed as DUS transformations of some baseline models, PGDUS transformation is the only solution. Different statistical properties such as moments, MGF, CF, quantile function, CGF, order statistic, and entropy of the PGDUSE distribution are derived. The parameter

Table 2.14: Blood Cancer Patients Dataset

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23
0.52	4.98	6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09
0.22	13.80	25.74	0.50	2.46	3.64	5.09	7.26	9.47	14.24
0.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81
0.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32
0.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
0.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01
0.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33
0.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64	17.36
0.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46	4.40	5.85
0.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37	12.02	2.02
0.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76	12.07
0.73	2.07	3.36	6.93	8.65	12.63	22.69	5.49		

Table 2.15: Findings for PGDUSL distribution

Model	**MLEs**	$\log \mathbf{L}$	**AIC**	**AICc**	**KS**	**p-value**
LD	$\hat{\lambda} = 15.2817$ $\hat{\theta} = 0.0074$	-414.98	833.960	834.056	0.094	0.208
DUSE	$\hat{\mu} = 0.1342$	-433.139	868.278	868.309	0.081	0.366
DUSL	$\hat{\lambda} = 6.471$ $\hat{\theta} = 0.0253$	-413.077	830.153	830.249	0.075	0.463
PGDUSL	$\hat{\alpha} = 3.842$ $\hat{\beta} = 0.0605$ $\hat{\theta} = 1.3984$	-411.019	828.039	828.2324	0.035	0.998

estimation has been done using the method of maximum likelihood. Monte Carlo simulations are carried out. Real data analysis is performed to show that the

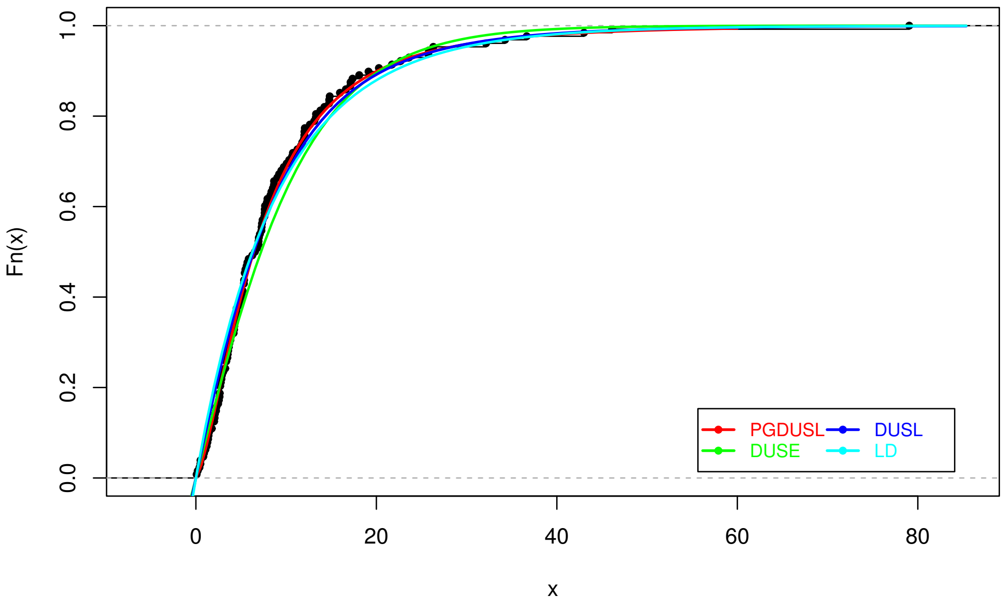


Figure 2.10: ECDF plot of the models for blood cancer patients dataset.

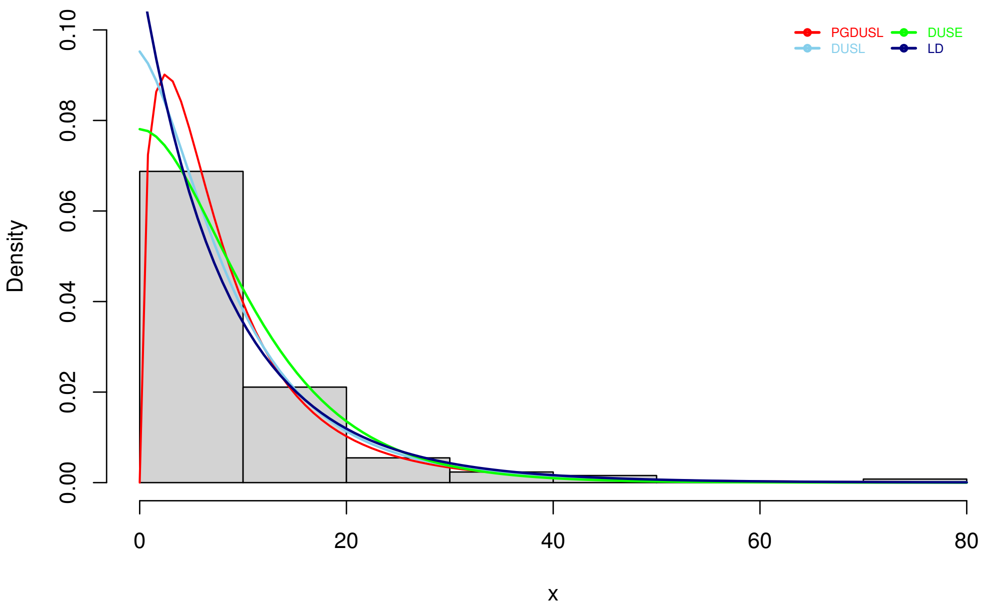


Figure 2.11: Estimated densities of the models for the blood cancer patients dataset.

proposed generalization of the DUS transformation using exponential distributions can be used effectively to provide better fits.

Similarly, the power generalized DUS transformations of Weibull and Lomax distributions have been proposed. Studies on fundamental properties like moments, MGF, CF, CGF, quantile function, distribution of order statistics, and Rényi entropy are also carried out. The parameter estimation has been given by the maximum likelihood method. By using the simulation study, it is observed that the estimates of the proposed distributions have a smaller bias and mean square error when the sample size is large. Real data applications have been performed to determine the applicability of the proposed model. Furthermore, a better fit is adjudged for the proposed model when compared with a few existing models. When conducting reliability analyses on a parallel system where each of the components has a specific DUS-transformed lifetime distribution, the PGDUS approach is highly useful.

CHAPTER 3

Exponential-Gamma $(3, \theta)$ Distribution: A Bathtub Shaped Failure Rate Model

3.1 Introduction

Modeling and analyzing lifetime data using mixture distribution is a prominent practice in many applied sciences, such as medicine, engineering, and finance. Mixture distributions are useful when dealing with lifetime data analysis. When a new component switches on for the first time, it may fail at the same instant, or it may fail due to overvoltage, jerking, or any such shocks. Failure due to random shocks can be modeled using an exponential distribution, while failure due to the degradation of components occurs. Failure time may be distributed as a Gamma distribution, Weibull distribution, or any other lifetime distribution if it is fitted to the data. When a group of lifetimes consists of lifetimes due to both types of failures, such as random failures and failures due to degradation, one should use a mixture.

A variety of distributions can be used to model lifetime data, though the failure rate functions of the majority of them do not exhibit bathtub shapes. However, many real-life systems demonstrate BFR functions. To address this discrepancy, distributions like the exponentiated Weibull by Pal et al.(2006), exponentiated gamma by Nadarajah and Gupta (2007), generalized Lindley by Nadarajah et al. (2011), and X-exponential by Chacko (2016) have been proposed to model lifetime data with bathtub-shaped failure rate models.

Models with bathtub-shaped failure rate functions apply to reliability analysis, particularly in reliability-related decision-making, cost analysis, and burn-in analysis. It is necessary to use exponential distributions when dealing with random failures and other lifetime distributions when dealing with failures due to ageing in such situations. The purpose of this chapter is to examine a mixture of an exponential distribution and a gamma distribution that has a BFR function. Real-world problems can be accurately modeled by this distribution.

The introduction of a mixture distribution uses gamma and exponential distributions in many different areas. This modeling strategy is useful when working with populations, systems, or datasets that have intrinsic differences in their properties. The exponential distribution is used to represent constant failure rates, whereas the gamma distribution, with a shape value of 3, describes wear-out failure mechanisms. The occurrence of various behaviors within a population or system can be explained by using these two distributions as a mixture. The fact that we can produce bathtub-shaped failure rate behavior for this combination distribution is a major concern. This is beneficial in reliability analysis, health research, financial modeling, quality control, and other fields.

This chapter is organized as follows. Section 3.2 considers the exponential-gamma $(\mathbf{3}, \theta)$ distribution. In section 3.3, various statistical properties of the exponential-gamma $(\mathbf{3}, \theta)$ distribution are derived. The estimation procedure is given in section 3.4. Section 3.5 provides a comprehensive simulation study. Additionally, section 3.6 provides data analysis. At the end of the chapter, a summary is given.

3.2 Exponential-Gamma $(\mathbf{3}, \theta)$ Distribution

A mixture of exponential (θ) and gamma $(3,\theta)$ distributions are considered. It is denoted as EGD(θ). The PDF of the mixture of the exponential (θ) and gamma $(3,\theta)$ distribution is as follows:

$$f(x;\theta) = p\, f_1(x;\theta) + (1-p)\, f_2(x;3,\theta),$$

where $p = \frac{\theta}{1+\theta}$, $f_1(x;\theta) = \theta e^{-\theta x}$ and $f_2(x;3,\theta) = \theta^3 \frac{x^2}{2} e^{-\theta x}$.
Then,

$$f(x;\theta) = \frac{\theta^2}{1+\theta}(1+\frac{\theta}{2}x^2)e^{-\theta x}, x > 0, \theta > 0. \tag{3.2.1}$$

The CDF corresponding to the EGD(θ) distribution is

$$F(x;\theta) = 1 - \frac{(\theta(x(\theta x+2)+2)+2)e^{-\theta x}}{2(1+\theta)}; x > 0, \theta > 0. \tag{3.2.2}$$

The Survival function associated with Eq.(3.2.2) is

$$\bar{F}(x;\theta) = 1 - F(x;\theta) = \frac{(\theta(x(\theta x+2)+2)+2)e^{-\theta x}}{2(1+\theta)}; x > 0, \theta > 0. \tag{3.2.3}$$

The first derivative of the PDF is

$$f'(x) = \frac{\theta^3 e^{-\theta x}}{1+\theta}\left(x - 1 - \frac{\theta x^2}{2}\right).$$

The second derivative of the PDF is

$$f''(x) = \frac{\theta^3 e^{-\theta x}}{1+\theta}\left(1 - 2\theta x + \theta + \frac{\theta^2 x^2}{2}\right).$$

The mode of $f(x)$ is the point $x = x_0$ satisfying $f'(x_0) = 0$. Here $f'(x_0) = 0$ is at the point $x_0 = \frac{1 \pm \sqrt{1-\frac{\theta}{2}}}{\theta}$, $f''(x) < 0$ for $0 < x < 1$ and $f''(x) > 0$ for $1 \leq x \leq 2$.

The shape of the PDF is given in figure 3.1 and 3.2.
From the above figures, it is apparent that the PDF can be decreasing or unimodal.
The HRF of EGD(θ) is given below.

$$h(x) = \frac{f(x,\theta)}{\bar{F}(x,\theta)} = \frac{2(1+\theta)\theta^2(1+\frac{\theta x^2}{2})}{(\theta(x(\theta x+2)+2)+2)}; x > 0, \theta > 0. \tag{3.2.4}$$

The first derivative of HRF is

$$h'(x) = 2(1+\theta)\theta^2 \frac{\theta x(\theta(x(\theta x+2)+2)+2) - 2\theta(\theta x+1)(1+\frac{\theta x^2}{2})}{(\theta(x(\theta x+2)+2)+2)^2}.$$

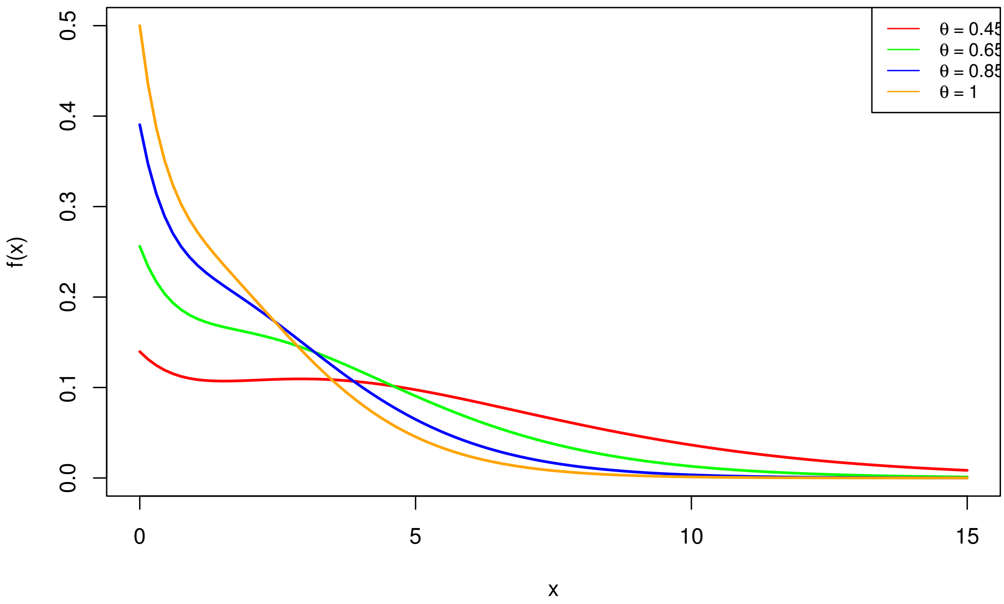


Figure 3.1: PDF plot for $\theta \leq 1$

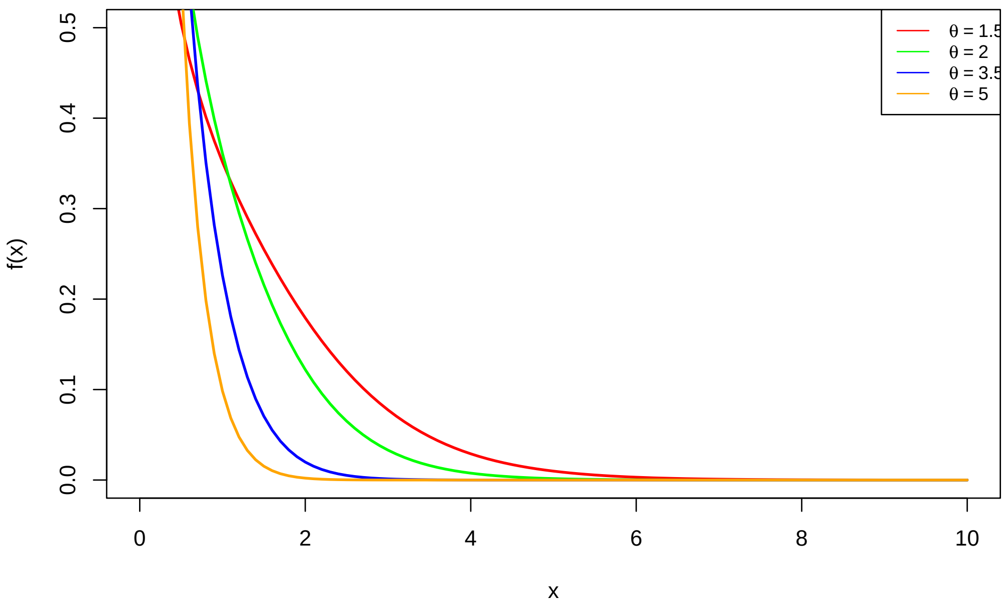


Figure 3.2: PDF plot for $\theta > 1$

The second derivative of HRF is given by

$$h''(x) = \frac{4\theta^3(\theta x + 1)(-\theta^2 x^2 + 6\theta - 2\theta x + 2)(1 + \theta)}{(\theta(x(\theta x + 2) + 2) + 2)^3}.$$

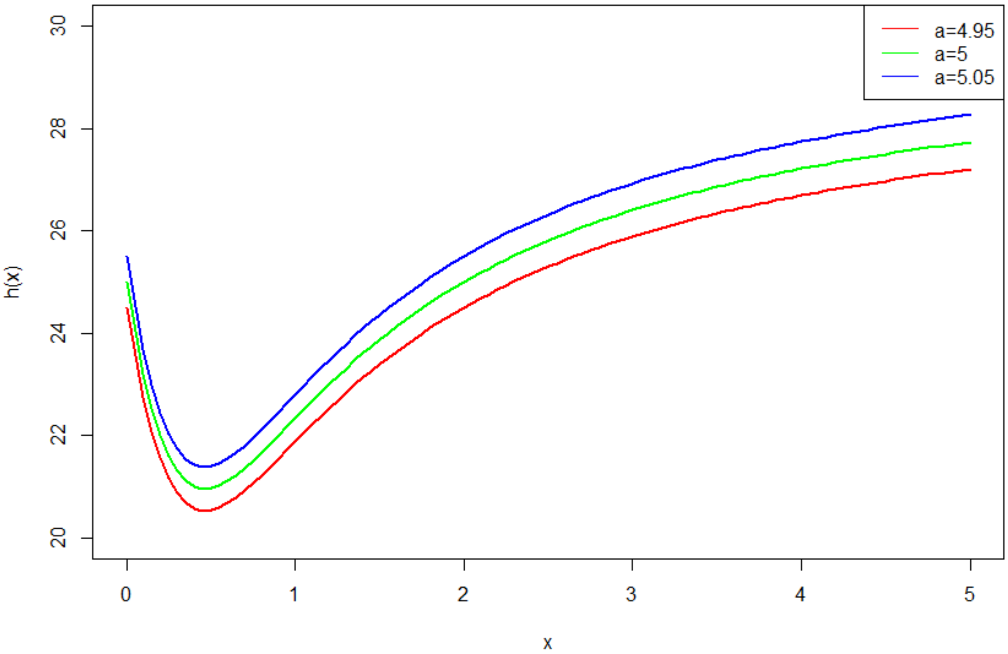


Figure 3.3: HRF plot of $EGD(\theta)$ for θ =4.95, 5, 5.15

The extremum of h(x) is the point $x = x_0$ satisfying $h'(x_0) = 0$, and these points correspond to a maximum or a minimum or a point of inflection according to $h''(x) < 0, h''(x) > 0$ and $h''(x) = 0$ respectively. Here $h'(x) = 0$ at the point $x_0 = \frac{-1+\sqrt{1+2\theta}}{\theta}$ and $h''(x) > 0$ for $\theta > 0$. So h(x) must attain a unique minimum at $x = x_0$.

Initially, the plot of $h(x)$ decreases monotonically and then increases, giving a bathtub shape. Fig.3.3 provides the HRFs of $EGD(\theta)$ for different parameter values.

3.3 Statistical Properties of EGD$(3, \theta)$

Here, the statistical measures for the $EGD(\theta)$ distribution, such as moments, skewness, kurtosis, MGF, CF, quantile function, median, Rènyi entropy, Lorenz curve, and Gini index are discussed.

3.3.1 Moments

In the statistical literature, the concept of moments is of paramount importance. We can measure the central tendency of a population by using moments. Moments also help in measuring the scatteredness, asymmetry, and peakedness of a curve for a particular distribution.

The rth raw moment (about the origin) of $EGD(\theta)$ is

$$\mu_r' = p\frac{r!}{\theta^r} + (1-p)\frac{\Gamma(r+3)}{2\theta^r} = \frac{2\theta r! + \Gamma(r+3)}{2(1+\theta)\theta^r}.$$

Therefore, the mean and variance of $EGD(\theta)$ are respectively given by

$$\mu = \frac{\theta + 3}{\theta(1+\theta)},$$

and

$$\sigma^2 = \frac{\theta^2 + 8\theta + 3}{\theta^2(1+\theta)^2}.$$

The skewness and kurtosis can be obtained using these raw moments as

$$Skewness = \frac{2\theta^3 + 30\theta^2 - 63\theta + 16}{\theta^2 + 8\theta + 3},$$

and

$$Kurtosis = \frac{9\theta^4 + 192\theta^3 + 306\theta^2 + 216\theta + 45}{(\theta^2 + 8\theta + 3)^2}.$$

3.3.2 Moment Generating Function and Characteristic Function

Let X has $EGD(\theta)$ distribution, then the MGF of X, $M_X(t) = E(e^{tX})$, is

$$M_X(t) = \frac{\theta^2}{1+\theta}\left(- \frac{(t-\theta)^2 + \theta}{(t-\theta)^3} \right),$$

for $t > 0$. Similarly, the CF of X becomes $\phi(t) = M_X(it)$,

$$\phi(t) = \frac{\theta^2}{1+\theta}\left(- \frac{(it-\theta)^2 + \theta}{(it-\theta)^3} \right),$$

where $i = \sqrt{-1}$.

3.3.3 Quantile Function and Median

Here, the quantile and median formulas of $EGD(\theta)$ distribution are determined. The quantile x_p of the $EGD(\theta)$ is given from

$$F(x_p) = p, 0 < p < 1$$

The 100 p^{th} percentile can be obtained as,

$$(\theta(x(\theta x + 2) + 2) + 2)e^{-\theta x} = 2(1 - p)(1 + \theta). \tag{3.3.1}$$

Setting $p = 0.5$ in Eq. (3.3.1), the median of $EGD(\theta)$ is obtained as follows.

$$(\theta(x(\theta x + 2) + 2) + 2)e^{-\theta x} = (1 + \theta).$$

The $x_{0.5}$ is the solution of the above monotone increasing function. Using different statistical software, the quantiles or percentiles can be obtained.

3.3.4 Rènyi Entropy

An important entropy measure is Rènyi entropy (Rènyi (1980)). If X has the $EGD(\theta)$ then Rènyi entropy is defined by

$$\Im_R(\nu) = \frac{1}{1-\nu} \log\Big\{\int f^{\nu}(x)dx\Big\},$$

where $\nu > 0$ and $\nu \neq 1$. Then we can calculate, for $EGD(\theta)$,

$$\int f^{\nu}(x)dx = \int_0^{\infty} \left\{\frac{\theta^2}{1+\theta} e^{-\theta x\left(1+\frac{\theta x^2}{2}\right)}\right\}^{\nu} dx$$

$$= \left(\frac{\theta^2}{1+\theta}\right)^{\nu} \int_0^{\infty} \left(1 + \frac{\theta x^2}{2}\right)^{\nu} e^{-\nu\theta x}$$

$$= \left(\frac{\theta^2}{1+\theta}\right)^{\nu} \sum_{k=0}^{\infty} \binom{\nu}{k} (-1)^k \int_0^{\infty} x^{2k} e^{-\nu\theta x} dx$$

$$= \left(\frac{\theta^2}{1+\theta}\right)^{\nu} \sum_{k=0}^{\infty} \binom{\nu}{k} (-1)^k \frac{\Gamma(2k+1)}{(\nu\theta)^{2k+1}}.$$

Therefore, Rènyi entropy is given by

$$\Im_R(\nu) = \frac{1}{1-\nu}\log\left\{\left(\frac{\theta^2}{1+\theta}\right)^{\nu}\sum_{k=0}^{\infty}\binom{\nu}{k}(-1)^k\frac{\Gamma(2k+1)}{(\nu\theta)^{2k+1}}\right\}$$

$$= \frac{\nu}{1-\nu}\log\left(\frac{\theta^2}{1+\theta}\right) + \frac{1}{1-\nu}\log\left\{\sum_{k=0}^{\infty}\binom{\nu}{k}(-1)^k\frac{\Gamma(2k+1)}{(\nu\theta)^{2k+1}}\right\}.$$

3.3.5 Lorenz Curve and Gini Index

The Lorenz curve and the Gini index have applications not only in economics but also in reliability.
The Lorenz curve is defined by

$$L(p) = \frac{1}{p}\int_0^q xf(x)dx$$

or equivalently,

$$L(p) = \frac{1}{p}\int_0^q xF^{-1}(x)dx,$$

where $p = E(X)$ and $q = F^{-1}(p)$.
The Gini index is given by

$$G = 1 - 2\int_0^1 L(p)dp.$$

If X has $EGD(\theta)$ then

$$L(p) = \frac{1}{p}\left[\frac{\theta+3}{\theta(\theta+1)} - \frac{(\theta(q(\theta(q(\theta q+3)+2)+6)+2)+6)e^{-\theta q}}{2\theta(1+\theta)}\right].$$

Gini Index is

$$G = 1 - \frac{2}{p\theta(1+\theta)}\left[\theta+3-\frac{(\theta(q(\theta(q(\theta q+3)+2)+6)+2)+6)e^{-\theta q}}{2}\right], \theta > 0.$$

3.3.6 Distribution of Maximum and Minimum

Let $X_1, X_2, \ldots, X_n$ be a simple random sample from $EGD(\theta)$. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics obtained from this sample. The

PDF of $X_{(r)}$ is given by,

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)}[F(x;\theta)]^{r-1}[1-F(x;\theta)]^{n-r}f(x;\theta),$$

where $F(x;\theta), f(x;\theta)$ are the CDF and PDF given by Eq. (3.2.2) and Eq. (3.2.1), respectively. That is,

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)}\left[1-\tfrac{(\theta(x(\theta x+2)+2)+2)e^{-\theta x}}{2(1+\theta)}\right]^{r-1}\left[\tfrac{(\theta(x(\theta x+2)+2)+2)e^{-\theta x}}{2(1+\theta)}\right]^{n-r}$$

$$\frac{\theta^2}{1+\theta}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}. \quad (3.3.2)$$

Then the PDF of the smallest and largest order statistics, $X_{(1)}$ and $X_{(n)}$, respectively, are

$$f_1(x) = \frac{1}{B(1,n)}\left[\frac{(\theta(x(\theta x+2)+2)+2)}{2(1+\theta)}\right]^{n-1}\frac{\theta^2}{1+\theta}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}$$

and

$$f_n(x) = \frac{1}{B(n,1)}\left[1-\frac{(\theta(x(\theta x+2)+2)+2)}{2(1+\theta)}\right]^{n-1}\frac{\theta^2}{1+\theta}\left(1+\frac{\theta}{2}x^2\right)e^{-\theta x}.$$

The CDF of $X_{(r)}$ is

$$F_{r:n}(x) = \sum_{j=r}^{n}\binom{n}{j}\left[1-\frac{(\theta(x(\theta x+2)+2)+2)}{2(1+\theta)}\right]^{j}\left[\frac{(\theta(x(\theta x+2)+2)+2)}{2(1+\theta)}\right]^{n-j}. \quad (3.3.3)$$

Then the CDF of the smallest and largest order statistics $X_{(1)}$ and $X_{(n)}$, respectively, are

$$F_1(x) = 1-\left[\frac{(\theta(x(\theta x+2)+2)+2)}{2(1+\theta)}\right]^{n}, \theta > 0$$

and

$$F_n(x) = \left[1 - \frac{(\theta(x(\theta x + 2) + 2) + 2)}{2(1 + \theta)}\right]^n, \theta > 0.$$

These distributions can be used in reliability operations.

3.4 Parametric Estimation

In this section, point estimation of the unknown parameter of the $EGD(\theta)$ is described by using the method of maximum likelihood for complete sample data, as given below.

3.4.1 Maximum Likelihood Estimation

The likelihood function of the $EGD(\theta)$ distribution is

$$L = \prod_{i=1}^{n} f(x_i; \theta) = \prod_{i=1}^{n} \frac{\theta^2(1 + \frac{\theta}{2}x_i^2)e^{-\theta x}}{1 + \theta}$$

The log-likelihood function is,

$$\log L(x_i; \theta) = 2n \log \theta - n \log(1 + \theta) + \sum_{i=1}^{n} \left[\log\left(1 + \frac{\theta x_i^2}{2}\right) - \theta x_i\right].$$

The first partial derivatives of the log-likelihood function with respect to θ is

$$\frac{\partial L}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{1 + \theta} + \sum_{i=1}^{n} \left(\frac{x_i^2}{2(1 + \frac{\theta x_i^2}{2})} - x_i\right) \tag{3.4.1}$$

Setting the left side of the above equation to zero, the likelihood equation as a system of nonlinear equations in θ is obtained. Solving this system in θ gives the MLE of θ. It is easy to obtain numerically by using a statistical software package like the *nlm* package in R programming with arbitrary initial values.

The Fisher information about $\theta, I(\theta)$, is

$$I(\theta) = E\left\{-\frac{\partial^2}{\partial \theta^2} \log f(X; \theta)\right\} = E\left(\frac{2}{\theta^2} - \frac{1}{(1 + \theta)^2} + \frac{x^4}{4} \frac{1}{(1 + \frac{\theta x^2}{2})^2}\right)$$

$$= \frac{2}{\theta^2} - \frac{1}{(1+\theta)^2} + E\left\{\frac{x^4}{4}\frac{1}{(1+\frac{\theta x^2}{2})^2}\right\}.$$

Then the asymptotic $100(1-\alpha)\%$ confidence interval for θ is given by

$$\hat{\theta} \pm Z_{\alpha/2}\frac{I^{-1/2}(\hat{\theta})}{\sqrt{n}}.$$

3.5 Simulation Study

A simulation study is conducted to illustrate the performance of the accuracy of the estimation method. The following scheme is used:

1. Specify the value of the parameter θ.
2. Specify the sample size n.
3. Generate a random sample with size n from $EGD(\theta)$.
4. Using the estimation method used in this chapter, calculate the point estimate of the parameter θ.
5. Repeat steps 3-4, N=1000 times.
6. Calculate the bias and the MSE.

3.6 Applications

Data analysis is provided to see how the new model works. The data set is taken from Klein and Berger (1997). It shows survival data on the death times of 26 psychiatric inpatients admitted to the University of Iowa hospital during the years 1935-1948.

Different distributions were used, such as ED, EED, and $EGD(\theta)$, to analyze the data. The estimate(s) of the unknown parameter(s), corresponding KS test statistic, and Log L values for three different models are given in table 3.3. The AIC (see Akaike(1974)), BIC, and CAIC are presented in the following table 3.4.

Table 3.3 shows the parameter MLEs, KS test statistic value with p-value, and log-likelihood values of the fitted distributions, and table 3.4 shows the values of AIC, BIC, and CAIC. The values in tables 3.3 and 3.4 indicate that the $EGD(\theta)$

Table 3.1: Simulation study for $\theta = 1, 1.5, 1.85$.

θ	n	Bias	MSE
1	50	-0.0009	3.6485×10^{-05}
	100	0.0004	1.5573×10^{-05}
	500	1.311×10^{-05}	8.599×10^{-08}
	1000	3.6889×10^{-05}	1.491×10^{-09}
1.5	50	-0.0007	2.637×10^{-05}
	100	-0.0006	3.393×10^{-05}
	500	-3.906×10^{-06}	7.628×10^{-09}
	1000	-3.823×10^{-05}	1.462×10^{-06}
1.85	50	0.0017	0.0002
	100	0.0009	8.593×10^{-05}
	500	0.0002	1.410×10^{-05}
	1000	3.296×10^{-05}	1.086×10^{-06}

Table 3.2: The survival data on the death times of Psychiatric inpatients

1	1	2	22	30	28	32	11	14	36	31	33	33
37	35	25	31	22	26	24	35	34	30	35	40	39

distribution is a strong competitor to other distributions used here for fitting the dataset.

P-P plot for ED, EED and $EGD(\theta)$ are given in Figure 3.4 which shows that $EGD(\theta)$ model is more plausible than ED and EED models.

Table 3.3: The estimates, K-S test statistic and log-likelihood for the dataset

Model	**Estimates**	**KS**	**Log L**	**p value**
ED	$\hat{\theta}$ =0.0378	0.3728	-111.1302	0.0015
EED	$\hat{a}$ =1.79724674,$\hat{b}$ =0.0525	0.3146	-108.9871	0.0116
EGD	$\hat{\theta}$ =0.1050	0.2613	-104.5856	0.0574

Table 3.4: AIC, BIC, and CAIC of the models based on the dataset

Model	**AIC**	**BIC**	**CAIC**
ED	224.2604	225.5185	226.5185
EED	221.9741	224.4903	226.4903
EGD	211.1713	212.4294	213.4294

3.7 Summary

A bathtub-shaped failure rate model, Exponential-Gamma$(3, \theta)$ distribution, is discussed, and its properties are studied. Moments, skewness, kurtosis, MGF, CF, Rènyi entropy, Lorenz curve, Gini index, and the distribution of maximum and minimum order statistics are obtained. A simulation study is conducted to illustrate the accuracy of the estimation method that has been obtained using maximum likelihood estimators. The application of $EGD(\theta)$ to real data shows that the new distribution is effective in providing a better fit than the exponential and exponentiated exponential distributions.

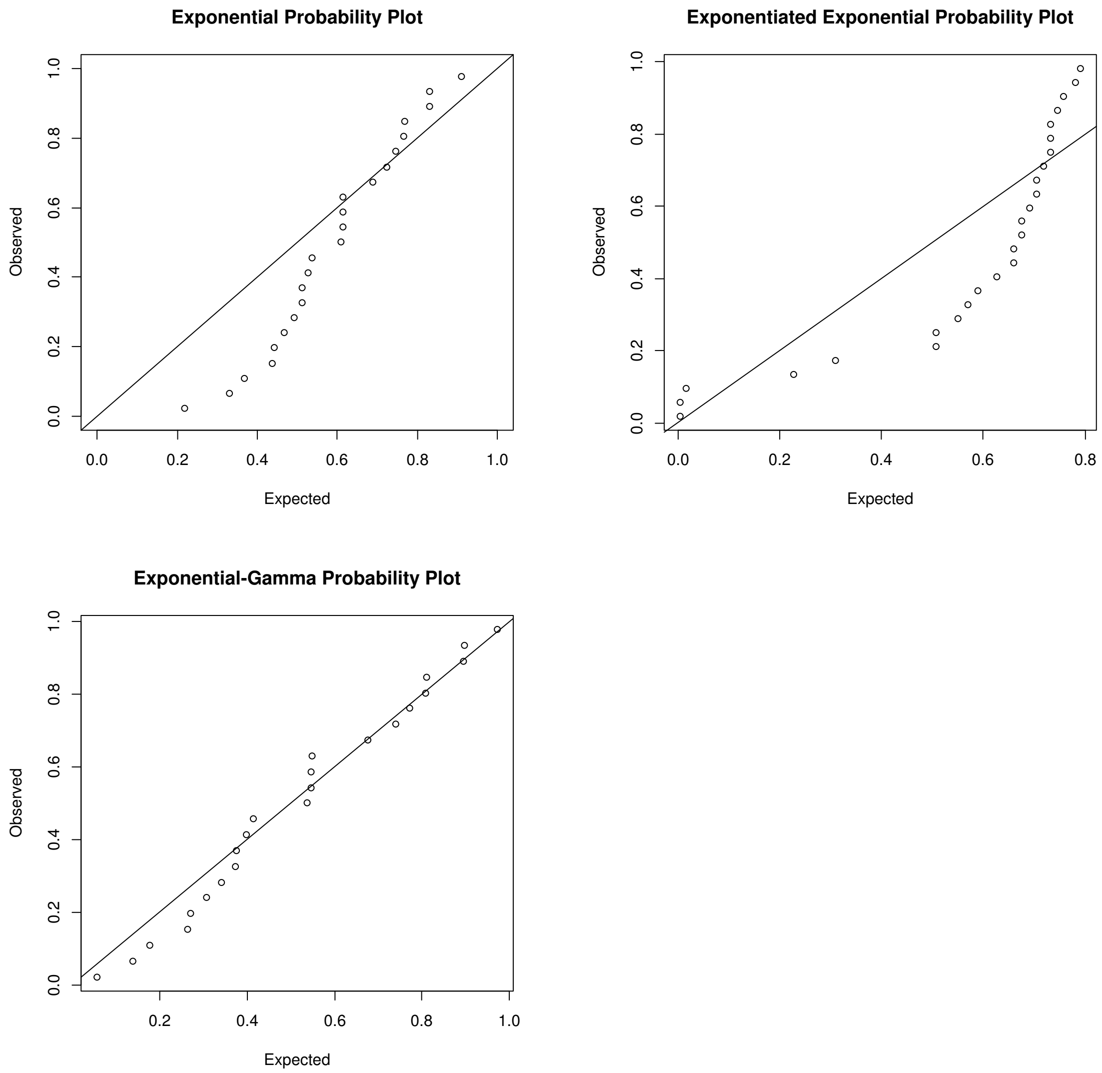


Figure 3.4: P-P Plots

CHAPTER 4

Generalized ν-Birnbaum Saunders Distribution

4.1 Introduction

Motivated by problems with vibration in commercial aircraft causing fatigue in the materials, the two-parameter BS distribution, also known as the fatigue life distribution, was proposed by Birnbaum and Saunders (1969a). The model was developed based on the impression that failure is due to the development and growth of a dominant crack. The BS distribution is now a natural model in many instances where the accumulation of a specific factor forces a quantifiable characteristic to exceed a critical threshold. A few examples of instances in which this distribution can be used are (i) heat-induced migration of metallic flaws in nano-circuits; (ii) ingestion of toxic chemicals from industrial waste by humans; (iii) pollution in the atmosphere as a result of an accumulation of pollutants over time; (iv) accumulation of deleterious substances in the lungs from air pollution; (v) events such as earthquakes and tsunamis occurring naturally, and so on. The BS distribution has two parameters modifying its shape and scale: a failure rate with an upside-down bathtub shape and a close relation to the normal distribution; see Leiva (2015).

The CDF of a two-parameter BS random variable T can be written as

$$F_T(t;\alpha,\beta) = \begin{cases} \Phi\left[\dfrac{1}{\alpha}\left(\left(\dfrac{t}{\beta}\right)^{\dfrac{1}{2}} - \left(\dfrac{\beta}{t}\right)^{\dfrac{1}{2}}\right)\right] & \text{if } t > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{4.1.1}$$

with $\alpha > 0$ and $\beta > 0$ being respectively the shape and scale parameters and $\Phi(.)$ is the standard normal CDF. The corresponding PDF of the BS model can be expressed in terms of the PDF of the standard normal distribution and is given by

$$f_T(t;\alpha,\beta) = \begin{cases} \dfrac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\dfrac{\beta}{t}\right)^{\dfrac{1}{2}} + \left(\dfrac{\beta}{t}\right)^{\dfrac{3}{2}}\right] e^{-\dfrac{1}{2\alpha^2}\left(\dfrac{t}{\beta}+\dfrac{\beta}{t}-2\right)} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.2}$$

It is known that the density function of the BS distribution is unimodal, and although the hazard rate is not an increasing function of t, the average hazard rate is nearly a non-decreasing function of t (Mann et al., (1974)).

Often, it is very likely to observe a three-phase behavior of HF in the case of studying the life cycle of an industrial product or the entire life cycle of a biological entity. For example, non-monotone hazard rates involving a U-shaped (bathtub-shaped) pattern are exhibited in the case of the age-specific death rate in human life tables. The core motivation behind developing a more flexible distribution is its capability to model the underlying monotonic and non-monotonic failure rate behavior of the observed data.

In this chapter, a distribution called the ν-Birnbaum Saunders (BS) distribution is discussed, which generalizes the BS model. It is noted here that the BS distribution only has a decreasing or upside-down bathtub shape for its hazard function. It is important to note that the shape of the distribution always depends on the power of the random variable, thus facilitating the development of more flexible models. Chacko et al. (2015) considered a generalization of the BS distribution, incorporating a new shape parameter exhibiting both monotonic and non-monotonic failure rate behaviors, but statistical inference has not been given. Since the estimation of parameters is essential for using any distribution, this chapter provides some structural properties of the distribution and the method of estimation. A discussion on maximum likelihood estimation of the parameters is given and derived the observed information matrix. The use of the distribution is justified by three real-life data sets: the industrial devices data set reported by Aarset (1987), exceedances of flood peaks data given in Choulakian and Stephens (2001), and the insurance data reported in Andrews and Herzberg (2012).

Several extensions and generalizations of the BS distribution are studied by many researchers, including its bivariate and multivariate extensions. The rest of the chapter is organized as follows: In Section 4.2, the $\nu - BS$ distribution, its structural properties, moments, quantiles, and order statistics are given. Also, the estimation procedure is given using the method of maximum likelihood. In addition to this, an extensive simulation study is carried out along with two real-life applications. Section 4.3 is devoted to the bivariate $\nu - BS$ distribution. In section 4.4, the multivariate $\nu - BS$ distribution is defined. The summary is given in the final section.

4.2 Univariate ν-Birnbaum Saunders Distribution

In this section, an extension of the BS distribution is considered, motivated by the work of Chacko et al. (2015), who call this extended version of the BS distribution ν-BS distribution. The study of ν-BS distribution is motivated by three real-life data examples-industrial devices data set, exceedances of flood peaks data, and insurance data. In order to investigate the fitness of the data to the ν-BS distribution, we have to estimate the parameters. So estimation of the parameters of ν-BS distribution is considered in this chapter.

4.2.1 Cumulative Distribution Function

The CDF of a $\nu - BS$ random variable T is given by

$$F(t;\alpha,\beta,\nu) = \begin{cases} \Phi\Big(\dfrac{1}{\alpha}\Big\{\Big(\dfrac{t}{\beta}\Big)^{\nu} - \Big(\dfrac{t}{\beta}\Big)^{-\nu}\Big\}\Big) & \text{if } t > 0, \alpha, \beta, \nu > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2.1}$$

where $\Phi(.)$ is the standard normal CDF. Here, $\alpha > 0$ and $\beta > 0$ are respectively, the shape parameter and the scale parameter. Note that the parameters α and β in Eq. (4.2.1) are governed by the proposed shape parameter $\nu > 0$. One can obtain the BS distribution in its particular case when $\nu = \frac{1}{2}$.

4.2.2 Probability Density Function

For a random variable T with CDF defined in Eq.(4.2.1), the corresponding PDF is given by

$$f(t;\alpha,\beta,\nu) = \begin{cases} \frac{\nu}{\alpha\beta\sqrt{2\pi}} \, e^{-\frac{1}{2\alpha^2}\left[\left(\frac{t}{\beta}\right)^{2\nu}+\left(\frac{t}{\beta}\right)^{-2\nu}-2\right]}\left[\left(\frac{t}{\beta}\right)^{\nu-1}+\left(\frac{t}{\beta}\right)^{-(\nu+1)}\right] & \text{if } t > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2.2}$$

From now on, the notation $T \sim BS(\alpha, \beta, \nu)$ is used to denote a univariate $\nu-$BS random variable T with parameters α, β, and ν. The PDF in Figure 4.1 has been plotted for different values of the parameters. From the plot, it can be seen that the PDF is unimodal in nature.

4.2.3 Hazard Function

The following section discusses the shape characteristics of the HRF of a BS random variable. With $T \sim BS(\alpha, \beta, \nu)$, the HRF of T is given by

$$h_T(t;\alpha,\beta,\nu) = \frac{f(t;\alpha,\beta,\nu)}{\bar{F}_T(t;\alpha,\beta,\nu)}$$

It is possible to choose $\beta = 1$ without loss of generality since the HRF's form does not depend on the scale parameter β.

$$h_T(t;\alpha,1,\nu) = \frac{\frac{1}{\alpha\sqrt{2\pi}}\ \epsilon'_\nu(t)\ e^{-\frac{1}{2\alpha^2}\epsilon_\nu^2(t)}}{\Phi(-\frac{\epsilon_\nu(t)}{\alpha})} \tag{4.2.3}$$

where $\epsilon_\nu(t) = (t)^\nu - (t)^{-\nu}$, $\epsilon'_\nu(t) = \frac{\nu}{t}\Big((t)^\nu - (t)^{-\nu}\Big)$ and $\epsilon''_\nu(t) = \frac{\nu}{t^2}\Big((\nu-1)t^\nu - (\nu+1)t^{-\nu}\Big)$.

Kundu et al. (2008) then showed that the HRF in Eq. (4.2.3) is always

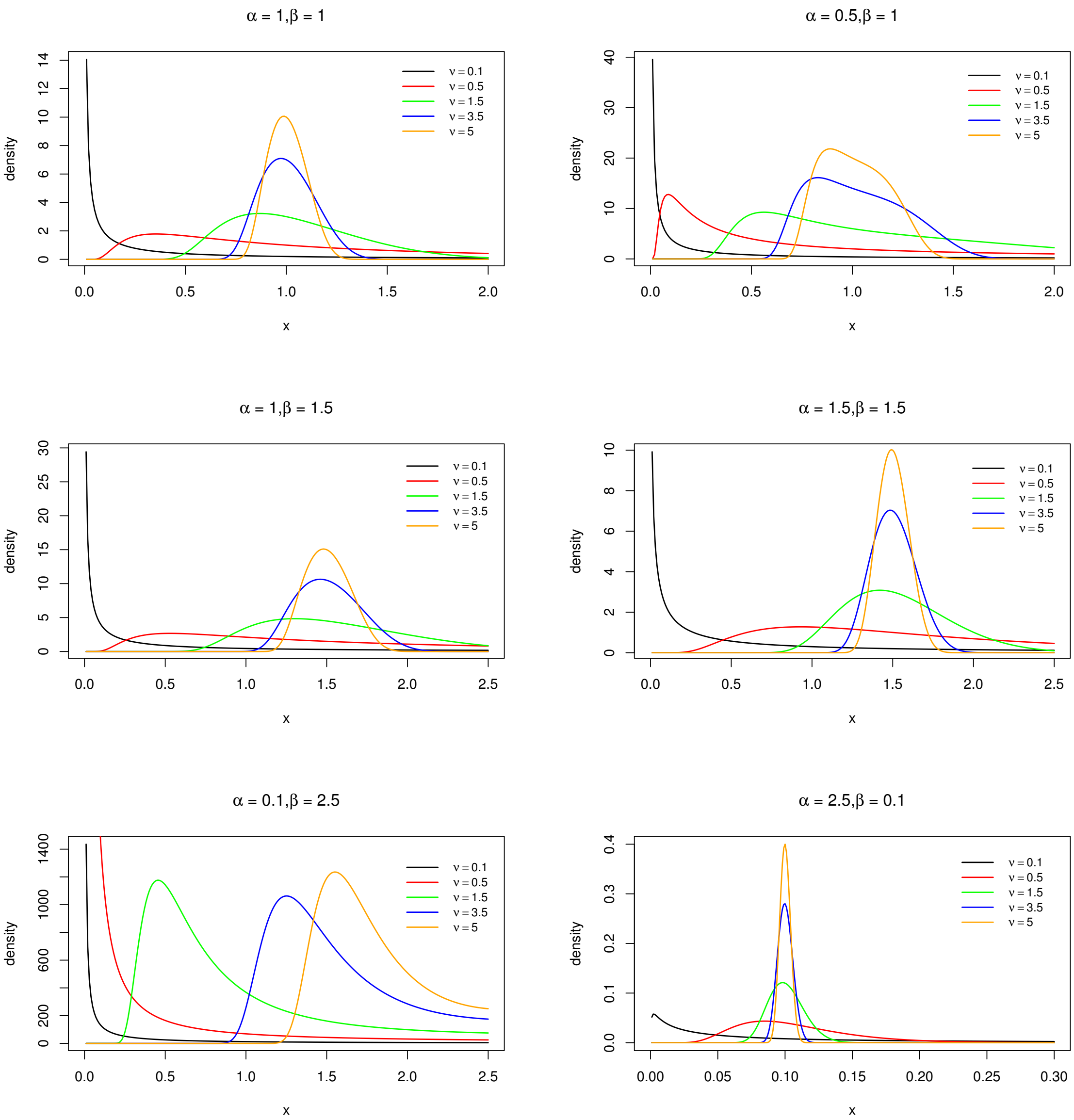


Figure 4.1: Probability density function plots

unimodal. The plots of the HF of $BS(\alpha, \beta, \nu)$ in Eq.(4.2.3) for different values of α and ν, are presented in Figure 4.2. Whenever $0 < \nu < 1$, from (4.2.3) it can shown that $\ln(h_T(t; \alpha, 1, \nu)) \to 1/2\alpha^2$ as $t \to \infty$.

Moments

If $T \sim BS(\alpha, \beta, \nu)$ (T has a ν-BS distribution with parameters α, β and ν), the moments of the random variable T can be obtained by making the following transformation:

$$Z = \frac{1}{\alpha}\left[\left(\frac{T}{\beta}\right)^{\nu} - \left(\frac{T}{\beta}\right)^{-\nu}\right]$$

or

$$T = \frac{\beta}{2^{1/\nu}}\left[\alpha Z + \sqrt{4 + (\alpha Z)^2}\right]^{1/\nu} = \beta\left[\frac{W}{\beta}\right]^{1/2\nu} \tag{4.2.4}$$

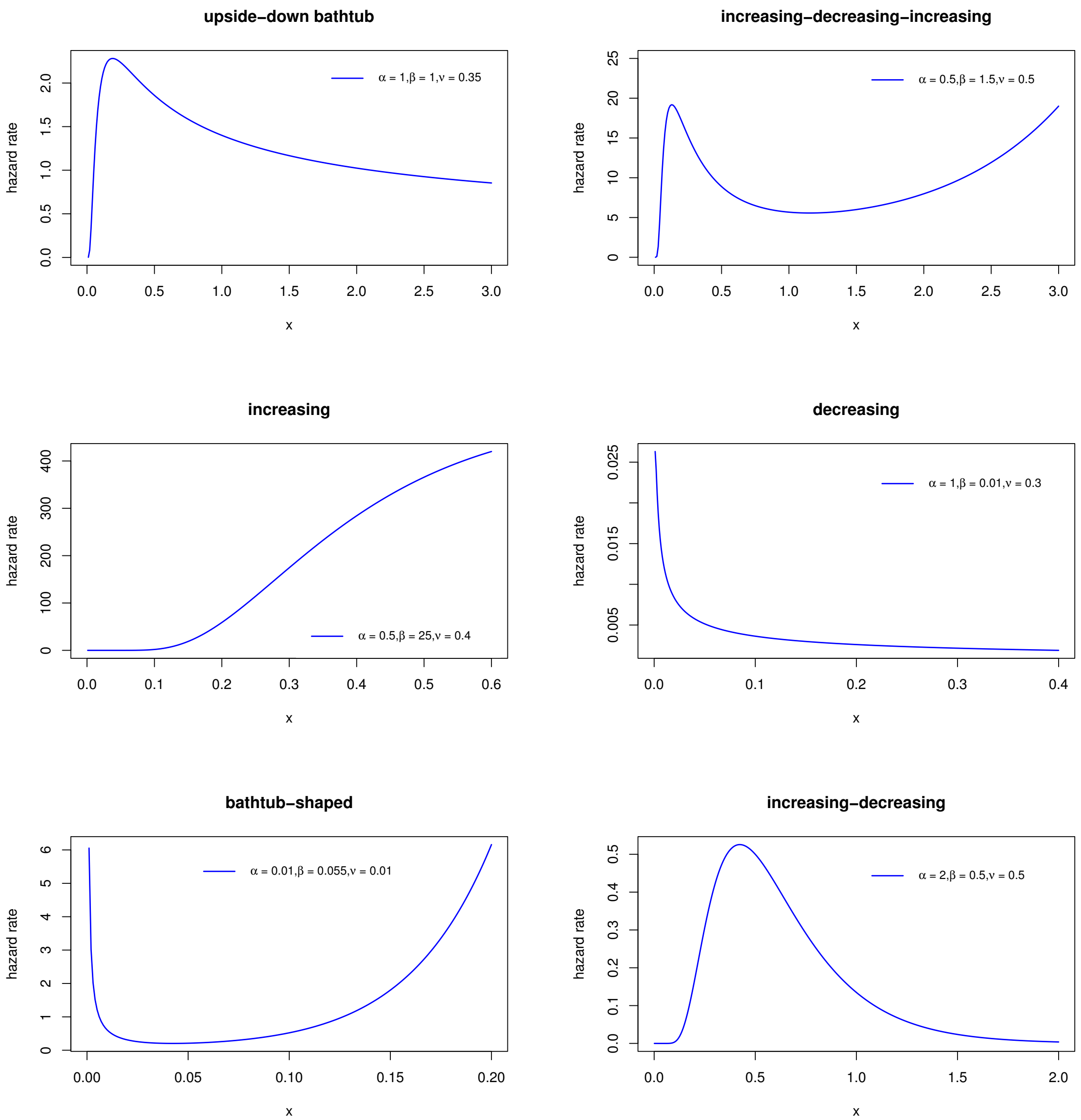


Figure 4.2: Failure rate function plots for different parameter values.

or

$$T^{2\nu} = \beta^{2\nu-1} W \tag{4.2.5}$$

where $W = \frac{\beta}{4}\Big[\alpha Z + \sqrt{1+(\alpha Z)^2}\Big]^2 \sim BS(\alpha, \beta)$ and $Z \sim N(0,1)$. Hence,

$$\begin{aligned} E(T^r) &= \beta^r E\left[\left(\frac{W}{\beta}\right)^{\frac{r}{2\nu}}\right] \\ &= \beta^{r-\frac{r}{2\nu}} E\left[W^{\frac{r}{2\nu}}\right] \end{aligned}$$

Now in case if $r/2\nu$ is an integer then

$$E(T^r) = \beta^r \sum_{j=1}^{r/\nu} \binom{r/2\nu}{2j} \sum_{i=0}^{j} \frac{(r/\nu - 2j + 2i)!}{2^{r/2\nu - j + i}(r/2\nu - j + i)!} \left(\frac{\alpha}{2}\right)^{r/\nu - 2j + 2i} \tag{4.2.6}$$

(see Leiva et al. (2009) in this regard). Rieck (1999) also obtained $E(T^r)$, for fractional values of $r/2\nu$, in terms of the Bessel function, from the MGF of $E(\ln(W))$. For $r = 2\nu$, then

$$E(T^{2\nu}) = \beta^{2\nu-1} E(W) = \frac{\beta^{2\nu}}{2}(\alpha^2 + 2). \tag{4.2.7}$$

If $T \sim BS(\alpha, \beta, \nu)$, then it can be easily shown that $T^{-1} \sim BS(\alpha, \beta^{-1}, \nu^{-1})$ (T has a ν-BS distribution with parameters α, β^{-1} and ν^{-1}). Therefore, for integer r, it can be readily obtained from Eq. (4.2.6) that

$$E(T^{-r}) = \beta^{-r} \sum_{j=1}^{r\nu} \binom{r\nu/2}{2j} \sum_{i=0}^{j} \frac{(r\nu - 2j + 2i)!}{2^{r\nu/2 - j + i}(r\nu/2 - j + i)!} \left(\frac{\alpha}{2}\right)^{r\nu - 2j + 2i}. \tag{4.2.8}$$

For $r = 2\nu$, then

$$E(T^{-2\nu}) = \beta^{-2\nu+1} E(W^{-1}) = \frac{\beta^{-2\nu}}{2}(\alpha^2 + 2). \tag{4.2.9}$$

Quantiles

Quantiles can be obtained as a solution to the equation $F_T(t_q) = q$, where t_q is the qth quantile. Hence,

$$\Phi\left[\frac{1}{\alpha}\left\{\left(\frac{t_q}{\beta}\right)^{\nu} - \left(\frac{\beta}{t_q}\right)^{\nu}\right\}\right] = q.$$

Now, solving the above equation, the qth quantile $(0 < q < 1)$ can be written as

$$t_q = \frac{\beta}{2^{\frac{1}{\nu}}}\left(\alpha z_q + \sqrt{(\alpha z_q)^2 + 4}\right)^{\frac{1}{\nu}}, \tag{4.2.10}$$

where $z_q = \Phi^{-1}(q)$ is the qth quantile of a standard normal random variable. Then, using the ν- BS quantile function, that is, the inverse transform method, a generator of random numbers for the ν-BS distribution is summarized in the following Algorithm 1.

Algorithm 1 : Generator of random numbers from $\nu - BS$ distribution.

1: Generate a random number z from $Z \sim N(0,1)$.
2: Set values for α, β and ν of $T \sim BS(\alpha,\beta,\nu)$.
3: Compute a random number t from $T \sim BS(\alpha,\beta,\nu)$ by using Eq. (4.2.10) conducting to

$$t = \frac{\beta}{2^{1/\nu}}\left[\alpha z + \sqrt{4 + (\alpha z)^2}\right]^{1/\nu}.$$

4: Repeat steps 1 to 3 until the required amount of random numbers to be completed.

Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. The density function $f_{p:n}(t)$ of the p-th order statistic $T_{p:n}$, for $p = 1,\ldots,n$, from independent identically distributed $BS(\alpha,\beta,\nu)$ random variables $T_1,\ldots,T_n$ is given by

$$f_{p:n}(t) = \frac{f(t)}{B(p, n-p+1)} F(t)^{p-1}[1 - F(t)]^{n-p}.$$

For convenience, let us consider the Eq.(4.2.1) and Eq.(4.2.2) as

$$F(t) = \Phi(\mu_t) \tag{4.2.11}$$

where $\mu_t = \frac{1}{\alpha}\epsilon_\nu(\frac{t}{\beta})$ and

$$f(t) = \phi(\mu_t)M_t \tag{4.2.12}$$

where $M_t = d\mu_t/dt$ and $\phi(.)$ is the standard normal density function. As a result of substituting Eq.(4.2.11) and Eq.(4.2.12) into the above expression,

$$f_{p:n}(t) = \frac{\phi(\mu_t)M_t}{B(p, n-p+1)}[\Phi(\mu_t)]^{p-1}[1-\Phi(\mu_t)]^{n-p}.$$

The above PDF can be expressed in terms of the binomial expansion as

$$f_{p:n}(t) = \frac{\phi(\mu_t)M_t}{B(p, n-p+1)}\sum_{k=0}^{n-p}(-1)^k\binom{n-p}{k}[\Phi(\mu_t)]^{p+k-1}.$$

Thus, this PDF of the $BS(\alpha, \beta, \nu)$ order statistics can be reduced to

$$f_{p:n}(t) = \sum_{k=0}^{n-p} m_k f(t),\ t > 0 \tag{4.2.13}$$

where $m_{i+1} = \frac{(-1)^k\binom{n-p}{k}[\Phi(\mu_t)]^{p+k-1}}{B(p,n-p+1)}$ and $f(t)$ is in Eq.(4.2.12). As a result, the PDF Eq.(4.2.13) of $BS(\alpha, \beta, \nu)$ order statistics can be viewed as a linear combination of the $BS(\alpha, \beta, \nu)$ density functions. In this way, many mathematical properties of $BS(\alpha, \beta, \nu)$ order statistics, such as moments and the generating function, can be determined from the $BS(\alpha, \beta, \nu)$ distribution.

4.2.4 Estimation and Testing of Hypothesis

In this Section, the estimation methodologies for the unknown parameters in the case of the ν-BS distribution are first discussed. The likelihood ratio (LR) test is then discussed in this setup.

Point Estimation

The point estimation of the parameters of the ν-BS distribution by the method of maximum likelihood is considered.

1. Complete data case: Let $\boldsymbol{T} = \{T_1, T_2, ..., T_n\}$ be a random sample of size n

and $\boldsymbol{\theta} = (\alpha, \beta, \nu)$ be the unknown parameter vector. Based on the random sample, $\widehat{\boldsymbol{\theta}}$, the MLE of $\boldsymbol{\theta}$, can be obtained by maximizing the log-likelihood function. The associated likelihood and the log-likelihood function are respectively given by

$$L(\boldsymbol{\theta}|\boldsymbol{t}) = \left(\frac{\nu}{\alpha\beta\sqrt{2\pi}}\right)^n e^{-\frac{1}{2\alpha^2}\sum_{i=1}^n\left[\left(\frac{t_i}{\beta}\right)^{2\nu}+\left(\frac{t_i}{\beta}\right)^{-2\nu}-2\right]} \prod_{i=1}^n \left[\left(\frac{t_i}{\beta}\right)^{\nu-1} + \left(\frac{t_i}{\beta}\right)^{-(\nu+1)}\right], \tag{4.2.14}$$

and

$$l(\boldsymbol{\theta}|\boldsymbol{t}) = n\ln\nu - n\ln\alpha - n\ln\beta - \frac{n}{2}\ln(2\pi) - \frac{1}{2\alpha^2}\sum_{i=1}^n\left[\left(\frac{t_i}{\beta}\right)^{2\nu} + \left(\frac{t_i}{\beta}\right)^{-2\nu} - 2\right]$$
$$+\sum_{i=1}^n \log\left[\left(\frac{t_i}{\beta}\right)^{\nu-1} + \left(\frac{t_i}{\beta}\right)^{-(\nu+1)}\right], \tag{4.2.15}$$

where $\boldsymbol{t} = \{t_1,\, t_2,...,t_n\}$ is the observed sample. The components of the score vector $\boldsymbol{U(\theta)} = (U_\alpha, U_\beta, U_\nu)^T$ are

$$U_\alpha = \frac{-n}{\alpha} + \frac{1}{\alpha^3}\sum_{i=1}^n\left[\left(\frac{t_i}{\beta}\right)^{2\nu} + \left(\frac{\beta}{t_i}\right)^{2\nu} - 2\right]$$

$$U_\beta = \frac{n}{\beta} + \frac{\nu}{\alpha^2}\sum_{i=1}^n\left[\frac{\beta^{2\nu-1}}{t_i^{2\nu}} - \frac{t_i^{2\nu}}{\beta^{2\nu+1}}\right] + (\nu+1)\sum_{i=1}^n \frac{\frac{\beta^\nu}{t_i^{\nu+1}} - \frac{t_i^{\nu-1}}{\beta^\nu}}{(\frac{t_i}{\beta})^{\nu-1} + (\frac{\beta}{t_i})^{\nu+1}}$$

$$U_\nu = \tfrac{n}{\nu} - \tfrac{1}{\alpha^2}\textstyle\sum_{i=1}^n\left[\left(\frac{t_i}{\beta}\right)^{2\nu}\log\left(\frac{t_i}{\beta}\right) + \left(\frac{\beta}{t_i}\right)^{2\nu}\log\left(\frac{\beta}{t_i}\right)\right] + \sum_{i=1}^n \frac{(\frac{t_i}{\beta})^{\nu-1}\log(\frac{t_i}{\beta}) + (\frac{\beta}{t_i})^{\nu+1}\log(\frac{\beta}{t_i})}{(\frac{t_i}{\beta})^{\nu-1} + (\frac{\beta}{t_i})^{\nu+1}}$$

Setting these equations to zero, $\boldsymbol{U(\theta)} = 0$, and solving them simultaneously yields $\widehat{\boldsymbol{\theta}}$ of the three parameters. From the score equation $U_\alpha = 0$, it can be written as

$$\widehat{\alpha} = \widehat{\alpha}(\beta,\nu) = \left\{\frac{1}{n}\sum_{i=1}^n\left[\left(\frac{t_i}{\beta}\right)^{2\nu} + \left(\frac{\beta}{t_i}\right)^{2\nu} - 2\right]\right\}^{\frac{1}{2}}. \tag{4.2.16}$$

Plugging in $\widehat{\alpha}$ replacing α in the log-likelihood function $l(\boldsymbol{\theta}|\boldsymbol{t})$, the profile log-likelihood function of β and ν is obtained first and then maximized using some numerical routine to obtain $\widehat{\beta}$ and $\widehat{\nu}$. Finally, $\widehat{\alpha} = \widehat{\alpha}(\widehat{\beta}, \widehat{\nu})$ is obtained.

2. Multicensored data case: More often, censored data occur in lifetime data analysis. Some basic mechanisms of censoring are well known in the literature as, for example, Type-I and Type-II censoring. The survival function of the ν-BS distribution has a simple convenient form and hence this distribution can be employed in analyzing censored data. In this context, the general case of multicensored data is considered. Suppose there are $n = n_0 + n_1 + n_2$ units of which n_0 are known to have failed at the times $t_1, \ldots, t_{n_0}$; n_1 are known to have failed in the interval $[s_{i-1}, s_i]$ for $i = 1, \ldots, n_1$; and n_2 units have survived at least till a time r_i $(i = 1, ..., n_2)$ but not observed any longer. It is to note here that Type-I and Type-II censoring are contained as particular cases of multicensoring. The log-likelihood function of $\boldsymbol{\theta} = (\alpha, \beta, \nu)$ for this multicensored data takes the following form:

$$\begin{aligned}
l(\boldsymbol{\theta}|\boldsymbol{t}) &\propto n_0 \ln \nu - n_0 \ln(\alpha\beta) - \frac{1}{2\alpha^2} \sum_{i=1}^{n_0} \left[\left(\frac{t_i}{\beta}\right)^{2\nu} + \left(\frac{t_i}{\beta}\right)^{-2\nu} - 2 \right] \\
&+ \sum_{i=1}^{n_0} \log \left[\left(\frac{t_i}{\beta}\right)^{\nu-1} + \left(\frac{t_i}{\beta}\right)^{-(\nu+1)} \right] \\
&+ \sum_{i=1}^{n_2} \log \left[1 - \Phi\left(\frac{1}{\alpha} \left\{ \left(\frac{r_i}{\beta}\right)^{\nu} - \left(\frac{r_i}{\beta}\right)^{-\nu} \right\} \right) \right] \\
&+ \sum_{i=1}^{n_1} \log \left[\Phi\left(\frac{1}{\alpha} \left\{ \left(\frac{s_i}{\beta}\right)^{\nu} - \left(\frac{s_i}{\beta}\right)^{-\nu} \right\} \right) \right] - \sum_{i=1}^{n_1} \log \left[\Phi\left(\frac{1}{\alpha} \left\{ \left(\frac{s_{i-1}}{\beta}\right)^{\nu} - \left(\frac{s_{i-1}}{\beta}\right)^{-\nu} \right\} \right) \right].
\end{aligned} \tag{4.2.17}$$

The MLEs are obtained by maximizing the above log-likelihood function with respect to unknown parameters. It is not possible to obtain any of the MLEs as a function of one or others. One requires either carrying out a three-dimensional maximization of the objective function $l(\boldsymbol{\theta}|\boldsymbol{t})$ in Eq. (4.2.17) or obtaining the score vector and solving them to obtain $\boldsymbol{\theta}$.

Interval Estimation

Assuming the asymptotic normality of the MLEs, the CIs for $\boldsymbol{\theta}$ are computed using the observed information matrix $I = \left(\left(\frac{\partial l(\boldsymbol{\theta}|\boldsymbol{t})}{\partial\theta_i\partial\theta_j}\right)\right)$, $i, j = 1, 2, 3$, where $l(\boldsymbol{\theta}|\boldsymbol{t})$ is the log-likelihood function as defined in Eq.(4.2.15). The $100(1-\gamma)\%$ asymptotic CIs for $\boldsymbol{\theta}$ are respectively given by $\widehat{\alpha} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{11}}$, $\widehat{\beta} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{22}}$, $\widehat{\nu} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{33}}$ where V_{ij} is the $(i, j)-$th element of the inverse of the observed Fisher information matrix I. This interval estimation method is quite useful for its computational ease and provides coverage probabilities close to the nominal value.

Testing of Hypothesis

In this context, it is worthwhile to mention that the LR statistic often turns out to be useful for testing the goodness-of-fit of the ν-BS model and for comparing it with the usual BS model. One can easily check if the fit using the ν-BS model is statistically "superior" to a fit using the BS model for a given data set by computing

$$w = 2\{l(\widehat{\alpha}, \widehat{\beta}, \widehat{\nu}|\boldsymbol{t}) - l(\widetilde{\alpha}, \widetilde{\beta}, 0.5|\boldsymbol{t})\},$$

where $\widehat{\alpha}, \widehat{\beta}, \widehat{\nu}$ are the unrestricted MLEs and $\widetilde{\alpha}, \widetilde{\beta}$ are the restricted estimates. Also, the LR statistic is asymptotically distributed under the null model as χ^2 distribution with 1 degree of freedom. Further, the LR test rejects the null hypothesis if $w > \eta_n$, where η_n denotes the upper $100\eta\%$ point of the χ^2 distribution with 1 degree of freedom.

4.2.5 Simulation Study

In this section, a simulation study is performed with various sample sizes and parameter values to assess the effectiveness of the proposed estimation methodology. For illustration purposes, different sample sizes are considered $(n = 40, 60, 80, 100, 120)$ and the parameter values are taken as $\alpha = 2, \beta = 1, \nu = 1.5$. Based on the likelihood principle, the average estimates (AEs), MSEs, and biases for each unknown model parameter are computed. When it comes to the interval estimation problem, it is noted that the exact distribution of the MLEs is not possible to compute. Hence, interval estimates are computed in terms of asymptotic CIs. All the results are based on 5000 replications and are available in Table 4.1.

Some of the observations are quite evident from the results obtained in Table

4.1. As the sample size increases, the AEs approach the true values of the model parameters in all cases, and the corresponding MSEs decrease. In the case of the results associated with the interval estimates, the performance of asymptotic CIs is quite satisfactory in terms of coverage probabilities (CPs). With the increase in sample sizes, the average lengths (ALs) of all the model parameters decrease, which is quite expected.

Table 4.1: MLEs, MSEs, Biases, CPs and ALs for $\nu - BS$ model with $\alpha = 2, \beta = 1$ and $\nu = 1.5$

n	MLEs	MSE	Bias	CP	AL
40	$\hat{\alpha}$= 2.3355	2.0985	0.6355	0.9875	4.5109
	$\hat{\beta}$=1.0004	0.0053	0.0004	0.9154	0.2310
	$\hat{\nu}$=1.7261	0.3938	0.2261	0.9028	1.9249
60	$\hat{\alpha}$=2.2167	1.1183	0.4167	0.9930	3.9053
	$\hat{\beta}$=1.0047	0.0039	0.0047	0.9321	0.2139
	$\hat{\nu}$=1.6515	0.2552	0.1515	0.9261	1.7610
80	$\hat{\alpha}$=2.1666	0.7898	0.3666	0.9883	3.2447
	$\hat{\beta}$=1.0016	0.0026	0.0016	0.9201	0.1855
	$\hat{\nu}$=1.6483	0.1769	0.1483	0.9298	1.4733
100	$\hat{\alpha}$=2.1259	0.5568	0.2259	0.9710	2.8059
	$\hat{\beta}$=1.0025	0.0019	0.0025	0.9171	0.1666
	$\hat{\nu}$=1.5887	0.1248	0.0887	0.9411	1.3035
120	$\hat{\alpha}$= 2.1930	0.4671	0.1930	0.9710	2.4835
	$\hat{\beta}$=1.0015	0.0016	0.0015	0.9271	0.1537
	$\hat{\nu}$=1.5749	0.1001	0.0749	0.9461	1.1789

4.2.6 Real Life Applications

In the following, applications of the ν-BS distribution to real data are presented for illustrative purposes. In order to show how well the ν-BS distribution can be applied to real-life phenomena, three real-life data sets are used- industrial devices data given by Aarset (1987), exceedances of flood peaks data given in Choulakian and Stephens (2001), and insurance data reported in Andrews and Herzberg (2012).

Industrial devices data

At first, industrial devices' real-life data set are considered (see Aarset (1987) in this respect) which is given in Table 4.2. This data set represents the lifetimes of 50 industrial devices put on life tests at time zero. In real data applications, several authors studied this data set for different statistical models since it presents a bathtub-shaped failure rate, see for example, Ahmed (2014) and Kayal et al. (2017). A detailed summary of these data is provided in Table 4.3.

Table 4.2: Industrial devices data

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

The MLEs of all the model parameters are computed based on the principle of maximum likelihood. Despite our inability to theoretically verify the unimodality of the profile log-likelihood function of β and ν, the contour plot in Figure 4.3(a) indicates that the function is indeed unimodal. The K-S distance is also reported along with the p-value for the goodness of fit. It is observed that both the BS distribution and $\nu-$BS distribution fit the data well. However, based on the Maximum log-likelihood (MLL) value, K-S distance, and AIC value, it can be seen

Table 4.3: Descriptive statistics: Industrial devices data

Mean	Median	Variance	Skewness	Kurtosis	Minimum	Maximum
35.8800	34.0000	861.6100	-0.1400	1.4100	0.1000	83.0000

that the proposed $\nu-$BS distribution outperforms the BS distribution. All the associated results are listed in Table 4.4. The LR statistic to test the hypothesis H_0 : BS against H_1 : $\nu-$BS is 52.6200 (p-value < 0.01). Thus, using any usual significance level, the null hypothesis is rejected in favor of the $\nu-$BS distribution, i.e., the $\nu-$BS distribution is significantly better than the BS distribution.

Table 4.4: MLEs (standard errors in parentheses), K-S distance, p-values, MLL values, and AIC values: industrial devices Aarset data set

Distribution	Estimates			K-S distance	p-value	MLL	AIC
BS(α, β, ν)	31.9352	3.8157	1.2286	0.1543	0.8356	-227.1600	460.3200
	(18.9847)	(0.4530)	(0.1769)				
BS(α, β)	2.7455	7.1877		0.1783	0.7798	-253.4700	510.9400
	(0.2982)	(1.5499)					

Exceedances of flood peaks data

For our second real-life illustration, a data set corresponding to the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada is considered. The data consist of 72 exceedances for the years 1958–1984, rounded to one decimal place (see Choulakian and Stephens (2001) in this respect) and are given in Table 4.5. Table 4.6 gives a descriptive summary of these data.

The MLEs of all the model parameters are computed based on the principle of maximum likelihood. Despite our inability to theoretically verify the unimodality of the profile log-likelihood function of β and ν, the contour plot in Figure 4.3(b) indicates that the function is indeed unimodal. The K-S distance is also reported

Table 4.5: Exceedances of flood peaks data

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	1.9	13.0	12.0	9.3
1.4	18.7	8.5	25.5	11.6	14.1	22.1	1.1	2.5	14.4	1.7	37.6
0.6	2.2	39.0	0.3	15.0	11.0	7.3	22.9	1.7	0.1	1.1	0.6
9.0	1.7	7.0	20.1	0.4	2.8	14.1	9.9	10.4	10.7	30.0	3.6
5.6	30.8	13.3	4.2	25.5	3.4	11.9	21.5	27.6	36.4	2.7	64.0
1.5	2.5	27.4	1.0	27.1	20.2	16.8	5.3	9.7	27.5	2.5	27.0

Table 4.6: Descriptive statistics: exceedances of flood peaks data

Mean	**Median**	**Variance**	**Skewness**	**Kurtosis**	**Minimum**	**Maximum**
12.2000	9.5000	151.2200	1.4700	5.8900	0.1000	64.0000

along with the p-value for the goodness of fit. It is observed that both the BS distribution and $\nu-$ BS distribution fit the data well. However, based on the MLL value, K-S distance, and AIC value, it can be seen that the proposed $\nu-$ BS distribution outperforms the BS distribution. All the associated results are listed in Table 4.10. The LR statistic to test the hypothesis H_0 : BS against H_1 : $\nu-$BS is 50.3400 (p-value < 0.01). Thus, the null hypothesis is rejected in favor of the $\nu-$BS distribution using any usual significance level. Therefore, the $\nu-$BS distribution is significantly better than the BS distribution based on the LR statistic.

Insurance data

Finally, the data representing Swedish third-party motor insurance for 1977 for one of several geographical zones are considered. The data were compiled by a Swedish committee on the analysis of risk premiums in motor insurance. The data points

Table 4.7: MLEs (standard errors in parentheses), K-S distance, p-values, MLL values, and AIC values: exceedances of flood peaks data set

Distribution	Estimates			K-S distance	p-value	MLL	AIC
BS(α, β, ν)	1.0897	5.1582	0.3481	0.1404	0.5996	-230.8600	467.7200
	(0.9356)	(1.4117)	(0.2447)				
BS(α, β)	1.7583	4.4179		0.1457	0.5470	-256.0300	516.2300
	(0.1477)	(0.6497)					

are the aggregate payments by the insurer in thousand Skr (Swedish currency). The data set was originally reported in Andrews and Herzberg (2012) and is as provided in Table 4.8. Table 4.9 gives a descriptive summary of these data.

Table 4.8: Insurance data

5014	5855	6486	6540	6656	6656
7212	7541	7558	7797	8546	9345
11762	12478	13624	14451	14940	14963
15092	16203	16229	16730	18027	18343
19365	21782	24248	29069	34267	38993

Table 4.9: Descriptive statistics: Insurance data

Mean	Median	Variance	Skewness	Kurtosis	Minimum	Maximum
14525.7300	14037.5000	69927726	1.3016	1.6004	5014	38993

MLEs of all the model parameters are computed based on the principle of maximum likelihood. Despite our inability to theoretically verify the unimodality of the profile log-likelihood function of β and ν, the contour plot in Figure 4.3(c) indicates that the function is indeed unimodal. The K-S distance is also reported along with the p-value for the goodness of fit. It is observed that both the BS distribution and $\nu-$ BS distribution fit the data well. However, based on the MLL value, K-S distance, and AIC value, it can be seen that the proposed $\nu-$ BS distribution outperforms the BS distribution. All the associated results are listed in Table 4.10. The LR statistic to test the hypothesis H_0 : BS against H_1 : $\nu-$BS is 7.1230 (p-value = 0.0076 < 0.01). Thus, the null hypothesis is rejected in favor of the $\nu-$BS distribution using any usual significance level. Therefore, the $\nu-$BS distribution is significantly better than the BS distribution based on the LR statistic.

Table 4.10: MLEs (standard errors in parentheses), K-S distance, p-values, MLL values, and AIC values: insurance data set

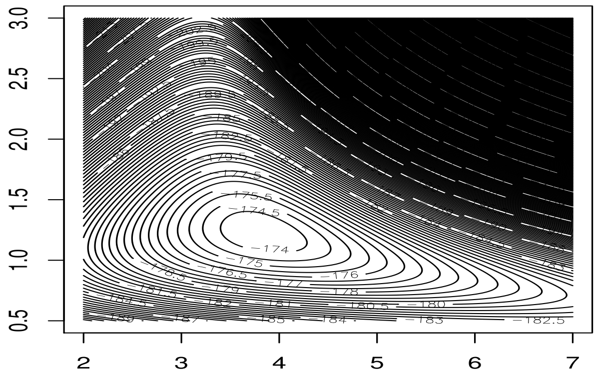
Distribution	Estimates			K-S distance	p-value	MLL	AIC
BS(α,β,ν)	2.4285	1.3219	1.6654	0.1305	0.7052	-16.7831	39.5662
	(1.4121)	(0.1069)	(0.5985)				
BS(α,β)	0.5595	1.2559		0.1385	0.6130	-20.3446	44.6892
	(0.0722)	(0.1233)					

4.3 Bivariate ν- Birnbaum Saunders Distribution

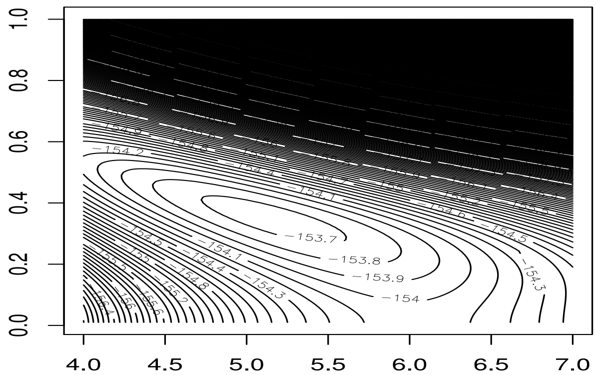
In this section, a new generalized form of BVBS distribtion is proposed and call it a ν-BVBS distribution.

4.3.1 CDF, PDF, and HRF of ν- BVBS Distribution

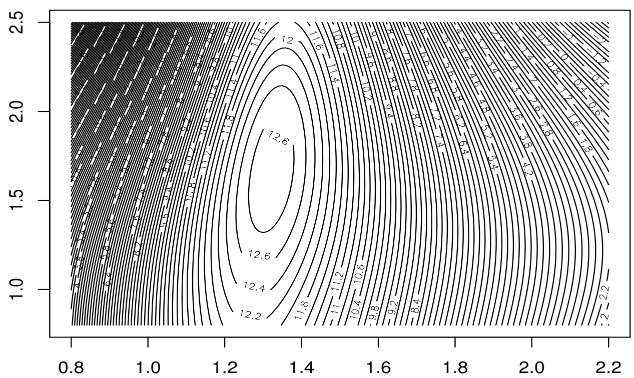
The joint CDF of a ν-BVBS random vector (T_1, T_2) with parameters $\alpha_1, \beta_1, \nu_1, \alpha_2, \beta_2, \nu_2$, and ρ can be written as



(a)



(b)



(c)

Figure 4.3: Contour plot of β and ν in (a)industrial devices data, (b)exceedances of flood peaks data and (c)insurance data using $\nu-$ Birnbaum Saunders distribution

$$F(t_1,t_2)=\Phi_2\left[\frac{1}{\alpha_1}\left(\left(\frac{t_1}{\beta_1}\right)^{\nu_1}-\left(\frac{\beta_1}{t_1}\right)^{\nu_1}\right),\frac{1}{\alpha_2}\left(\left(\frac{t_2}{\beta_2}\right)^{\nu_2}-\left(\frac{\beta_2}{t_2}\right)^{\nu_2}\right);\rho\right];t_1>0,t_2>0 \tag{4.3.1}$$

Here $\alpha_1 > 0, \beta_1 > 0, \alpha_2 > 0, \beta_2 > 0, -1 < \rho < 1$ and $\Phi_2(.;\rho)$ is CDF of standard BV normal vector (z_1, z_2) with correlation coefficient ρ. One can obtain the BVBS distribution in its particular case when $\nu_1 = \nu_2 = \frac{1}{2}$. For a BV random vector (T_1, T_2) with CDF as in Eq. (4.3.1), the corresponding joint PDF is given by

$$f_{T_1,T_2}(t_1,t_2)=\frac{\nu_1\nu_2}{2\pi\alpha_1\alpha_2\beta_1\beta_2\sqrt{1-\rho^2}}\left[\left(\frac{t_1}{\beta_1}\right)^{\nu_1-1}+\left(\frac{\beta_1}{t_1}\right)^{\nu_1+1}\right]\left[\left(\frac{t_2}{\beta_2}\right)^{\nu_2-1}+\left(\frac{\beta_2}{t_2}\right)^{\nu_2+1}\right]$$

$$exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{1}{\alpha_1^2}\left(\left(\frac{t_1}{\beta_1}\right)^{\nu}_1-\left(\frac{\beta_1}{t_1}\right)^{\nu}_1\right)^2+\frac{1}{\alpha_2^2}\left(\left(\frac{t_2}{\beta_2}\right)^{\nu}_2-\left(\frac{\beta_2}{t_2}\right)^{\nu}_2\right)^2\right.\right.$$

$$\left.\left.-\frac{2\rho}{\alpha_1\alpha_2}\left[\left(\frac{t_1}{\beta_1}\right)^{\nu}_1-\left(\frac{\beta_1}{t_1}\right)^{\nu}_1\right]\left[\left(\frac{t_2}{\beta_2}\right)^{\nu}_2-\left(\frac{\beta_2}{t_2}\right)^{\nu}_2\right]\right]\right\}$$

4.3.2 Properties of ν-BVBS Distribution

1. If $(T_1, T_2) \sim$ BVBS $(\alpha_1, \beta_1, \nu_1, \alpha_2, \beta_2, \nu_2, \rho)$ then it can be easily shown that its marginals, $T_i, \sim \nu - BS(\alpha_i, \beta_i, \nu_i)$

2. If $(T_1, T_2) \sim \nu - BS(\alpha_1, \beta_1, \nu_1, \alpha_2, \beta_2, \nu_2, \rho)$ then

   - $(T_1^{-1}, T_2^{-1}) \sim \nu - BS(\alpha_1, \frac{1}{\beta_1}, \nu_1, \alpha_2, \frac{1}{\beta_2}, \nu_2, \rho)$
   - $(T_1^{-1}, T_2) \sim \nu - BS(\alpha_1, \frac{1}{\beta_1}, \nu_1, \alpha_2, \beta_2, \nu_2, \rho)$
   - $(T_1, T_2^{-1}) \sim \nu - BS(\alpha_1, \beta_1, \nu_1, \alpha_2, \frac{1}{\beta_2}, \nu_2, \rho)$

4.4 Multivariate ν- Birnbaum Saunders Distribution

Along the same lines as the univariate and bivariate $\nu-$BS distribution, the multivariate $\nu-$BS distribution can be defined. First, let us recall the definition of the multivariate BS distribution [see Eq. 1.2.4].

Then the multivariate $\nu-$BS distribution is as follows:

Definition 4.4.1. *Let $\underline{\alpha}, \underline{\beta} \in \mathbb{R}^m$, where $\underline{\alpha} = (\alpha_1, \cdots, \alpha_m)^T$ and $\underline{\beta} = (\beta_1, \cdots, \beta_m)^T$, with $\alpha_1 > 0, \beta_i > 0$ for $i = 1, 2, \cdots, m$. Let $\mathbf{\Gamma}$ be a $m \times m$ positive definite correlation matrix. Then, the random vector $\underline{T} = (T_1, \cdots, T_m)^T$ is said to have a m-variate BS distribution with parameters $(\underline{\alpha}, \underline{\beta}, \mathbf{\Gamma}, \nu)$ if it has the joint CDF as*

$$\begin{aligned} P(\underline{T} \leq \underline{t}) &= P(T_1 \leq t_1, \cdots, T_m \leq t_m) \\ &= \Phi_m\left[\frac{1}{\alpha_1}\left(\left(\frac{t_1}{\beta_1}\right)^{\nu} - \left(\frac{\beta_1}{t_1}\right)^{\nu}\right), \cdots, \frac{1}{\alpha_m}\left(\left(\frac{t_m}{\beta_m}\right)^{\nu} - \left(\frac{\beta_m}{t_m}\right)^{\nu}\right); \Gamma\right] \end{aligned}$$

for $t_1 > 0, \cdots, t_m > 0$ and $0 < \nu < 1$. Here, for $\underline{u} = (u_1, \cdots, u_m)^T$, $\Phi_m(\underline{u}; \mathbf{\Gamma})$ denotes the joint CDF of a standard normal vector $\underline{Z} = (Z_1, \cdots, Z_m)^T$ with correlation matrix $\mathbf{\Gamma}$.

4.5 Summary

This chapter considers the univariate, bivariate, and multivariate $\nu-$ Birnbaum Saunders distributions and mainly focuses on the univariate case. Several interesting and useful properties are studied in detail. The point estimates of the model parameters of the univariate $\nu-$ Birnbaum Saunders distribution are obtained by employing the maximum likelihood principle. In order to obtain interval estimates, asymptotic CIs are computed using the observed information matrix. In an extensive simulation study, both estimation methodologies were thoroughly explored. Applications of the $\nu-$BS distribution to three real data sets are given to show that the $\nu-$ Birnbaum Saunders distribution provides consistently better modeling than the BS distribution. This extension is intended to attract a broad range of applications to the literature on fatigue life distributions.

CHAPTER 5

Inference for $\mathbf{R = P[X > Y]}$ based on the Exponential-Gamma $(\mathbf{3}, \lambda)$ Distribution

5.1 Introducion

Stress-strength (SS) reliability analysis is an important area of reliability analysis. Strength can be considered as a random variable. In light of the uncertainty in the operating environment of the unit, the stress applied to it should also be considered as a random variable. Let X represent a unit's strength, and Y represent the random stress that the operational environment imposes on the unit. $R = P(X > Y)$ is SS reliability (R).

It is easy to compute R if the stress and strength are assumed to or fitted to have some well-known statistical distribution. At the same time, if the fitted probability distributions have more parameters, then the problem becomes complicated. In such situations, one has to estimate SS reliability if the values of parameters are not available. Estimating the reliability of SS models is essential to determining strength and stress levels. The estimation of SS reliability is more complicated for single-component and multi-component systems. The problem of estimating reliability for single-component SS models is well documented in the literature.

A variety of censoring schemes have been employed in the literature to analyze

SS reliability. Based on the Gumbel copula under the type-I progressively hybrid censoring scheme, Bai et al. (2018) assessed the reliability of the multi-component SS model. Abravesh et al. (2019) assessed SS reliability with classical and Bayesian estimation methods based on type-II censored Pareto distributions. Byrnes et al. (2019) used progressively first failure-censored samples to estimate R for the Burr type XII distribution. Under progressive type II censoring, Zhang et al. (2019) examined the reliability of the generalized Rayleigh distribution. The inference of multicomponent SS reliability under progressive Type II censoring is presented by Jha et al. (2020), in which stress and strength variables have common unit Gompertz distributions. Karimi Ezmareh and Yari (2022) studied the inference of SS reliability for the Gompertz distribution using a type II censoring scheme.

The exponential-gamma $(3, \lambda)$ distribution studied in Chapter 3, which has a bathtub-shaped failure rate function, is used to analyze SS reliability.In this chapter, the exponential-gamma distribution $(3, \lambda)$ is denoted by $EGD(3, \lambda)$ or simply EGD.Specifically, $EGD(3, \lambda)$ has the PDF

$$f(x) = \frac{\lambda^2}{1+\lambda}\left(1 + \frac{\lambda}{2}x^2\right)e^{-\lambda x}, x > 0, \lambda > 0. \tag{5.1.1}$$

It should be noted that $EGD(3, \lambda)$ is a mixture of exponential distribution with a scale parameter of λ and gamma distribution with a shape parameter of 3 and a scale parameter of λ with mixing proportion $\frac{\lambda}{1+\lambda}$. It has been relatively unexplored whether SS reliability can be estimated when stress and strength vary independently following an $EGD(3, \lambda)$ distribution. This motivates the estimation of stress-strength reliability using $EGD(3, \lambda)$.

Consider two independent random variables X and Y from the $EGD(3, \lambda)$ with different parameters λ_1 and λ_2. This chapter focuses on the estimation of the parameter $R = P(X > Y)$ while stress and strength have $EGD(3, \lambda)$ distribution under type-II censoring. Typically, the problem of estimating R arises when dealing with the reliability of a component of strength X subjected to a load or stress Y. The component will fail if the stress exceeds its threshold level. As a result, R can be viewed as a measure of reliability.

The type II censoring method is briefly explained. Suppose that $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_m$ are independent random samples drawn from X and Y random variables, respectively. Consider the ordered statistics of these samples to be $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ and $y_{(1)}, y_{(2)}, \ldots, y_{(m)}$. The x_i's and y_i's are collected until failure occurs at r_1 and r_2 (where r_1 is less than or equal to n and r_2 is less than or equal to m).

Our goal in this chapter is to estimate the SS reliability when both stress and strength follow EGD with different parameters λ_1 and λ_2 under the type II censoring scheme. Section 5.2 considers the SS reliability of $EGD(3, \lambda)$. In section 5.3, the MLE of R using type-II censoring, the asymptotic distribution, and the CI for the MLE of R are obtained. An extensive simulation study is presented in section 5.4. Section 5.5 presents the results of the analysis of real data. In the final section, a summary is given.

5.2 Stress-Strength Reliability of $EGD(3, \lambda)$ Distribution

In this section, SS reliability is estimated using the EGD distribution. The general mathematical expression of SS reliability for the independent random variables X and Y is given by

$$R = \int_{-\infty}^{\infty} f_X(x)\, F_Y(x)\, dx,$$

where $f_X(x)$ and $F_Y(x)$ are the marginal PDF of X and marginal CDF of Y, respectively.

Consider X and Y as independent random variables having the EGD distribution with parameters λ_1 and λ_2, respectively. Suppose $X \sim EGD(3, \lambda_1)$ and $Y \sim EGD(3, \lambda_2)$. Then, SS reliability is

$$R = \int_0^{\infty} \frac{\lambda_1^2}{(1+\lambda_1)} \left(1 + \frac{\lambda_1}{2} x^2\right) e^{-\lambda_1 x} \left[1 - \frac{(\lambda_2(x(\lambda_2 x + 2) + 2) + 2) e^{-\lambda_2 x}}{2(1+\lambda_2)}\right] dx$$

$$= \tfrac{\lambda_1^2}{2(1+\lambda_1)(1+\lambda_2)} \int_0^{\infty} \left(1 + \tfrac{\lambda_1 x^2}{2}\right) e^{-\lambda_1 x} [2(1+\lambda_2) - (\lambda_2^2 x^2 + 2x\lambda_2 + 2\lambda_2 + 2) e^{-\lambda_2 x}] dx$$

$$= \frac{\lambda_1^2}{(1+\lambda_1)} \int_0^{\infty} \left(1 + \frac{\lambda_1}{2} x^2\right) e^{-\lambda_1 x} dx$$

$$- \tfrac{\lambda_1^2}{2(1+\lambda_1)(1+\lambda_2)} \int_0^{\infty} \left(1 + \tfrac{\lambda_1}{2} x^2\right) e^{-(\lambda_1+\lambda_2)x} (\lambda_2^2 x^2 + 2x\lambda_2 + 2\lambda_2 + 2) dx$$

$$=1-\left[\frac{\lambda_1^2(\lambda_2^2+\lambda_1\lambda_2+\lambda_1)}{(1+\lambda_1)(1+\lambda_2)(\lambda_1+\lambda_2)^3}+\frac{\lambda_1^2\lambda_2}{(1+\lambda_1)(1+\lambda_2)(\lambda_1+\lambda_2)^2}+\frac{\lambda_1^2}{(1+\lambda_1)(\lambda_1+\lambda_2)}\right.$$

$$\left.+\frac{6\lambda_1^3\lambda_2^2}{(1+\lambda_1)(1+\lambda_2)(\lambda_1+\lambda_2)^5}+\frac{3\lambda_1^3\lambda_2}{(1+\lambda_1)(1+\lambda_2)(\lambda_1+\lambda_2)^4}\right]$$

$$=\frac{\lambda_2(10\lambda_1^2\lambda_2^2+5\lambda_1\lambda_2^3+\lambda_2^4+12\lambda_1^2\lambda_2^3+6\lambda_1\lambda_2^4)}{(1+\lambda_1)(1+\lambda_2)(\lambda_1+\lambda_2)^5}$$

$$+\frac{\lambda_2(3\lambda_1^4\lambda_2+10\lambda_1^3\lambda_2^2+\lambda_2^5+\lambda_1^5\lambda_2+4\lambda_1^4\lambda_2^2+6\lambda_1^3\lambda_2^3+4\lambda_1^2\lambda_2^4+\lambda_1\lambda_2^5)}{(1+\lambda_1)(1+\lambda_2)(\lambda_1+\lambda_2)^5} \quad (5.2.1)$$

This expression evaluates if the values of parameters are available. But in practice, it is not available. Hence, one has to estimate the parameters to determine reliability.

5.3 Maximum Likelihood Estimator of R

Let us suppose that $X_{(1)}, X_{(2)}, \ldots, X_{(r_1)}$ is a type II censored sample from $EGD(3, \lambda_1)$ and $Y_{(1)}, Y_{(2)}, \ldots, Y_{(r_2)}$ is a type II censored sample from $EGD(3, \lambda_2)$. The two samples are assumed to be independent. The joint likelihood function is

$$L=\frac{n!\,m!}{(n-r_1)!(m-r_2)!}\frac{\lambda_1^{2r_1}}{(1+\lambda_1)^{r_1}}e^{-\lambda_1\sum_{k=1}^{r_1}x_{(k)}}\frac{\lambda_2^{2r_2}}{(1+\lambda_2)^{r_2}}e^{-\lambda_2\sum_{l=1}^{r_2}y_{(l)}}\left(\frac{1}{2(1+\lambda_1)}\right)^{n-r_1}\left(\frac{1}{2(1+\lambda_2)}\right)^{m-r_2}$$

$$\prod_{l=1}^{r_2}\left(1+\frac{\lambda_2}{2}y_{(l)}^2\right)\left[(\lambda_2(y_{(r_2)}(\lambda_2 y_{(r_2)}+2)+2)+2)e^{-\lambda_2 y_{(r_2)}}\right]^{m-r_2}$$

$$\prod_{k=1}^{r_1}\left(1+\frac{\lambda_1}{2}x_{(k)}^2\right)\left[(\lambda_1(x_{(r_1)}(\lambda_1 x_{(r_1)}+2)+2)+2)e^{-\lambda_1 x_{(r_1)}}\right]^{n-r_1}. \quad (5.3.1)$$

The log-likelihood associated with the above equation is given by

$$\log L=\log(n!)+\log(m!)-\log((n-r_1)!)-\log((m-r_2)!)+2r_1\log(\lambda_1)+2r_2\log(\lambda_2)$$

$$-r_1\log(1+\lambda_1)-r_2\log(1+\lambda_2)-\lambda_2\sum_{l=1}^{r_2}y_{(l)}-(n-r_1)\log(2(1+\lambda_1))$$

$$- (m - r_2) \log(2(1 + \lambda_2)) + \sum_{k=1}^{r_1} \log\left(1 + \frac{\lambda_1}{2} x_{(k)}^2\right) + \sum_{l=1}^{r_2} \log\left(1 + \frac{\lambda_2}{2} y_{(l)}^2\right)$$

$$- \lambda_1 \sum_{k=1}^{r_1} x_{(k)} + (n - r_1) \log(\lambda_1(x_{(r_1)}(\lambda_1 x_{(r_1)} + 2) + 2) + 2) - (n - r_1)\lambda_1 x_{(r_1)}$$

$$+ (m - r_2) \log(\lambda_2(y_{(r_2)}(\lambda_2 y_{(r_2)} + 2) + 2) + 2) - (m - r_2)\lambda_2 \, y_{(r_2)}$$

The first derivative of the above log-likelihood equation with respect to the unknown parameters λ_1 and λ_2 are respectively given by

$$\frac{\partial \log L}{\partial \lambda_1} = \frac{2r_1}{\lambda_1} - \frac{r_1}{1 + \lambda_1} - \sum_{k=1}^{r_1} x_{(k)} - \frac{(n - r_1)}{1 + \lambda_1} - (n - r_1)x_{(r_1)} + \sum_{k=1}^{r_1} \frac{x_{(k)}^2}{(2 + \lambda_1 x_{(k)}^2)}$$

$$+ 2(n - r_1)\frac{(1 + x_{(r_1)} + \lambda_1 x_{(r_1)}^2)}{(\lambda_1(x_{(r_1)}(\lambda_1 x_{(r_1)} + 2) + 2) + 2)}$$

$$\frac{\partial \log L}{\partial \lambda_2} = \frac{2r_2}{\lambda_2} - \frac{r_2}{1 + \lambda_2} - \sum_{l=1}^{r_2} y_{(l)} - \frac{(m - r_2)}{(1 + \lambda_2)} - (m - r_2)y_{(r_2)} + \sum_{l=1}^{r_2} \frac{y_{(l)}^2}{(2 + \lambda_2 y_{(l)}^2)}$$

$$+ 2(m - r_2)\frac{(1 + y_{(r_2)} + \lambda_2 y_{(r_2)}^2)}{(\lambda_2(x_{(r_2)}(\lambda_2 y_{(r_2)} + 2) + 2) + 2)}.$$

The second derivative of the above log-likelihood equation with respect to the unknown parameters λ_1 and λ_2 are respectively given by

$$\frac{\partial^2 \log L}{\partial \lambda_1^2} = \frac{n}{(1+\lambda_1)^2} - \frac{2r_1}{\lambda_1^2} - \sum_{k=1}^{r_1} \frac{x_{(k)}^4}{(2+\lambda_1 x_{(k)}^2)^2} - 2(n - r_1)\frac{(x_{(r_1)}(\lambda_1 x_{(r_1)}+2)(\lambda_1 x_{(r_1)}^2+2)+2)}{(\lambda_1(x_{(r_1)}(\lambda_1 x_{(r_1)}+2)+2)+2)^2}.$$

$$\frac{\partial^2 \log L}{\partial \lambda_2^2} = \frac{m}{(1+\lambda_2)^2} - \frac{2r_2}{\lambda_2^2} - \sum_{l=1}^{r_2} \frac{y_{(l)}^4}{(2+\lambda_2 y_{(l)}^2)^2} - 2(m-r_2)\frac{(y_{(r_2)}(\lambda_2 y_{(r_2)}+2)(\lambda_2 y_{(r_2)}^2+2)+2)}{(\lambda_2(y_{(r_2)}(\lambda_2 y_{(r_2)}+2)+2)+2)^2}.$$

Using Eq.(5.2.1), MLE of SS reliability, $\hat{R}_{ML}$, can be calculated as follows:

$$\hat{R}_{ML} = \frac{\hat{\lambda}_2(10\hat{\lambda}_1^2\hat{\lambda}_2^2+5\hat{\lambda}_1\hat{\lambda}_2^3+\hat{\lambda}_2^4+12\hat{\lambda}_1^2\hat{\lambda}_2^3+6\hat{\lambda}_1\hat{\lambda}_2^4+3\hat{\lambda}_1^4\hat{\lambda}_2+10\hat{\lambda}_1^3\hat{\lambda}_2^2+\hat{\lambda}_2^5+\hat{\lambda}_1^5\hat{\lambda}_2+4\hat{\lambda}_1^4\hat{\lambda}_2^2+6\hat{\lambda}_1^3\hat{\lambda}_2^3+4\hat{\lambda}_1^2\hat{\lambda}_2^4+\hat{\lambda}_1\hat{\lambda}_2^5)}{(1+\hat{\lambda}_1)(1+\hat{\lambda}_2)(\hat{\lambda}_1+\hat{\lambda}_2)^5}. \tag{5.3.2}$$

Asymptotic Distribution and Confidence Intervals

The asymptotic distribution and confidence interval (CI) for the MLE of R are given in this section. Let us represent the Fisher information matrix of $\lambda = (\lambda_1, \lambda_2)$ as $I(\lambda)$. In order to obtain the asymptotic variance of the MLE of R, $\hat{R}_{ML}$, use $I(\lambda)$ in Eq.(5.3.2), where

$$I(\lambda) = E\begin{bmatrix} -\frac{\partial^2 \log L}{\partial \lambda_1^2} & -\frac{\partial^2 \log L}{\partial \lambda_1 \partial \lambda_2} \\ \\ -\frac{\partial^2 \log L}{\partial \lambda_2 \partial \lambda_1} & -\frac{\partial^2 \log L}{\partial \lambda_2^2} \end{bmatrix}.$$

The asymptotic normality of R is obtained by using the following definition

$$d(\lambda) = \left(\frac{\partial R}{\partial \lambda_1}, \frac{\partial R}{\partial \lambda_2}\right)' = (d_1, d_2)'$$

where

$$\frac{\partial R}{\partial \lambda_1} = -\frac{\lambda_1\lambda_2^2(\lambda_1^5+(4\lambda_2+6)\lambda_1^4+(+\lambda_2^2+20\lambda_2+3)\lambda_1^3+(4\lambda_2^3+24\lambda_2^2+48\lambda_2)\lambda_1^2+(\lambda_2^4+12\lambda_2^3+21\lambda_2^2+30\lambda_2)\lambda_1+2\lambda_2^4+6\lambda_2^3)}{(1+\lambda_1)^2(1+\lambda_2)(\lambda_1+\lambda_2)^6}$$

and

$$\frac{\partial R}{\partial \lambda_2} = \frac{\lambda_1^2\lambda_2(\lambda_2^5+2(2\lambda_1+3)\lambda_2^4+(6\lambda_1^2+20\lambda_1)\lambda_2^3+4\lambda_1(\lambda_1^2+6\lambda_1+12)\lambda_2^2+\lambda_1(\lambda_1^3+12\lambda_1^2+21\lambda_1+30)\lambda_2+2\lambda_1^4+6\lambda_1^3}{(1+\lambda_1)^2(1+\lambda_2)(\lambda_1+\lambda_2)^6}$$

As a result, the asymptotic distribution of $\hat{R}_{ML}$ can be represented as

$$\sqrt{n+m}(\hat{R}_{ML} - R) \rightarrow^{d} N(0, d'(\lambda)\, I^{-1}(\lambda)\, d(\lambda)).$$

We obtain the asymptotic variance of $\hat{R}_{ML}$ as follows:

$$AV(\hat{R}_{ML}) = \frac{1}{n+m} d'(\lambda)\, I^{-1}(\lambda)\, d(\lambda)$$

$$= V(\hat{\lambda_1})d_1^2 + V(\hat{\lambda_1})d_2^2 + 2d_1 d_2(\hat{\lambda_1}\hat{\lambda_1}).$$

Asymptotic $100(1-\omega)\%$ CI for R can be obtained as

$$\hat{R}_{ML} \pm Z_{\omega/2}\sqrt{AV(\hat{R}_{ML})}$$

where $Z_{\omega/2}$ is the upper $\omega/2$ quantile of the standard normal distribution. To assess the efficiency of the estimators, a simulation study is carried out and given in next section.

5.4 Simulation Study

This section presents some results related to the performance of estimators of R using the Newton-Raphson method. For this purpose, 1000 samples are generated using independent $EGD(3, \lambda_1)$ and $EGD(3, \lambda_2)$ distributions for various sample sizes under type II censoring scheme. The parameter values, (λ_1, λ_2), used in this study were (0.5, 1.5), (1, 1.5), and (1.5, 0.5). Corresponding to these parameter values, R values are 0.8391, 0.6405, and 0.1609, respectively.

Tables 5.1- 5.3 provided estimates of R based on the MLE method along with average biases, mean square errors (MSEs), and 95% CIs. From these simulation results, biases and MSEs decrease with increasing sample size (n, m).

Table 5.1: MLE, average(Avg) bias, and MSEs of different estimators of R when $\lambda_1 = 0.5$ and $\lambda_2 = 1.5$.

(n,m)	$(\mathbf{r_1}, \mathbf{r_2})$	Avg Bias	MSEs	95% CI	Estimates
(15,15)	(15,15)	0.0124	0.0080	(0.7361, 0.9523)	$\hat{\lambda}_1$=0.5124
		0.0878	0.1124		$\hat{\lambda}_2$=1.5878
	(14,14)	0.0193	0.0091	(0.6832, 0.9927)	$\hat{\lambda}_1$=0.5193
		0.0589	0.1044		$\hat{\lambda}_2$=1.5589
	(12,12)	0.0215	0.0105	(0.7259, 0.9563)	$\hat{\lambda}_1$=0.5215
		0.0940	0.1412		$\hat{\lambda}_2$=1.5940
(25,25)	(25,25)	0.0113	0.0046	(0.7499, 0.9290)	$\hat{\lambda}_1$=0.5113
		0.0430	0.0557		$\hat{\lambda}_2$=1.5430
	(23,23)	0.0114	0.0046	(0.7648, 0.9185)	$\hat{\lambda}_1$=0.5114
		0.0621	0.0643		$\hat{\lambda}_2$=1.5621
	(21,21)	0.0135	0.0057	(0.7713, 0.9072)	$\hat{\lambda}_1$=0.5135
		0.0493	0.0642		$\hat{\lambda}_2$=1.5493
(30,30)	(30,30)	0.0088	0.0040	(0.7615, 0.9197)	$\hat{\lambda}_1$=0.5089
		0.0440	0.0454		$\hat{\lambda}_2$=1.5440
	(28,28)	0.0089	0.0041	(0.7638, 0.9123)	$\hat{\lambda}_1$=0.5099
		0.0231	0.0442		$\hat{\lambda}_2$=1.5231
	(25,25)	0.0087	0.0048	(0.7584, 0.9201)	$\hat{\lambda}_1$=0.5087
		0.0324	0.0511		$\hat{\lambda}_2$=1.5324

Table 5.2: MLE, Avg bias, and MSEs of different estimators of R when $\lambda_1 = 1$ and $\lambda_2 = 1.5$.

(n,m)	$(\mathbf{r_1}, \mathbf{r_2})$	Avg Bias	MSEs	95% CI	Estimates
(15,15)	(15,15)	0.0322	0.0382	(0.4666, 0.8290)	$\hat{\lambda}_1$=1.0322
		0.0878	0.1124		$\hat{\lambda}_2$=1.5878
	(14,14)	0.0477	0.0447	(0.3602, 0.9138)	$\hat{\lambda}_1$=1.0477
		0.0589	0.1044		$\hat{\lambda}_2$=1.5589
	(12,12)	0.0479	0.0481	(0.4690, 0.8167)	$\hat{\lambda}_1$=1.0479
		0.0885	0.1253		$\hat{\lambda}_2$=1.5885
(25,25)	(25,25)	0.0281	0.0218	(0.4877, 0.7925)	$\hat{\lambda}_1$=1.0281
		0.0430	0.0557		$\hat{\lambda}_2$=1.5430
	(23,23)	0.0280	0.0222	(0.5142, 0.7738)	$\hat{\lambda}_1$=1.0280
		0.0621	0.0643		$\hat{\lambda}_2$=1.5621
	(21,21)	0.0332	0.0279	(0.5269, 0.7526)	$\hat{\lambda}_1$=1.0332
		0.0493	0.0642		$\hat{\lambda}_2$=1.5493
(30,30)	(30,30)	0.0208	0.0173	(0.5233, 0.7624)	$\hat{\lambda}_1$=1.0208
		0.0447	0.0470		$\hat{\lambda}_2$=1.5447
	(28,28)	0.0242	0.0202	(0.5100, 0.7700)	$\hat{\lambda}_1$=1.0242
		0.0362	0.0463		$\hat{\lambda}_2$=1.5380
	(25,25)	0.0254	0.0209	(0.5195, 0.7626)	$\hat{\lambda}_1$=1.0254
		0.0434	0.0535		$\hat{\lambda}_2$=1.5434

Table 5.3: MLE, Avg bias, and MSEs of different estimators of R when $\lambda_1 = 1.5$ and $\lambda_2 = 0.5$.

(n,m)	**(r_1, r_2)**	**Avg Bias**	**MSEs**	**95% CI**	**Estimates**
(15,15)	(15,15)	0.0618	0.0913	(0.0564, 0.2635)	$\hat{\lambda}_1$=1.5618
		0.0151	0.0080		$\hat{\lambda}_2$=0.5151
	(14,14)	0.0682	0.1229	(0.0472, 0.2696)	$\hat{\lambda}_1$=1.5682
		0.0132	0.0078		$\hat{\lambda}_2$=0.5132
	(12,12)	0.0826	0.1231	(0.0369, 0.2811)	$\hat{\lambda}_1$=1.5826
		0.0186	0.0099		$\hat{\lambda}_2$=0.5186
(25,25)	(25,25)	0.0406	0.0516	(0.0820, 0.2431)	$\hat{\lambda}_1$=1.5369
		0.0141	0.0048		$\hat{\lambda}_2$=0.5141
	(23,23)	0.0387	0.0620	(0.0881, 0.2340)	$\hat{\lambda}_1$=1.5381
		0.0111	0.0050		$\hat{\lambda}_2$=0.5111
	(21,21)	0.0464	0.0659	(0.0700, 0.2506)	$\hat{\lambda}_1$=1.5464
		0.0117	0.0056		$\hat{\lambda}_2$=0.5117
(30,30)	(30,30)	0.0369	0.0463	(0.0923, 0.2264)	$\hat{\lambda}_1$=1.5406
		0.0078	0.0035		$\hat{\lambda}_2$=0.5078
	(28,28)	0.0381	0.0478	(0.0873, 0.2304)	$\hat{\lambda}_1$=1.5387
		0.0060	0.0040		$\hat{\lambda}_2$=0.5060
	(25,25)	0.0312	0.0525	(0.0989, 0.2245)	$\hat{\lambda}_1$=1.5312
		0.0107	0.0042		$\hat{\lambda}_2$=0.5107

5.5 Applications

To check the applicability of the model we considered the dataset used by Sonker et al. (2023) which is extracted from the dataset available in Andrews and Herzberg (2012) and contains information on Kevlar pressure vessels' stress rupture life under constant pressure. With $r_1 = r_2 = 19$, Type II right censoring is performed on the complete dataset.The data are presented as follows.

X : 6121, 11604, 9711, 9106, 11026, 17568, 1921, 4921, 10861, 11214, 11608, 5956, 1337, 10205, 11745, 2322, 16179, 14110, 7501, 8666.

Y : 1942, 17568, 3629, 11362, 4006, 14496, 6068, 7886, 5905, 6473, 11895, 4012, 13670, 10396, 17092, 8108, 1051, 5445, 5817, 5620.

When the EGD model is fitted to the data, it can be seen that the model fits the data quite well. Similarly, the following data is fitted with the Lindley (LD) model, and it can be observed that the EGD model provides a better fit to the data than the LD model. Since, the EGD model has minimum CVM and KS values, and maximum p-values.

The MLE for parameters λ_1 and λ_2, Cramer-Von Mises (CVM) and the Kolmogorov-Smirnov (K-S) tests are given in Table 5.4 and 5.5. As a result, the MLE of R of EGD model is $\hat{R} = 0.5002$, and the 95% CI for R is (0.2805,0.7199).

Table 5.4: MLE, CVM, and KS goodness of fit tests for X data

Data	Estimates	CVM (p-value)	KS (p-value)
LD	0.0002	0.1883 (0.2929)	0.1836 (0.4873)
EGD	0.0003	0.1297 (0.4614)	0.1683 (0.5967)

Table 5.5: MLE, CVM, and KS goodness of fit tests for Y data

Data	Estimates	CVM (p-value)	KS (p-value)
LD	0.0002	0.1883 (0.2929)	0.1836 (0.4872)
EGD	0.0003	0.1299 (0.4607)	0.1680 (0.5989)

5.6 Summary

There have been several well-developed estimation techniques for SS models with single components that follow well-known lifetime distributions. The EGD model is found to be a better model than some existing models. In this chapter, the problem of estimating SS reliability with an EGD distribution in a single-component SS model for independent stress and strength random variables under type-II censoring is discussed in detail. The MLE of SS reliability, $\hat{R}_{ML}$, is obtained. The extensive simulation revealed that the MSE and average biases caused by estimates approach zero when sample sizes are increased. The analysis is conducted on real-life datasets and compares the EGD model with the Lindley model. The EGD model is found to be a good fit, and it can be used for SS reliability analysis.

CHAPTER 6

A Simple Step-Stress Analysis of Type II Gumbel Distribution

6.1 Introduction

Technology in the modern world evolves faster than ever before. As it gets better, every industry gains. Ultimately, we gain from their results because they lead to better products and services. Since its inception, the market has been and always will be competitive. As a result, producers compete to offer their clients the highest quality products possible. Failure time within a specific timeframe under normal operating conditions cannot be estimated because product quality is constantly advancing. Early failures using ALT methodologies are encouraged in this instance. Using this method, we put more stress than usual on promoting early failures. It lowers the price and enhances the quality of the product.

A type of ALT called step-stress life testing allows the experimenter to gradually increase the stress levels at predetermined intervals throughout the test. 'n' identical units are placed on a life-testing experiment at a starting stress level in a set-up for a multiple-step stress model. The stress level then continued to rise at pre-defined intervals. If there are only two degrees of stress, the model is known as the simple step-stress model.

A model that links the distributions under various stress levels is needed to analyze failure time data from any SSALT experiment. The cumulative exposure model (Sedyakin (1966)), its generalizations (Bagdonavičius (1978)), the proportional hazard model (Cox (1992)), tampered random variable model (Goel (1972)), tampered failure rate model (Bhattacharyya and Soejoeti (1989)), and Khamis-Higgins model (Khamis and Higgins (1998)) are the most frequently used models in the literature.

Here, a failure rate-based model with pre-fixed but arbitrarily chosen failure rates at various stress levels is used (see Kateri and Kamps (2015, 2017)). It is assumed that the HRF of the distribution for the step stress approach is as follows, where S_1 and S_2 denote the stress levels and T denotes the time at which the stress changes.

$$h(x) = \begin{cases} h_1(x) & \text{if } 0 < x \leq T \\ \\ h_2(x) & \text{if } T < x < \infty \quad . \end{cases} \tag{6.1.1}$$

The corresponding CDF is,

$$F(x) = \begin{cases} F_1(x) & \text{if } 0 < x \leq T \\ \\ 1 - \dfrac{1 - F_1(T)}{1 - F_2(T)}(1 - F_2(x)) & \text{if } T < x < \infty \quad . \end{cases} \tag{6.1.2}$$

SSALT setups with type II Gumbel lifetime distributions are rarely examined with regard to inference procedures. Dutta et al. (2023) used Gumbel type II distribution for the simple step-stress life test based on a tampered random variable model under type-II censoring.
This chapter discusses the estimation problem for the Type-II Gumbel distribution utilizing Type-II censoring in the failure rate-based SSALT model. The SSALT model with Type II Gumbel distribution under Type II censoring has been developed to comprehensively assess the reliability and failure characteristics of products or systems exposed to stress testing, which is seldom explored. A type II Gumbel distribution is selected as it is capable of modeling rare but catastrophic failures.

In addition to exploring how increasing stress levels affect the failure rate of a product, the step-stress approach also assists in understanding how the product performs under various environmental conditions. The use of type II censoring, with its periodic inspections and data collection, is an effective and efficient way to conduct long-term tests. As a result of this combination of techniques, organizations can make informed decisions regarding product design, warranty policies, and maintenance strategies by gaining insight into how stress and aging can affect failure behavior and the life expectancy of products.
The baseline lifetime X is distributed according to the Type II Gumbel distribution, whose PDF and CDF are, respectively,

$$f_i^*(x;\beta,\theta) = \begin{cases} \beta_i\theta_i x^{-(\beta_i+1)} e^{-(\theta_i x^{-\beta_i})} & \text{if } x>0,\ \beta>0,\ \theta>0 \\ \\ 0 & \text{otherwise} \end{cases}, \tag{6.1.3}$$

and

$$F_i^*(x;\beta,\theta) = \begin{cases} e^{-(\theta_i x^{-\beta_i})} & \text{if } x>0,\ \beta>0,\ \theta>0 \\ \\ 0 & \text{otherwise} \end{cases}, \tag{6.1.4}$$

where β, θ are the shape and scale parameters, respectively. The HRF is given by

$$h_i^*(x;\beta,\theta) = \begin{cases} \dfrac{\beta_i\theta_i x^{-(\beta_i+1)} e^{-(\theta_i x^{-\beta_i})}}{1-e^{-(\theta_i x^{(-\beta_i)})}} & \text{if } x>0,\ \beta>0, \theta>0 \\ \\ 0 & \text{otherwise} \end{cases}. \tag{6.1.5}$$

Depending on the parameter values, the Type-II Gumbel distribution's HRF decreases or takes the shape of a UBFR. The Type-II Gumbel distribution is highly adaptable to represent meteorological occurrences, reliability analysis, and life testing, as well as in medical and epidemiological applications because of these shapes of HRF.

6.2 Model Description

Under a Type-II censoring scheme, a simple SSALT model with two stress levels, S_1, and S_2, is analyzed. In the life testing experiment, n identical units are first placed at the stress level S_1. At the pre-determined time T $(0 < T < \infty)$, the stress level is increased to a higher level S_2, and the experiment ends when the rth failure occurs (r is a pre-determined integer $\leq n$).

Let n_i be the number of units that fail at $S_i (i = 1, 2)$. The following ordered failure time data given below are observed using this notation.

$$\Im = \{x_{1:n} < ... < x_{n_1:n} < T < x_{n_1+1:n} < ... < x_{r:n}\}, \tag{6.2.1}$$

where $r = n_1 + n_2$.

Assume that the lifetime distributions of the experimental units at stress levels S_1 and S_2 are Type-II Gumbel distributions, with differences in both the shape and scale parameters. To relate the CDFs of lifetime distributions at two successive stress levels to the CDFs of the lifetime under the used conditions, the assumptions from the SSALT model based on failure rate are used.

To peruse the failure time data, the HRF h(t), the CDF G(t), and the associated PDF g(t) of the lifetime of an experimental unit under the assumption of the failure rate-based SSALT model are respectively given by

$$h(x) = \begin{cases} \dfrac{\beta_1\theta_1 x^{-(\beta_1+1)} e^{-(\theta_1 x^{-\beta_1})}}{1 - e^{-(\theta_1 x^{(-\beta_1)})}} & \text{if } 0 < x \leq T \\ \dfrac{\beta_2\theta_2 x^{-(\beta_2+1)} e^{-(\theta_2 x^{-\beta_2})}}{1 - e^{-(\theta_2 x^{(-\beta_2)})}} & \text{if } T < x < \infty, \end{cases} \tag{6.2.2}$$

$$G(x) = \begin{cases} e^{-(\theta_1 x^{-\beta_1})} & \text{if } 0 < x \leq T \\ 1 - \dfrac{e^{-\theta_1 T^{-\beta_1}}}{e^{-\theta_2 T^{-\beta_2}}} e^{-(\theta_2 x^{-\beta_2})} & \text{if } T < x < \infty, \end{cases} \tag{6.2.3}$$

$$g(x) = \begin{cases} \beta_1\theta_1 x^{-(\beta_1+1)} e^{-(\theta_1 x^{-\beta_1})} & \text{if } 0 < x \le T \\ \\ \dfrac{\beta_2\theta_2 e^{-\theta_1 T^{-\beta_1}}}{e^{-\theta_2 T^{-\beta_2}}} x^{-(\beta_2+1)} e^{-(\theta_2 x^{-\beta_2})} & \text{if } T < x < \infty. \end{cases} \tag{6.2.4}$$

6.3 Maximum Likelihood Estimation

The MLEs of the unknown parameters $\beta_1, \theta_1, \beta_2$, and θ_2 are determined here using the likelihood function based on the observed type-II censored data in Eq.(6.2.1).

If $X_{1:n} < \cdots < X_{r:n}$ denotes the ordered Type-II censored sample from any continuous CDF $F^*(.)$, PDF $f^*(.)$, then the likelihood function of this censored sample can be stated as follows:

$$L(\boldsymbol{\theta}) = \frac{n!}{(n-r)!}\left\{\prod_{k=1}^{n} f_X(x_{k:n})\right\}\{1 - F_X(x_{r:n})\}^{n-r}, 0 < x_{1:n} < ... < x_{r:n} < \infty,$$

where $\boldsymbol{\theta}$ is the vector representing model's parameters.

Let $\boldsymbol{\theta} = (\beta_1, \theta_1, \beta_2, \theta_2)$ denotes the set of unknown parameters. Using the type-II censored data in Eq.(6.2.1) of failure time from the Type II Gumbel distribution with differences in the shape and scale parameters at each of the two stress levels and assuming a failure rate based simple SSALT model, the likelihood function is obtained as

$$L(\boldsymbol{\theta}|\Im) = \frac{n!}{(n-r)!}\beta_1^{n_1}\theta_1^{n_1}\beta_2^{n_2}\theta_2^{n_2}\prod_{k=1}^{n_1} x_{k:n}^{-(\beta_1+1)}\prod_{k=n_1+1}^{r} x_{k:n}^{-(\beta_2+1)}\prod_{k=1}^{n_1} e^{-\theta_1 x_{k:n}^{-\beta_1}}$$

$$\prod_{k=n_1+1}^{r} e^{-\theta_2 x_{k:n}^{-\beta_2}}\left(\frac{e^{\theta_1 T^{-\beta_1}}}{e^{-\theta_2 T^{-\beta_2}}}\right)^{n-n_1}(e^{-\theta_1 x_{r:n}^{-\beta_2}})^{n-r}. \tag{6.3.1}$$

The associated log-likelihood function $\ell(\boldsymbol{\theta})$ of the observed data is given by

$$\ell(\boldsymbol{\theta}) = \psi_1(\beta_1, \theta_1) + \psi_2(\beta_2, \theta_2), \tag{6.3.2}$$

where

$$\psi_1(\beta_1,\theta_1) = \ln n! + \ln(n-r)! + n_1 \ln\beta_1 + n_1 \ln\theta_1 - (\beta_1+1)\sum_{k=1}^{n_1} \ln x_{k:n}$$

$$- \theta_1 \sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} + (n-n_1)\ln[1-e^{-\theta_1 T^{-\beta_1}}], \quad (6.3.3)$$

and

$$\psi_2(\beta_2,\theta_2) = n_2 \ln\beta_2 + n_2 \ln\theta_2 - (\beta_2+1)\sum_{k=n_1+1}^{r} \ln x - \theta_2 \sum_{k=n_1+1}^{r} x_{k:n}^{-\beta_2}$$

$$- (n-n_1)\ln[1-e^{-\theta_2 T^{-\beta_2}}] + (n-r)\ln[1-e^{-\theta_2 x_{r:n}^{-\beta_2}}]. \quad (6.3.4)$$

Hence, $\hat{\boldsymbol{\theta}}$ can be obtained by maximizing the log-likelihood function Eq.(6.3.2) over the region Θ. The Eq.(6.3.2) can be written as the sum of two equations Eq.(6.3.3) and Eq.(6.3.4). Differentiating Eq.(6.3.2) with respect to $\beta_1, \theta_1, \beta_2, \theta_2$ respectively and equating them to zero, the normal equations are obtained as

$$\frac{\partial \ell}{\partial \beta_1} = \frac{n_1}{\beta_1} - \sum_{k=1}^{n_1} \ln x_{k:n} + \theta_1 \sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} \ln x_{k:n} + (n-n_1)\frac{\theta_1 T^{-\beta_1} \ln T e^{-\theta_1 T^{-\beta_1}}}{1-e^{-\theta_1 T^{-\beta_1}}}, \quad (6.3.5)$$

$$\frac{\partial \ell}{\partial \theta_1} = \frac{n_1}{\theta_1} - \sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} + (n-n_1)\frac{T^{-\beta_1} e^{-\theta_1 T^{-\beta}}}{1-e^{-\theta_1 T^{-\beta_1}}}, \quad (6.3.6)$$

$$\frac{\partial \ell}{\partial \beta_2} = \frac{n_2}{\beta_2} - \sum_{k=n_1+1}^{r} \ln x_{k:n} - \theta_2 \Sigma_{k=n_1+1}^{r} x_{k:n}^{-\beta_2} \ln x_{k:n} - (n-n_1)\frac{\theta_2 \ln T e^{-\theta_2 T^{-\beta_2}}}{1-e^{-\theta_2 T^{-\beta_2}}}$$

$$+ (n-r)\frac{\theta_2 \ln x_{r:n} e^{-\theta_2 x_{r:n}^{-\beta_2}}}{1-e^{-\theta_2 x_{r:n}^{-\beta_2}}}, \quad (6.3.7)$$

and

$$\frac{\partial \ell}{\partial \theta_2} = \frac{n_2}{\theta_2} - \sum_{k=n_1+1}^{r} x_{k:n}^{-\beta_2} - (n-n_1)\frac{T^{-\beta_2} e^{-\theta_2 T^{-\beta_2}}}{1-e^{-\theta_2 T^{-\beta_2}}} + (n-r)\frac{x_{r:n}^{-\beta_2} e^{-\theta_2 x_{r:n}^{-\beta_2}}}{1-e^{-\theta_2 x_{r:n}^{-\beta_2}}}. \quad (6.3.8)$$

Multiplying Eq.(6.3.6) with $\theta_1 \ln T$, we get

$$n_1 \ln T - n_1 \ln T \sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} + (n-n_1)\frac{\theta_1 \ln T\, T^{-\beta_1} e^{-\theta_1 T^{-\beta}}}{1-e^{-\theta_1 T^{-\beta_1}}} = 0 \tag{6.3.9}$$

Substracting Eq.(6.3.9) from Eq.(6.3.5) and simplifying, we get

$$\theta_1 = \frac{\frac{n_1}{\beta_1} - n_1 \ln T + n_1 \ln T \sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} - \sum_{k=1}^{n_1} \ln x_{k:n}}{\sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} \ln x_{k:n}} \tag{6.3.10}$$

6.4 Interval Estimation

In this section, a method for constructing CIs for the unknown parameters $\beta_1, \theta_1, \beta_2$, and θ_2 are presented. The exact CIs of the unknown parameters cannot be obtained because the closed forms of the MLEs do not exist. The asymptotic CIs are provided, assuming the MLEs are asymptotically normal.

6.4.1 Asymptotic Confidence Intervals

Using the observed Fisher information matrix, a method is presented that assumes asymptotic normality of the MLEs to obtain the CIs for $\beta_1, \theta_1, \beta_2$, and θ_2. For large sample sizes, this method is useful due to its simplicity in computation.

To begin with, we need to obtain explicit expressions for the elements of the Fisher information matrix $I(\boldsymbol{\theta})$. The elements of $I(\boldsymbol{\theta})$ are

$$\frac{\partial^2 \ell}{\partial \beta_1^2} = -\frac{n_1}{\beta_1^2} - \theta_1 \sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} (\ln x_{k:n})^2 - \frac{(n-n_1)\theta_1 (\ln T)^2 T^{-\beta_1}[(1-\theta_1 T^{-\beta_1})e^{\theta_1 T^{-\beta_1}} - 1]}{(e^{\theta_1 T^{-\beta_1}} - 1)^2}$$

$$\frac{\partial^2 \ell}{\partial \theta_1 \partial \beta_1} = \sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} \ln x_{k:n} - \frac{(n-n_1)\ln T\, T^{-\beta_1}[(\theta_1 T^{-\beta_1} - 1)e^{\theta_1 T^{-\beta_1}} + 1]}{(e^{\theta_1 T^{-\beta_1}} - 1)^2}$$

$$\frac{\partial^2 \ell}{\partial \theta_1^2} = -\frac{n_1}{\theta_1^2} - (n-n_1)T^{-2\beta_1} \frac{e^{\theta_1 T^{-\beta_1}}}{(e^{\theta_1 T^{-\beta_1}} - 1)^2}$$

$$\frac{\partial^2 \ell}{\partial \beta_1 \partial \theta_1} = -\sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} \ln x_{k:n} + (n-n_1)\ln T \frac{T^{-\beta_1}[(1-\theta_1 T^{-\beta_1})e^{\theta_1 T^{-\beta_1}} - 1]}{(e^{\theta_1 T^{-\beta_1}} - 1)^2}$$

$$\frac{\partial^2 \ell}{\partial \beta_2^2} = -\frac{n_2}{\beta_2^2} + \theta_2 \sum_{k=n_1+1}^{r} x_{k:n}^{-\beta_2} (\ln x_{k:n})^2 - (n-n_1)\,\theta_2^2\,(\ln T)^2 \frac{T^{-\beta_2} e^{\theta_2 T^{-\beta_2}}}{(e^{\theta_2 T^{-\beta_2}}-1)^2} + (n-r)\theta_2^2\,(\ln x_{r:n})^2\, \frac{e^{\theta_2 x_{r:n}^{-\beta_2}}}{x_{r:n}^{\beta_2}(e^{\theta_2 x_{r:n}^{-\beta_2}}-1)^2}$$

$$\frac{\partial^2 \ell}{\partial \theta_2 \partial \beta_2} = -\sum_{k=n_1+1}^{r} x_{k:n}^{-\beta_2} \ln x_{k:n} + \frac{(n-n_1)\ln T\,[(\theta_2 T^{-\beta_2}-1)e^{\theta_2 T^{-\beta_2}}+1]}{(e^{\theta_2 T^{-\beta_2}}-1)^2} - (n-r)\frac{\ln x_{r:n}\,[(\theta_2 x_{r:n}^{-\beta_2}-1)e^{\theta_2 x_{r:n}^{-\beta_2}}+1]}{(e^{\theta_2 x_{r:n}^{-\beta_2}}-1)^2}$$

$$\frac{\partial^2 \ell}{\partial \theta_2^2} = -\frac{n_2}{\theta_2^2} + (n-n_1)\frac{T^{-2\beta_2} e^{\theta_2 T^{-\beta_2}}}{(e^{\theta_2 T^{-\beta_2}}-1)^2} - (n-r)\frac{x_{r:n}^{-2\beta_2} e^{\theta_2 x_{r:n}^{-\beta_2}}}{(e^{\theta_2 x_{r:n}^{-\beta_2}}-1)^2}$$

$$\frac{\partial^2 \ell}{\partial \beta_2 \partial \theta_2} = \sum_{k=n_1+1}^{r} x_{k:n}^{-\beta_2} \ln x_{k:n} - (n-n_1)\theta_2^2 \frac{(\ln T)^2\, T^{-\beta_2} e^{\theta_2 T^{-\beta_2}}}{(e^{\theta_2 T^{-\beta_2}}-1)^2} + (n-r)\theta_2^2 \frac{(\ln x_{r:n})^2\, x_{r:n}^{-\beta_2} e^{\theta_2 x_{r:n}^{-\beta_2}}}{(e^{\theta_2 x_{r:n}^{-\beta_2}}-1)^2}.$$

Then, the $100(1-\alpha)\%$ asymptotic CIs for $\beta_1, \theta_1, \beta_2$, and θ_2 are, respectively

$(\hat{\beta}_1 \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{11}})$, $(\hat{\theta}_1 \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{22}})$, $(\hat{\beta}_2 \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{33}})$, and $(\hat{\theta}_2 \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{44}})$,

where V_{ij} represents the (i,j)th element in the inverse of the Fisher information matrix I and z_p is the p-th upper percentile of a standard normal distribution.

6.5 Summary

This study introduces a simple step stress life testing model with type-II Gumbel lifetime distribution. A flexible failure-rate based SSALT model is considered based on type-II censoring. The point estimate of parameters using the maximum likelihood method is described under the notion of a failure rate-based model.

CHAPTER 7

Conclusion and Future Directions

7.1 Conclusion

A number of methods are available in the statistical literature to propose new distributions based on baseline distributions. In statistical distribution theory, adding a parameter to a family of distribution functions is a very common practice. In the context of data analysis, adding a parameter can greatly enhance the flexibility of a class of distribution functions. The DUS transformation to any of the lifetime distributions proved to be an alternative approach to getting more flexible models without adding new parameters. As in the generalized exponential distribution, we can find the distribution of the parallel system in which we can use the DUS transformed distribution instead of the simple exponential distribution. A generalization is made by taking power to the DUS transformation, which is quite desirable since the DUS transformation of any non-monotonic failure rate model leads to new better models without increasing parameters. Three new distributions are introduced based on this generalized transformation, the PGDUS transformation, using the baseline distributions, exponential, Weibull, and Lomax. PGDUS distribution is a distribution of $max(X_1, X_2, \ldots, X_n)$, where $(X_1, X_2, \ldots, X_n)$ follows DUS-transformed distributions. The PGDUS approach is highly useful when performing reliability analyses on a parallel system whose components have DUS-transformed lifetime distributions. The new distributions are studied in detail and investigated for their properties.

The use of statistical distributions plays a significant role in solving a variety of real-life problems. The use of mixture distributions is unavoidable since, in many real-life situations, instead of using a particular model, we have to use mixture models. The importance of mixture statistical distributions in reliability analysis led us to study a statistical distribution with a bathtub-shaped failure rate function called the exponential-gamma $(3, \theta)$ distribution. The distribution is studied in detail and has several properties.

Birnbaum-Saunders distributions can be applied in a variety of contexts when fatigue failure occurs. In particular, BS distributions have been applied to failure models in random environments characterized by stationary Gaussian processes. In addition, the BS fatigue life distribution can be used to efficiently model wear-out and cumulative damage situations. The genesis of this model makes it evident that fatigue processes are best modeled by this distribution. Many different fields have used BS distributions. As an example, in the earth sciences, particularly rain precipitation; in acceptance sampling and quality control; in warranty claim prediction; and in medicine. Additionally, the BS distribution is closely related to the inverse Gaussian distribution, which makes it an ideal distribution for use in actuarial science, demography, agriculture, economics, finance, toxicology, hydrology, environmental sciences, and wind energy. Further generalizations to the univariate BS distribution are considered. ν-BS distribution is one of the generalizations of BS distribution. But the estimation procedures for the ν-BS distribution were not available in the literature. In order to examine the usefulness of the ν-BS distribution, a detailed study is required.

Among the univariate, bivariate, and multivariate cases of the ν-BS distribution, the univariate case is discussed in detail. An in-depth study is conducted on some of their properties. By applying the maximum likelihood principle, point estimates for the univariate ν-BS distribution are obtained. Asymptotic CIs are calculated using the observed information matrix to obtain interval estimates. A comprehensive simulation study was conducted to examine the validity of the model. A comparison of the $\nu-$BS distribution using three real data sets is presented to demonstrate that the ν-BS distribution consistently provides better modeling than the BS distribution.

Statistical analysis of the SS reliability of a component consists of analyzing how the strength of the component and the stresses placed upon it interact.It is pertinent to consider the stress conditions of the environment in which it operates when determining an item's dependability or feasibility. Therefore, uncertainty regarding the actual level of environmental stress should be considered random. Stress and strength are both treated as random variables in the stress-strength model. Based on the simple stress-strength model, X represents the stress the operating environment places on the unit, and Y represents the unit's strength. The strength of a unit is greater than the stress, so it can perform the intended function. A unit's reliability can be defined as the probability that it will be able to withstand a specified level of stress. In reliability engineering and survival analysis, this model has been found to be of increasing use.

In the development of SS reliability models with single components that follow well-known lifetime distributions, several well-developed estimation techniques have appeared in literature. The EGD is a novel model, a mixture of exponential and gamma distributions. The SS reliability for independent stress and strength random variables that follow the EGD distribution under type-II censoring is discussed in detail in a single-component SS model. The MLE of SS reliability is obtained, and simulations have been done extensively. A comparison is made between the EGD model and the Lindley model based on real-life datasets. A good fit was found for the EGD model, and it can be used when analyzing SS reliability.

For high-reliability products or materials, a prolonged testing period is usually required. The use of ALTs can expedite the testing process. In ALT testing, products are subjected to harsher conditions than they would be under normal use conditions, which reduces their life expectancy. For a variety of reasons, including operational failures, device malfunctions, expense, and time constraints, ALTs may contain censored data.

A simple step stress life testing model with type-II Gumbel lifetime distribution is introduced. A flexible failure rate-based SSALT model is considered based on type-II censoring in this model. Under the concept of a failure rate-based model, point estimates of parameters are described by using the maximum likelihood method. The interval estimation is also derived.

All the research outputs would have given new insights to the reliability and survival analysis researchers.

7.2 Future Directions

Some of the research works that can be addressed in future is given below:

- New distributions using PGDUS transformation can be introduced based on other existing distributions as baseline distributions like inverse Kumaraswamy, inverse Weibull, KM transformation distributions etc.

- Stress-strength reliability estimation of the proposed PGDUS transformed distributions can be addressed.

- Bivariate and multivariate extensions to the proposed BS distribution can be explored in detail.

- Regression models and diagnostics can be developed based on the ν-Birnbaum-Saunders distribution in both uncensored and censored data.

- Bayesian approach to the stress-strength model given can be studied.

- The SSALT models can be designed that utilize type-II Gumbel distributions under censoring schemes such as type-I censoring, hybrid censoring, progressive censoring, etc., using different models such as cumulative exposure model, proportional hazard model, Khamis-Higgins model, etc.

- Inference for the SSALT model in the presence of competing risk model can be introduced.

List of Publications

1. Thomas, B., and Chacko, V. M. (2023). Power Generalized DUS Transformation in Weibull and Lomax Distributions. *Reliability: Theory & Applications*, 18(1 (72)), 368-384.

2. Thomas, B., and Chacko, V. M. (2022). Power generalized DUS Transformation of Exponential Distribution. *International Journal of Statistics and Reliability Engineering*, 9(1), 150-157.

3. Thomas, B., and Chacko, V. M. (2020). Exponential-Gamma $(3, \theta)$ Distribution and its Applications. *Reliability: Theory & Applications*, 15(3(58)), 49-61.

4. Chacko V. M. and Beenu Thomas (2019), On a Bathtub Shaped Failure Rate Model, published in proceedings of National Seminar on Statistical Approaches in Data Science, ISBN 978-81-935819-2-6.

5. Chacko, V. M., Deepthi, K. S., Thomas, B., and Rajitha, C. (2018). Weibull-Lindley Distribution: A Bathtub Shaped Failure Rate Model. *Reliability: Theory & Applications*, 13(4 (51)), 9-20.

6. Chacko, V. M., Thomas, B., and Deepthi, K. S. (2017). A 'One Parameter' Bathtub Shaped Failure Rate Distribution. *Reliability: Theory & Applications*, 12(3 (46)), 38-43.

List of Presentations

1. Presented the paper entitled "A New Generalization of the DUS Transformation with Applications to the Weibull and Lomax Distributions" in the National Seminar on Stochastic Modelling organized by the Department of Statistics, St. Thomas College (Autonomous), Thrissur, during 22-23 February 2023.

2. Presented the paper entitled "Power Generalized DUS Transformation in Weibull and Lomax Distributions" at the International Conference on Statistical Sciences & Stochastic Modelling (ICSSSM) organized by the Department of Statistics, University of Calicut, during 16-17 February 2023.

3. Presented the paper entitled "Mixture Distribution of Exponential and Gamma Distributions and its Applications" in a National Seminar on Recent Trends in Statistical Sciences (RTSS) organized by the Department of Statistics, University of Kerala, Thiruvananthapuram, during March 7-9, 2019.

4. Presented the paper entitled "Mixed Distribution of Exponential and Gamma" in a National Seminar organized by the Department of Statistics, St. Thomas' College, Thrissur, during February 6-7, 2019.

5. Presented the paper entitled "Exponential - Gamma Distribution" at Two Day International Conference on Mathematics in collaboration with the International Multidisciplinary Research Foundation (IMRF) organized by the

Department of Mathematics, St. Thomas' College (Autonomous), Thrissur during 29-30 June 2018.

6. Presented the paper entitled "A new bathtub-shaped failure rate distribution" at the National Seminar on Innovative Approaches in Statistics organized by the Department of Statistics, St. Thomas' College (Autonomous), Thrissur, during February 15-17, 2018.

7. Presented the paper entitled "A 'One Parameter' Bathtub Shaped Failure Rate Distribution" at the International Conference on IPECS (2018) organized by the Department of Statistics, Pondicherry University, during January 29-31, 2018.

Bibliography

Aarset, M. V. (1987). How to identify a bathtub hazard rate. *IEEE Transactions on Reliability*, *36*(1), 106–108.

Abdel-Ghaly, A., Aly, H., & Abdel-Rahman, E. (2016a). Adapting the extended Neyman's smooth test to be used in accelerated failure time models. *Journal of Data Science*, *14*(2), 271–293.

Abdel Ghaly, A. A., Aly, H. M., & Salah, R. N. (2016b). Different estimation methods for constant stress accelerated life test under the family of the exponentiated distributions. *Quality and Reliability Engineering International*, *32*(3), 1095–1108.

Abdel-Hamid, A. H., & Al-Hussaini, E. K. (2011). Inference for a progressive stress model from Weibull distribution under progressive type-II censoring. *Journal of Computational and Applied Mathematics*, *235*(17), 5259–5271.

Abravesh, A., Ganji, M., & Mostafaiy, B. (2019). Classical and bayesian estimation of stress-strength reliability in type II censored Pareto distributions. *Communications in Statistics-Simulation and Computation*, *48*(8), 2333–2358.

Adamidis, K., & Loukas, S. (1998). A lifetime distribution with decreasing failure rate. *Statistics and Probability Letters*, *39*(1), 35–42.

Ahmed, E. A. (2014). Bayesian estimation based on progressive type-II censoring from two-parameter bathtub-shaped lifetime model: an Markov chain Monte Carlo approach. *Journal of Applied Statistics*, *41*(4), 752–768.

Ahmed, S. E., Castro-Kuriss, C., Flores, E., Leiva, V., & Sanhueza, A. (2010). A truncated version of the Birnbaum-Saunders distribution with an application in financial risk. *Pakisthan Journal of Statistics*, *26*, 293-311.

Ahsan, S., Lemma, T. A., & Gebremariam, M. A. (2020). Reliability analysis of gas turbine engine by means of bathtub-shaped failure rate distribution. *Process Safety Progress*, *39*, e12115.

Akman, O., Sansgiry, P., & Minnotte, M. (1999). *On the estimation of reliability based on mixture of inverse Gaussian distribution applied statistical science IV*. Nova Science Publishers, Hauppauge.

Al Abbasi, J. N., Khaleel, M. A., Abdal-hammed, M. K., Loh, Y. F., & Ozel, G. (2019). A new uniform distribution with bathtub-shaped failure rate with simulation and application. *Mathematical Sciences*, *13*, 105–114.

Alamri, O. A., Abd El-Raouf, M., Ismail, E. A., Almaspoor, Z., Alsaedi, B. S., Khosa, S. K., & Yusuf, M. (2021). Estimate stress-strength reliability model using Rayleigh and half-normal distribution. *Computational Intelligence and Neuroscience*, *2021*.

AL-Hussaini, E. K., Abdel-Hamid, A. H., & Hashem, A. F. (2015). One-sample bayesian prediction intervals based on progressively type-II censored data from the half-logistic distribution under progressive stress model. *Metrika*, *78*, 771–783.

Al-Hussaini, E. K., & Sultan, K. S. (2001). Ch. 5. reliability and hazard based on finite mixture models. *Handbook of Statistics*, *20*, 139–183.

Al-Mutairi, D., Ghitany, M., & Kundu, D. (2013). Inferences on stress-strength reliability from Lindley distributions. *Communications in Statistics-Theory and Methods*, *42*(8), 1443–1463.

Aly, H. M., & Bleed, S. O. (2013). Bayesian estimation for the generalized logistic distribution type-II censored accelerated life testing. *International Journal of Contemporary Mathematical Sciences*, *8*(20), 969–986.

Anakha, K. K., & Chacko, V. M. (2021). Dus-Kumaraswamy distribution: A bathtub shaped failure rate model. *International Journal of Statistics and Reliability Engineering*, *8*(3), 359–367.

Andrews, D., & Herzberg, A. (2012). Data: a collection of problems from many fields for the student and research worker.

Bagdonavičius, V. (1978). Testing hypothesis of the linear accumulation of damages. *Probability Theory and its Applications*, *23*, 403–408.

Bai, D. S., Kim, M., & Lee, S. (1989). Optimum simple step-stress accelerated life tests with censoring. *IEEE Transactions on Reliability*, *38*(5), 528–532.

Bai, X., Shi, Y., Liu, Y., & Liu, B. (2018). Reliability estimation of multicomponent stress–strength model based on copula function under progressively hybrid censoring. *Journal of Computational and Applied Mathematics*, *344*, 100–114.

Bakouch, H. S., Jazi, M. A., Nadarajah, S., Dolati, A., & Roozegar, R. (2014). A lifetime model with increasing failure rate. *Applied Mathematical Modelling*, *38*(23), 5392–5406.

Balakrishnan, N. (2009). A synthesis of exact inferential results for exponential step-stress models and associated optimal accelerated life-tests. *Metrika*, *69*(2-3), 351.

Balakrishnan, N., Gupta, R. C., Kundu, D., Leiva, V., & Sanhueza, A. (2011). On some mixture models based on the Birnbaum–Saunders distribution and associated inference. *Journal of Statistical Planning and Inference*, *141*(7), 2175–2190.

Balakrishnan, N., & Han, D. (2008). Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under type-II censoring. *Journal of Statistical Planning and Inference*, *138*(12), 4172–4186.

Balakrishnan, N., & Kundu, D. (2019). Birnbaum-Saunders distribution: A review of models, analysis, and applications. *Applied Stochastic Models in Business and Industry*, *35*(1), 4–49.

Balakrishnan, N., Leiva, V., Sanhueza, A., & Vilca, F. (2009). Estimation in the Birnbaum-Saunders distribution based on scale-mixture of normals and the EM-algorithm. *SORT-Statistics and Operations Research Transactions*, 171–192.

Barlow, R. E., & Proschan, F. (1975). *Statistical theory of reliability and life testing: probability models* (Tech. Rep.). Florida State Univ Tallahassee.

Basak, I., & Balakrishnan, N. (2018). A note on the prediction of censored exponential lifetimes in a simple step-stress model with type-II censoring. *Calcutta Statistical Association Bulletin*, *70*(1), 57–73.

Benkhelifa, L. (2021). The Weibull Birnbaum-Saunders distribution and its applications. *Statistics, Optimization and Information Computing*, *9*(1), 61–81.

Bessler, S., Chernoff, H., & Marshall, A. W. (1962). An optimal sequential accelerated life test. *Technometrics*, *4*(3), 367–379.

Bhattacharyya, G., & Soejoeti, Z. (1981). Asymptotic normality and efficiency of modified least squares estimators in some accelerated life test models. *Sankhyā: The Indian Journal of Statistics, Series B*, 18–39.

Bhattacharyya, G. K., & Fries, A. (1982). Inverse Gaussian regression and accelerated life tests. *Lecture Notes-Monograph Series*, *2*, 101–117.

Bhattacharyya, G. K., & Soejoeti, Z. (1989). A tampered failure rate model for step-stress accelerated life test. *Communications in Statistics-Theory and*

Methods, *18*(5), 1627–1643.

Birnbaum, Z., & McCarty, R. (1958). A distribution-free upper confidence bound for Pr[Y < X], based on independent samples of X and Y. *The Annals of Mathematical Statistics*, 558–562.

Birnbaum, Z., et al. (1956). On a use of the Mann-Whitney statistic. In *Proceedings of the third berkeley symposium on mathematical statistics and probability* (Vol. 1, pp. 13–17).

Birnbaum, Z., & Saunders, S. C. (1968). A probabilistic interpretation of miner's rule. *SIAM Journal on Applied Mathematics*, *16*(3), 637–652.

Birnbaum, Z. W., & Saunders, S. C. (1969a). A new family of life distributions. *Journal of Applied Probability*, *6*(2), 319–327.

Birnbaum, Z. W., & Saunders, S. C. (1969b). Estimation for a family of life distributions with applications to fatigue. *Journal of Applied Probability*, *6*(2), 328–347.

Block, H. W., Li, Y., Savits, T. H., & Wang, J. (2008). Continuous mixtures with bathtub-shaped failure rates. *Journal of Applied Probability*, *45*(1), 260–270.

Byrnes, J. M., Lin, Y.-J., Tsai, T.-R., & Lio, Y. (2019). Bayesian inference of δ= P (X< Y) for Burr type XII distribution based on progressively first failure-censored samples. *Mathematics*, *7*(9), 794.

Cancho, V. G., Louzada-Neto, F., & Barriga, G. D. (2011). The Poisson-exponential lifetime distribution. *Computational Statistics and Data Analysis*, *55*(1), 677–686.

Castillo, G.-H. W., N O, & Bolfarine, H. (2011). Epsilon Birnbaum-Saunders distribution family: properties and inference. *Statistical Papers*, *52*, 871–883.

Chacko, V. (2016). X-exponential bathtub failure rate model. *Reliability: Theory and Applications*, *11*(4 (43)), 55–65.

Chacko, V. M., Mariya, J. P. V., & Paul, D. (2015). p-Birnbaum Saunders distribution: Applications to reliability and electronic banking habits. *Reliability: Theory and Applications*, *10*(1(36)), 70–77.

Chandra, N., Khan, M. A., & Gopal, G. (2017). Optimum quadratic step-stress accelerated life test plan for Weibull distribution under type-I censoring. *International Journal of System Assurance Engineering and Management*, *8*, 585–591.

Chen, S., & Gui, W. (2020). Statistical analysis of a lifetime distribution with a bathtub-shaped failure rate function under adaptive progressive type-II censoring. *Mathematics*, *8*(5), 670.

Chen, Z. (2000). A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. *Statistics and Probability Letters*, *49*(2), 155–161.

Chernoff, H. (1962). Optimal accelerated life designs for estimation. *Technometrics*, *4*(3), 381–408.

Choulakian, V., & Stephens, M. A. (2001). Goodness-of-fit tests for the generalized Pareto distribution. *Technometrics*, *43*(4), 478–484.

Church, J. D., & Harris, B. (1970). The estimation of reliability from stress-strength relationships. *Technometrics*, *12*(1), 49–54.

Cordeiro, G. M., Afify, A. Z., Yousof, H. M., Cakmakyapan, S., & Ozel, G. (2018). The Lindley Weibull distribution: properties and applications. *Anais da Academia Brasileira de Ciências*, *90*, 2579–2598.

Cordeiro, G. M., & Lemonte, A. J. (2011). The β-Birnbaum–Saunders distribution: An improved distribution for fatigue life modeling. *Computational Statistics and Data Analysis*, *55*(3), 1445–1461.

Cordeiro, G. M., Ortega, E. M., & da Cunha, D. C. C. (2013). The exponentiated generalized class of distributions. *Journal of Data Science*, *11*(1), 1–27.

Deepthi, K., & Chacko, V. (2020a). An upside-down bathtub-shaped failure rate model using a dus transformation of Lomax distribution. In *Stochastic models in reliability engineering* (pp. 81–100). CRC Press.

Deepthi, K. S., & Chacko, V. M. (2020b). Reliability estimation of stress-strength model using three parameter generalized Lindley distribution. *Advances and Applications in Statistics*, *65*(01), 69-89.

Desmond, A. (1985). Stochastic models of failure in random environments. *Canadian Journal of Statistics*, *13*(3), 171–183.

Dey, S., Ghosh, I., & Kumar, D. (2019). Alpha-power transformed Lindley distribution: properties and associated inference with application to earthquake data. *Annals of Data Science*, *6*, 623–650.

Dey, S., Raheem, E., & Mukherjee, S. (2017). Statistical properties and different methods of estimation of transmuted Rayleigh distribution. *Revista Colombiana* *de Estadística*, *40*(1), 165–203.

Díaz-García, J. A., & Leiva-Sánchez, V. (2005). A new family of life distributions based on the elliptically contoured distributions. *Journal of Statistical Planning and Inference*, *128*(2), 445–457.

Dutta, S., Sultana, F., & Kayal, S. (2023). Statistical inference for Gumbel type-II distribution under simple step-stress life test using type-II censoring. *Iranian Journal of Science*, *47*(1), 155–173.

El-Din, M. M., Abu-Youssef, S., Ali, N. S., & Abd El-Raheem, A. (2016). Parametric inference on step-stress accelerated life testing for the extension of exponential distribution under progressive type-II censoring. *Communications for Statistical Applications and Methods*, *23*(4), 269–285.

Escobar, L. A., & Meeker, W. Q. (2006). A review of accelerated test models. *Statistical Science*, 552–577.

Fierro, R., Leiva, V., Ruggeri, F., & Sanhueza, A. (2013). On a Birnbaum–Saunders distribution arising from a non-homogeneous Poisson process. *Statistics and Probability Letters*, *83*(4), 1233–1239.

Gauthami, P., & Chacko, V. (2021). Dus transformation of inverse Weibull distribution: an upside-down failure rate model. *Reliability: Theory and Applications*, *16*(2 (62)), 58–71.

Ghitany, M., Al-Mutairi, D. K., & Aboukhamseen, S. (2015). Estimation of the reliability of a stress-strength system from power Lindley distributions. *Communications in Statistics-Simulation and Computation*, *44*(1), 118–136.

Glaser, R. E. (1980). Bathtub and related failure rate characterizations. *Journal of the American Statistical Association*, *75*(371), 667–672.

Goel, P. K. (1972). Some estimation problems in the study of tampered random variables. *Technical Report no.50, Department of Statistics, Carnegie-Mellon University, Pittsburgh, Pennsylvania*, *1972*.

Gómez, O.-P. J. F., Héctor W, & Bolfarine, H. (2009). An extension of the generalized Birnbaum–Saunders distribution. *Statistics and Probability Letters*, *79*(3), 331–338.

Gupta, R. C., & Brown, N. (2001). Reliability studies of the skew-normal distribution and its application to a strength-stress model. *Communications in Statistics-Theory and Methods*, *30*(11), 2427–2445.

Gupta, R. D., & Kundu, D. (1999). Theory & methods: Generalized exponential distributions. *Australian & New Zealand Journal of Statistics*, *41*(2), 173–188.

Gupta, R. D., & Kundu, D. (2001). Exponentiated exponential family: An alternative to gamma and Weibull distributions. *Biometrical Journal: Journal of Mathematical Methods in Biosciences*, *43*(1), 117–130.

Gupta, R. D., & Kundu, D. (2009). A new class of weighted exponential distributions. *Statistics*, *43*(6), 621–634.

Hakamipour, N., & Rezaei, S. (2017). Optimizing the simple step stress accelerated life test with type I censored Frechet data. *REVSTAT-Statistical Journal*, *15*(1), 1–23.

Hamada, M. (2015). Bayesian analysis of step-stress accelerated life tests and its use in planning. *Quality Engineering*, *27*(3), 276–282.

Han, D., & Balakrishnan, N. (2010). Inference for a simple step-stress model with competing risks for failure from the exponential distribution under time constraint. *Computational Statistics and Data Analysis*, *54*(9), 2066–2081.

Han, D., & Kundu, D. (2014). Inference for a step-stress model with competing risks for failure from the generalized exponential distribution under type-I censoring. *IEEE Transactions on Reliability*, *64*(1), 31–43.

Hanagal, D. D. (1997). Note on estimation of reliability under bivariate Pareto stress-strength model. *Statistical Papers*, *38*, 453–459.

Hjorth, U. (1980). A reliability distribution with increasing, decreasing, constant and bathtub-shaped failure rates. *Technometrics*, *22*(1), 99–107.

Hu, C.-H., Plante, R. D., & Tang, J. (2012). Step-stress accelerated life tests: a proportional hazards–based non-parametric model. *IIE Transactions*, *44*(9), 754–764.

Jha, M. K., Dey, S., Alotaibi, R. M., & Tripathi, Y. M. (2020). Reliability estimation of a multicomponent stress-strength model for unit Gompertz distribution under progressive type II censoring. *Quality and Reliability Engineering International*, *36*(3), 965–987.

Jose, J., Xavier, T., & Drisya, M. (2019). Estimation of stress-strength reliability using Kumaraswamy half-logistic distribution. *Journal of Probability and Statistical Science*, *17*(2), 141–154.

Jose, J. K., Drisya, M., & Manoharan, M. (2022). Estimation of stress-strength reliability using discrete phase type distribution. *Communications in Statistics-Theory and Methods*, *51*(2), 368–386.

Kannan, N., & Kundu, D. (2020). Simple step-stress models with a cure fraction. *Brazilian Journal of Probability and Statistics*, *34*(1), 2-17.

Kao, J. H. (1959). A graphical estimation of mixed Weibull parameters in life-testing of electron tubes. *Technometrics*, *1*(4), 389–407.

Karimi Ezmareh, Z., Yari, G., et al. (2022). Statistical inference of stress-strength reliability of Gompertz distribution under type II censoring. *Advances in Mathematical Physics*, *2022*.

Kateri, M., & Kamps, U. (2015). Inference in step-stress models based on failure rates. *Statistical Papers*, *56*, 639–660.

Kateri, M., & Kamps, U. (2017). Hazard rate modeling of step-stress experiments. *Annual Review of Statistics and Its Application*, *4*, 147–168.

Kateri, M., Kamps, U., & Balakrishnan, N. (2011). Optimal allocation of change points in simple step-stress experiments under type-II censoring. *Computational Statistics and Data Analysis*, *55*(1), 236–247.

Kavya, P., & Manoharan, M. (2020). On a generalized lifetime model using DUS transformation. *Applied Probability and Stochastic Processes*, 281–291.

Kavya, P., & Manoharan, M. (2021). Some parsimonious models for lifetimes and applications. *Journal of Statistical Computation and Simulation*, *91*(18), 3693–3708.

Kayal, T., Tripathi, Y. M., Singh, D. P., & Rastogi, M. K. (2017). Estimation and prediction for Chen distribution with bathtub shape under progressive censoring. *Journal of Statistical Computation and Simulation*, *87*(2), 348–366.

Khamis, I. H., & Higgins, J. J. (1998). A new model for step-stress testing. *IEEE Transactions on Reliability*, *47*(2), 131–134.

Kim, C., & Bai, D.-S. (2002). Analyses of accelerated life test data under two failure modes. *International Journal of Reliability, Quality and Safety Engineering*, *9*(02), 111–125.

Klein, J. P., & Moeschberger, M. L. (2003). *Survival analysis: techniques for censored and truncated data* (Vol. 1230). Springer.

Kolmogorov, A. N. (1933). Grundbegriffe der wahrscheinlichkeitreichnung. *Ergebnisse der Mathematik*.

Komori, Y. (2006). Properties of the Weibull cumulative exposure model. *Journal of Applied Statistics*, *33*(1), 17–34.

Kotz, S., Leiva, V., & Sanhueza, A. (2010). Two new mixture models related to the inverse Gaussian distribution. *Methodology and Computing in Applied Probability*, *12*, 199–212.

Kotz, S., Pensky, M., et al. (2003). *The stress-strength model and its generalizations: theory and applications*. World Scientific.

Koutras, V. P. (2011). Two-level software rejuvenation model with increasing failure rate degradation. In *Dependable computer systems* (pp. 101–115).

Krishna, H., Dube, M., & Garg, R. (2019). Estimation of stress strength reliability of inverse Weibull distribution under progressive first failure censoring. *Austrian Journal of Statistics*, *48*(1), 14–37.

Krishnamoorthy, K., Mukherjee, S., Guo, H., et al. (2007). Inference on reliability in two-parameter exponential stress-strength model. *Metrika*, *65*(3), 261–273.

Kumar, D., Nassar, M., & Dey, S. (2022). Constant stress accelerated life test: Different methods of estimation under the exponentiated power Lindley distribution. *Strength of Materials*, *54*(3), 444–461.

Kumar, D., Singh, U., & Singh, S. K. (2015). A method of proposing new distribution and its application to bladder cancer patients data. *Journal of Statistics Applications and Probability Letters*, *2*(3), 235–245.

Kumar Mahto, A., Dey, S., & Mani Tripathi, Y. (2020). Statistical inference on progressive-stress accelerated life testing for the logistic exponential distribution under progressive type-II censoring. *Quality and Reliability Engineering International*, *36*(1), 112–124.

Kundu, D., Balakrishnan, N., & Jamalizadeh, A. (2010). Bivariate Birnbaum–Saunders distribution and associated inference. *Journal of Multivariate Analysis*, *101*(1), 113–125.

Kundu, D., Balakrishnan, N., & Jamalizadeh, A. (2013). Generalized multivariate Birnbaum–Saunders distributions and related inferential issues. *Journal of Multivariate Analysis*, *116*, 230–244.

Kundu, D., & Gupta, R. D. (2005). Estimation of P[Y< X] for generalized exponential distribution. *Metrika*, *61*(3), 291–308.

Kundu, D., & Gupta, R. D. (2006). Estimation of P [Y< X] for Weibull distributions.

IEEE Transactions on Reliability, *55*(2), 270–280.

Kundu, D., Kannan, N., & Balakrishnan, N. (2008). On the hazard function of Birnbaum–Saunders distribution and associated inference. *Computational Statistics and Data Analysis*, *52*(5), 2692–2702.

Kundu, D., & Raqab, M. Z. (2009). Estimation of R= P (Y< X) for three-parameter Weibull distribution. *Statistics and Probability Letters*, *79*(17), 1839–1846.

Kuş, C. (2007). A new lifetime distribution. *Computational Statistics and Data Analysis*, *51*(9), 4497–4509.

Lai, C., Xie, M., & Murthy, D. (2001). Ch. 3. bathtub-shaped failure rate life distributions. *Handbook of Statistics*, *20*, 69–104.

Lai, C. D., & Balakrishnan, N. (2009). *Continuous bivariate distributions*. Springer.

Lawless, J. F. (1982). *Statistical models and methods for lifetime data*. John Wiley & Sons.

Lee, E. T., & Wang, J. (2003). *Statistical methods for survival data analysis* (Vol. 476). John Wiley & Sons.

Leiva, V. (2015). The Birnbaum-Saunders distribution.

Leiva, V., Athayde, E., Azevedo, C., & Marchant, C. (2011). Modeling wind energy flux by a Birnbaum–Saunders distribution with an unknown shift parameter. *Journal of Applied Statistics*, *38*(12), 2819–2838.

Leiva, V., Marco, R., Balakrishnan, N., & Sanhueza, A. (2008). Lifetime analysis based on the generalized Birnbaum-Saunders distribution. *Computational Statistics and Data Analysis*, *52*(4), 2079–2097.

Leiva, V., Sanhueza, A., & Angulo, J. M. (2009). A length-biased version of the Birnbaum-Saunders distribution with application in water quality. *Stochastic Environmental Research and Risk Assessment*, *23*, 299–307.

Leiva, V., Santos-Neto, M., Cysneiros, F. J. A., & Barros, M. (2016). A methodology for stochastic inventory models based on a zero-adjusted Birnbaum-Saunders distribution. *Applied Stochastic Models in Business and Industry*, *32*(1), 74–89.

Lemonte, A. J. (2013a). A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. *Computational Statistics and Data Analysis*, *62*, 149–170.

Lemonte, A. J. (2013b). A new extension of the Birnbaum–Saunders distribution.

Brazilian Journal of Probability and Statistics, *27*, 133-149.

León, R. V., Ramachandran, R., Ashby, A. J., & Thyagarajan, J. (2007). Bayesian modeling of accelerated life tests with random effects. *Journal of quality technology*, *39*(1), 3–16.

Lomax, K. S. (1954). Business failures: Another example of the analysis of failure data. *Journal of the American Statistical Association*, *49*(268), 847–852.

Mann, N. R., Schafer, R. E., & Singpurwalla, N. D. (1974). Methods for statistical analysis of reliability and life data(book). *Research supported by the U. S. Air Force and Rockwell International Corp. New York, John Wiley and Sons, Inc., 1974. 573 p.*

Maurya, S. K., Kaushik, A., Singh, S. K., & Singh, U. (2017a). A new class of exponential transformed Lindley distribution and its application to yarn data. *International Journal of Statistics and Economics*, *18*(2), 135–151.

Maurya, S. K., Kaushik, A., Singh, S. K., & Singh, U. (2017b). A new class of distribution having decreasing, increasing, and bathtub-shaped failure rate. *Communications in Statistics-Theory and Methods*, *46*(20), 10359–10372.

Meeker, W. Q., & Escobar, L. A. (1998). Statistical methods for reliability data. *A. Wiley Interscience Publications*, 639.

Miller, R., & Nelson, W. (1983). Optimum simple step-stress plans for accelerated life testing. *IEEE Transactions on Reliability*, *32*(1), 59–65.

Mohie El-Din, M., Abu-Youssef, S., Ali, N. S., & Abd El-Raheem, A. (2017). Classical and bayesian inference on progressive-stress accelerated life testing for the extension of the exponential distribution under progressive type-II censoring. *Quality and Reliability Engineering International*, *33*(8), 2483–2496.

Mudholkar, G. S., & Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability*, *42*(2), 299–302.

Nadar, M., & Kızılaslan, F. (2014). Classical and bayesian estimation of P (X< Y) using upper record values from Kumaraswamy's distribution. *Statistical Papers*, *55*, 751–783.

Nadar, M., Kızılaslan, F., & Papadopoulos, A. (2014). Classical and bayesian estimation of P (Y< X) for Kumaraswamy's distribution. *Journal of Statistical Computation and Simulation*, *84*(7), 1505–1529.

Nadarajah, S., Bakouch, H. S., & Tahmasbi, R. (2011). A generalized Lindley distribution. *Sankhya B*, *73*, 331–359.

Nadarajah, S., & Gupta, A. K. (2007). The exponentiated gamma distribution with application to drought data. *Calcutta Statistical Association Bulletin*, *59*(1-2), 29–54.

Naderi, M., Mozafari, M., & Okhli, K. (2020). Finite mixture modeling via skew-Laplace Birnbaum–Saunders distribution. *Journal of Statistical Theory and Applications*, *19*(1), 49–58.

Nelson, W. (1980). Accelerated life testing-step-stress models and data analyses. *IEEE Transactions on Reliability*, *29*(2), 103–108.

Owen, W. J. (2006). A new three-parameter extension to the Birnbaum-Saunders distribution. *IEEE Transactions on Reliability*, *55*(3), 475–479.

Pal, A., Samanta, D., Mitra, S., & Kundu, D. (2021). A simple step-stress model for Lehmann family of distributions. *Advances in Statistics-Theory and Applications: Honoring the Contributions of Barry C. Arnold in Statistical Science*, 315–343.

Pal, M., Ali, M. M., & Woo, J. (2006). Exponentiated Weibull distribution. *Statistica*, *66*(2), 139–147.

Parsa, M., Borzadaran, G. M., & Roknabadi, A. R. (2014). On the change points of mean residual life and failure rate functions for some generalized gamma type distributions. *International Journal of Metrology and Quality Engineering*, *5*(1), 102.

Pascual, F. G., & Montepiedra, G. (2003). Model-robust test plans with applications in accelerated life testing. *Technometrics*, *45*(1), 47–57.

Proschan, F. (1963). Theoretical explanation of observed decreasing failure rate. *Technometrics*, *5*(3), 375–383.

Rajarshi, S., & Rajarshi, M. (1988). Bathtub distributions: A review. *Communications in Statistics-Theory and Methods*, *17*(8), 2597–2621.

Rao, G. S., Mbwambo, S., & Josephat, P. (2019). Estimation of stress–strength reliability from exponentiated inverse Rayleigh distribution. *International Journal of Reliability, Quality and Safety Engineering*, *26*(01), 1950005.

Raqab, M. Z., & Kundu, D. (2005). Comparison of different estimators of P [Y< X] for a scaled Burr type X distribution. *Communications in Statistics—Simulation* *and Computation®*, *34*(2), 465–483.

Raqab, M. Z., Madi, M. T., & Kundu, D. (2008). Estimation of P(Y< X) for the three-parameter generalized exponential distribution. *Communications in Statistics—Theory and Methods*, *37*(18), 2854–2864.

Rényi, A. (1961). On measures of entropy and information. In *Proceedings of the fourth berkeley symposium on mathematical statistics and probability, volume 1: Contributions to the theory of statistics* (Vol. 4, pp. 547–562).

Ross, S. M., Shanthikumar, J. G., & Zhu, Z. (2005). On increasing-failure-rate random variables. *Journal of Applied Probability*, *42*(3), 797–809.

Samanta, D., Koley, A., Gupta, A., & Kundu, D. (2020). Exact inference of a simple step-stress model with hybrid type-II stress changing time. *Journal of Statistical Theory and Practice*, *14*, 1–27.

Sanhueza, A., Leiva, V., & Balakrishnan, N. (2008). The generalized Birnbaum-Saunders distribution and its theory, methodology, and application. *Communications in Statistics—Theory and Methods*, *37*(5), 645–670.

Sarhan, A. M., & Kundu, D. (2009). Generalized linear failure rate distribution. *Communications in Statistics-Theory and Methods*, *38*(5), 642–660.

Saulo, H., Leão, J., & Bourguignon, M. (2012). The Kumaraswamy Birnbaum-Saunders distribution. *Journal of Statistical Theory and Practice*, *6*(4), 745–759.

Schmoyer, R. L. (1991). Nonparametric analyses for two-level single-stress accelerated life tests. *Technometrics*, *33*(2), 175–186.

Sedyakin, N. (1966). On one physical principle in reliability theory. *Technical Cybernetics*, *3*, 80–87.

Seo, K., & Pan, R. (2017). Data analysis of step-stress accelerated life tests with heterogeneous group effects. *IISE Transactions*, *49*(9), 885–898.

Sha, N., & Pan, R. (2014). Bayesian analysis for step-stress accelerated life testing using Weibull proportional hazard model. *Statistical Papers*, *55*, 715–726.

Shafiq, M., & Viertl, R. (2017). Bathtub hazard rate distributions and fuzzy life times. *Iranian Journal of Fuzzy Systems*, *14*(5), 31–41.

Sharma, V. K., Singh, S. K., Singh, U., & Agiwal, V. (2015). The inverse Lindley distribution: a stress-strength reliability model with application to head and neck cancer data. *Journal of Industrial and Production Engineering*, *32*(3), 162–173.

Shehla, R., & Khan, A. A. (2016). Reliability analysis using an exponential power model with bathtub-shaped failure rate function: a bayes study. *SpringerPlus*, *5*, 1–22.

Shrahili, M., & Kayid, M. (2022). Modeling extreme value data with an upside down bathtub-shaped failure rate model. *Open Physics*, *20*(1), 484–492.

Sindhu, T. N., Anwar, S., Hassan, M. K., Lone, S. A., Abushal, T. A., & Shafiq, A. (2023). An analysis of the new reliability model based on bathtub-shaped failure rate distribution with application to failure data. *Mathematics*, *11*(4), 842.

Sonker, P. K., Kumar, M., & Saroj, A. (2023). Stress–strength reliability models on power-Muth distribution. *International Journal of System Assurance Engineering and Management*, *14*(1), 173–195.

Surles, J., & Padgett, W. (2005). Some properties of a scaled Burr type X distribution. *Journal of Statistical Planning and Inference*, *128*(1), 271–280.

Tripathi, A., Singh, U., & Singh, S. K. (2021). Inferences for the DUS-exponential distribution based on upper record values. *Annals of Data Science*, *8*, 387–403.

Varghese, A. K., & Chacko, V. (2022). Estimation of stress-strength reliability for Akash distribution. *Reliability: Theory and Applications*, *17*(3 (69)), 52–58.

Vilca, S. A.-L. V., F, & Christakos, G. (2010). An extended Birnbaum–Saunders model and its application in the study of environmental quality in Santiago, Chile. *Stochastic Environmental Research and Risk Assessment*, *24*, 771–782.

Vilca-Labra, F., & Leiva-Sánchez, V. (2006). A new fatigue life model based on the family of skew-elliptical distributions. *Communications in Statistics—Theory and Methods*, *35*(2), 229–244.

Volodin, I. N., & Dzhungurova, O. A. (2000). On limit distributions emerging in the generalized Birnbaum-Saunders model. *Journal of Mathematical Sciences*, *99*, 1348–1366.

Wang, F. (2000). A new model with bathtub-shaped failure rate using an additive Burr XII distribution. *Reliability Engineering and System Safety*, *70*(3), 305–312.

Wang, J. (2020). Data analysis of step-stress accelerated life test with random group effects under Weibull distribution. *Mathematical Problems in Engineering*, *2020*, 1–11.

Wang, R., Sha, N., Gu, B., & Xu, X. (2014). Statistical analysis of a Weibull extension with bathtub-shaped failure rate function. *Advances in Statistics*, *2014*.

Wang, X., Yu, C., & Li, Y. (2015). A new finite interval lifetime distribution model for fitting bathtub-shaped failure rate curve. *Mathematical Problems in Engineering*, *2015*.

Wu, S.-J., Lin, Y.-P., & Chen, S.-T. (2008). Optimal step-stress test under type I progressive group-censoring with random removals. *Journal of Statistical Planning and Inference*, *138*(4), 817–826.

Xavier, T., & Jose, J. K. (2021a). A study of stress-strength reliability using a generalization of power transformed half-logistic distribution. *Communications in Statistics-Theory and Methods*, *50*(18), 4335–4351.

Xavier, T., & Jose, J. K. (2021b). Estimation of reliability in a multicomponent stress–strength model based on power transformed half-logistic distribution. *International Journal of Reliability, Quality and Safety Engineering*, *28*(02), 2150009.

Xie, M., & Lai, C. D. (1996). Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function. *Reliability Engineering and System Safety*, *52*(1), 87–93.

Xie, M., Tang, Y., & Goh, T. N. (2002). A modified Weibull extension with bathtub-shaped failure rate function. *Reliability Engineering and System Safety*, *76*(3), 279–285.

Yin, X.-K., & Sheng, B.-Z. (1987). Some aspects of accelerated life testing by progressive stress. *IEEE Transactions on Reliability*, *36*(1), 150–155.

Zhang, J.-P., & Wang, R.-T. (2009). Reliability life prediction of VFD by constant temperature stress accelerated life tests and maximum likelihood estimation. *Journal of Testing and Evaluation*, *37*(4), 316–320.

Zhang, T., Dwight, R., & El-Akruti, K. (2013). On a Weibull related distribution model with decreasing, increasing and upside-down bathtub-shaped failure rate. In *2013 proceedings annual reliability and maintainability symposium (rams)* (pp. 1–6).

Zhu, X., Balakrishnan, N., & Zhou, Y. (2020). Exact likelihood-ratio tests for a simple step-stress cumulative exposure model with censored exponential data. *Methodology and Computing in Applied Probability*, *22*, 497–509.