CHAPTER 2

A New Generalization to the DUS Transformation and its Applications

2.1 Introduction

Modeling and analysis of lifetime distributions have been extensively used in many fields of science, like engineering, medicine, survival analysis, and biostatistics. Fitting appropriate distributions is essential for proper data analysis. A search for distributions with a better fit is quite essential for data analysis in statistics and reliability engineering. With application to survival data analysis, Kumar et al. (2015) proposed a method called DUS transformation, which has received attention from many engineers and researchers in recent years. In terms of computation and interpretation, this transformation produces a parsimonious result since it does not include any new parameters other than those involved in the baseline distribution.

In the case where $F(x)$ is the CDF of the baseline distribution, the CDF of the DUS transformed distribution is as follows:

$$G(x) = \frac{1}{e-1}[e^{F(x)} - 1].$$

Maurya et al. (2017a) introduced the DUS transformation of the Lindley distribution. Tripathi et al. (2019) studied the DUS transformation of an exponential distribution and its inference based on the upper record values. Recent studies using the DUS transformation can be seen in the works of Deepthi and Chacko (2020a), Kavya and Manoharan (2020), Anakha and Chacko (2021), and Gauthami and Chacko (2021).

In this chapter, a new class of distribution is introduced using an exponentiated generalization of the DUS transformation, called the power generalized DUS (PGDUS) transformation. When we consider a parallel system, we have to apply power transformations to the distribution of components to get the system's distribution. Generalized exponential distribution was introduced by Gupta and Kundu (1999) which is the distribution of a parallel system when components are distributed exponentially. But when a researcher assumes an exponential distribution for its lifetime, only jerking, overvoltage, or any such random shocks are the cause of failure. It is a limitation. Why don't we go for any other lifetime distribution if the cause of failure is degradation? DUS transformation proved the advantage of getting an accurate model for the given data using baseline distributions like Weibull, Lomax, etc. Nevertheless, the question remains: how would the parallel system be distributed when components are distributed based on the DUS transformation of some baseline models? If we use exponential, Weibull, and Lomax distributions as baselines, what would be their distributional properties? An attempt to investigate the applicability of the exponentiated generalization of DUS transformation of some baseline models is addressed in this chapter. The use of other distributions as baseline distributions can be addressed by researchers.

This generalization improves the flexibility and accuracy of the model. The new PGDUS transformed distribution can be obtained as follows: Let X be a random variable with a baseline CDF $F(x)$ and the corresponding PDF $f(x)$. Then, the CDF of the PGDUS distribution is defined as:

$$G(x) = \left(\frac{e^{F(x)} - 1}{e - 1}\right)^{\theta}, \theta > 0, x > 0. \tag{2.1.1}$$

and the corresponding PDF is,

$$g(x) = \frac{\theta}{(e-1)^{\theta}}(e^{F(x)} - 1)^{\theta-1} e^{F(x)} f(x), \theta > 0, x > 0. \tag{2.1.2}$$

The associated survival function is,

$$S(x) = 1 - \left(\frac{e^{F(x)} - 1}{e - 1}\right)^{\theta}, \theta > 0, x > 0.$$

The corresponding HRF is,

$$h(x) = \frac{\theta f(x) e^{F(x)} (e^{F(x)} - 1)^{\theta - 1}}{(e - 1)^{\theta} - (e^{F(x)} - 1)^{\theta}}, \theta > 0, x > 0. \tag{2.1.3}$$

The primary motivation for this research stems from the significance of Eq. (2.1.1), as it is the distribution of failures in a parallel system with θ independent components. When researchers deal with parallel systems with components distributed as DUS-transformed lifetime distributions, the PGDUS transformation proves to be an inevitable tool. So the investigation of the PGDUS transformation of various lifetime distributions is relevant in the sense of the selection of appropriate lifetime models for parallel systems. In other words, it assists researchers in determining which distribution transformations best characterize the behavior of individual components in a parallel system, which has consequences for developing reliable systems and predicting their overall performance. As a result, this work is motivated by the need to improve our understanding of how different lifetime distributions can be effectively used in modeling and optimizing parallel systems, resulting in improved decision-making and reliability in a variety of engineering and scientific applications.

The remaining sections are arranged as follows. Section 2.2 introduces the PGDUS transformation of the exponential distribution. Section 2.3 presents the PGDUS transformation of the Weibull distribution, and Section 2.4 presents the PGDUS transformation of the Lomax distribution. The summary is given in section 2.5.

2.2 PGDUS Exponential Distribution

Here, the PGDUS transformation to the exponential distribution is considered. Consider the exponential distribution with parameter λ as the baseline distribution. Invoking the PGDUS transformation given in Eq.(2.1.1), the CDF of the PGDUS transformation of an exponential (PGDUSE) distribution is obtained as

$$G(x) = \left(\frac{e^{1-e^{-\lambda x}} - 1}{e - 1}\right)^{\theta}, \lambda > 0, \theta > 0, x > 0. \tag{2.2.1}$$

and the corresponding PDF is given by,

$$g(x) = \frac{\theta\lambda e^{1-\lambda x - e^{-\lambda x}}(e^{1-e^{-\lambda x}} - 1)^{\theta-1}}{(e-1)^{\theta}}, \lambda > 0, \theta > 0, x > 0. \tag{2.2.2}$$

Then, the associated HRF is,

$$h(x) = \frac{\theta\lambda e^{1-\lambda x - e^{-\lambda x}}(e^{1-e^{-\lambda x}} - 1)^{\theta-1}}{(e-1)^{\theta} - (e^{1-e^{-\lambda x}} - 1)^{\theta}}, \lambda > 0, \theta > 0, x > 0. \tag{2.2.3}$$

A PGDUSE distribution with parameters λ and θ is denoted by $PGDUSE(\lambda, \theta)$. Figure 2.1 shows that the density function of $PGDUSE(\lambda, \theta)$ distribution is likely to be unimodal. The HRF plot for different parameter values is given in Figure 2.2.

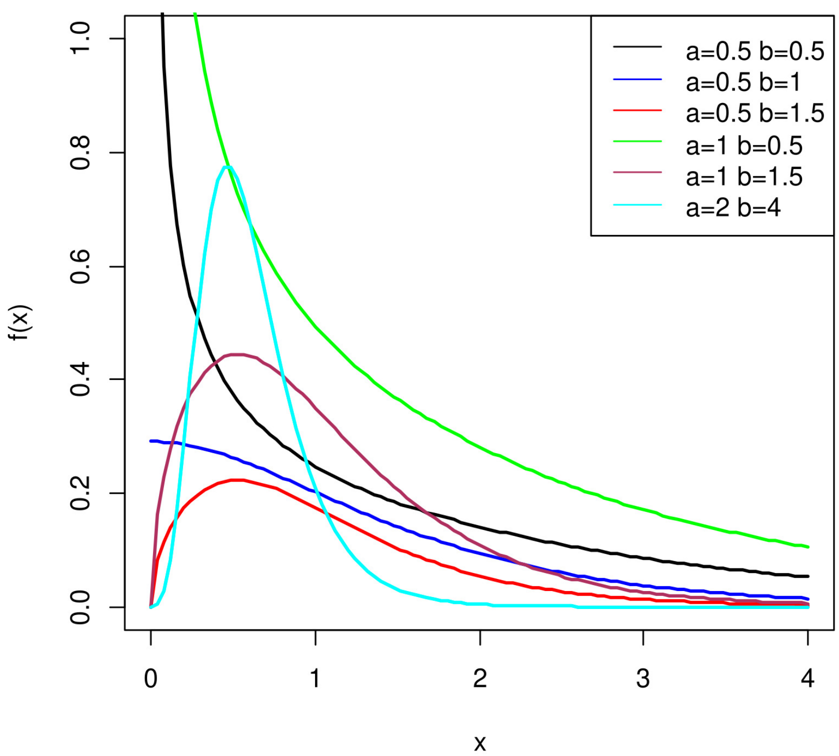

Figure 2.1: Density plot for PGDUSE

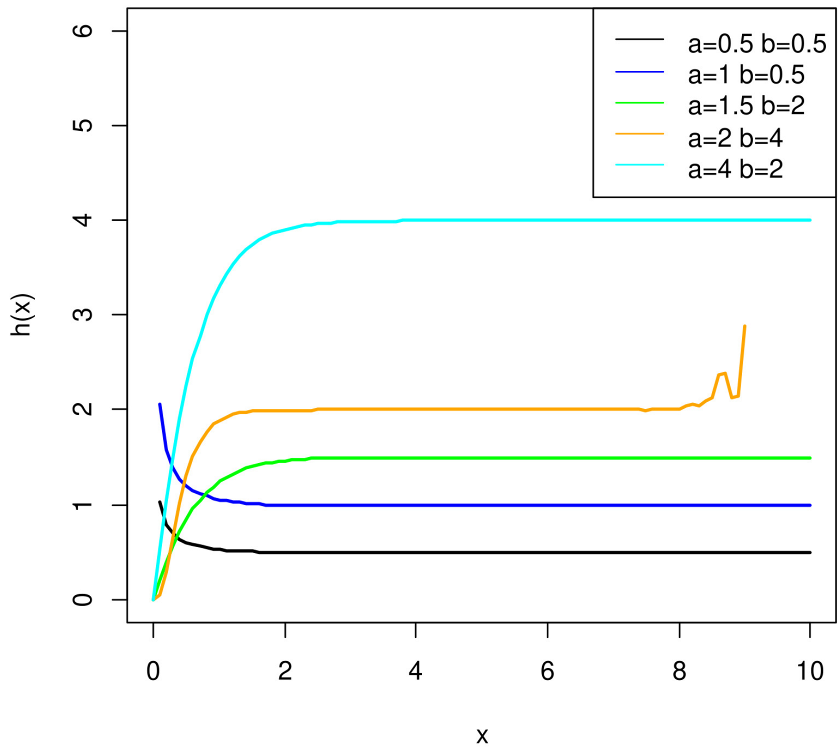

Figure 2.2: Failure rate plot for PGDUSE

2.2.1 Statistical Properties of PGDUSE Distribution

For a distribution, the statistical properties are inevitable. Here, a few statistical properties like moments, moment generating function (MGF), characteristic function (CF), cumulant generating function (CGF), quantile function (QF), order statistics, and entropy of the $PGDUSE(\lambda, \theta)$ distribution are derived.

Moments

The rth raw moment of the $PGDUSE(\lambda, \theta)$ distribution is given by

$$\mu'_r = E(X^r) \quad = \frac{\theta\lambda e}{(e-1)^{\theta}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\theta-1}{k} e^{\theta-k-1} (\theta-k)^m \frac{\Gamma(r+1)}{(\lambda+\lambda m)^{r+1}}.$$

By putting r=1, 2, 3... the raw moments can be viewed.

Moment Generating Function

The MGF of $PGDUSE(\lambda, \theta)$ distribution is given by

$$M_X(t) = \frac{\theta \lambda e}{(e-1)^{\theta}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\theta-1}{k} e^{\theta-k-1} \frac{(\theta-k)^m}{\lambda+\lambda m-t}.$$

Characteristic Function and Cumulant Generating Function

The characteristic function (CF) is given by

$$\phi_X(t) = \frac{\theta \lambda e}{(e-1)^{\theta}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\theta-1}{k} e^{\theta-k-1} \frac{(\theta-k)^m}{\lambda+\lambda m-it},$$

and the cumulant generating function (CGF) is given by

$$K_X(t) = \log\left(\frac{\theta \lambda e}{(e-1)^{\theta}}\right) + \log\left[\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\theta-1}{k} e^{\theta-k-1} \frac{(\theta-k)^m}{\lambda+\lambda m-it}\right]$$

where $i = \sqrt{-1}$ is the unit imaginary number.

Quantile Function

The qth quantile $Q(q)$ is the solution of the equation $G(Q(q)) = q$. Hence,

$$Q(q) = \frac{-1}{\lambda} \log(1 - \log(q^{\frac{1}{\theta}}(e-1)+1)).$$

The median is obtained by setting $q = 0.5$ in the above equation. Thus,

$$Median = \frac{-1}{\lambda} \log(1 - \log(0.5^{\frac{1}{\theta}}(e-1)+1)).$$

Order Statistic

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics corresponding to the random sample $X_1, X_2, \ldots, X_n$ of size n from the proposed $PGDUSE(\lambda, \theta)$ distribution. The PDF and CDF of rth order statistics of the proposed $PGDUSE(\lambda, \theta)$ distribution are given by

$$g_r(x) = \frac{n!\theta\lambda}{(r-1)!(n-r)!} \frac{e^{1-\lambda x - e^{-\lambda x}}(e^{1-e^{-\lambda x}}-1)^{\theta r-1}}{(e-1)^{2\theta}} \left[1 - \left(\frac{e^{1-e^{-\lambda x}}-1}{e-1}\right)^{\theta}\right]$$

and

$$G_r(x) = \sum_{i=1}^{n} \binom{n}{i} \left(\frac{e^{1-e^{-\lambda x}} - 1}{e-1} \right)^{\theta i} \left[1 - \left(\frac{e^{1-e^{-\lambda x}} - 1}{e-1} \right)^{\theta} \right]^{n-i}.$$

Then, the PDF and CDF of $X_{(1)}$ and $X_{(n)}$ are obtained by substituting $r = 1$ and $r = n$ respectively in $g_r(x)$ and $G_r(x)$. It is nothing but the distribution of minimum and maximum in series and parallel reliability systems, respectively.

Entropy

Entropy quantifies the measure of information or uncertainty. An important measure of entropy is Rényi entropy (1961). Rényi entropy is defined as

$$\beth_R(\delta) = \frac{1}{1-\delta} \log\left(\int g^{\delta}(x)dx \right),$$

where $\delta > 0$ and $\delta \neq 1$.

$$\int_0^{\infty} g^{\delta}(x)dx = \frac{\theta^{\delta}\lambda^{\delta}e^{\delta}}{(e-1)^{\theta\delta}} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\delta(\theta-1)}{k} (\delta\theta - k)^m e^{\delta\theta-\delta-k} \frac{1}{\lambda(\delta+m)}$$

The Rényi entropy takes the form

$$\beth_R(\delta) = \tfrac{1}{1-\delta} \log\left[\frac{\theta^{\delta}\lambda^{\delta}e^{\delta}}{(e-1)^{\theta\delta}} \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \binom{\delta(\theta-1)}{k} (\delta\theta - k)^m e^{\delta\theta-\delta-k} \frac{1}{\lambda(\delta+m)} \right]$$

$$= \tfrac{\delta}{1-\delta} \log\left[\frac{\theta\lambda e}{(e-1)^{\theta}} \right] + \tfrac{1}{1-\delta} \log\left[\sum_{k=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{k+m}}{m!} \tbinom{\delta(\theta-1)}{k} (\delta\theta - k)^m \frac{e^{\delta\theta-\delta-k}}{\lambda(\delta+m)} \right].$$

2.2.2 Estimation of PGDUSE Distribution

The estimation of parameters by the method of maximum likelihood is discussed. For this, consider a random sample of size n from $PGDUSE(\lambda, \theta)$ distribution. In this case, the likelihood function is given by,

$$L(x) = \prod_{i=1}^{n} g(x) = \prod_{i=1}^{n} \frac{\theta\lambda}{(e-1)^{\theta}} e^{1-\lambda x_i - e^{-\lambda x_i}} (e^{1-e^{-\lambda x_i}} - 1)^{\theta-1}.$$

Then the log-likelihood function becomes,

$$\log L = n\log\theta + n\log\lambda - \theta n\log(e-1) - \lambda\sum_{i=1}^{n} x_i + n - \sum_{i=1}^{n} e^{-\lambda x_i} + (\theta-1)\sum_{i=1}^{n}\log(e^{1-e^{-\lambda x_i}} - 1).$$

The maximum likelihood estimators (MLEs) are obtained by maximizing the log-likelihood concerning the unknown parameters λ and θ.

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} x_i e^{-\lambda x_i} + (\theta-1)\sum_{i=1}^{n}\frac{x_i e^{1-\lambda x_i - e^{-\lambda x_i}}}{e^{1-e^{-\lambda x_i}} - 1}.$$

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - n\log(e-1) + \sum_{i=1}^{n}\log(e^{1-e^{-\lambda x_i}} - 1).$$

These non-linear equations can be numerically solved through statistical software like R using arbitrary initial values. In the case of asymptotic normal MLEs, the confidence interval(CI)s for λ and θ are calculated by computing the observed information matrix given by

$$I = \begin{pmatrix} \frac{\partial^2 \log L}{\partial \lambda^2} & \frac{\partial^2 \log L}{\partial \lambda \partial \theta} \\ \frac{\partial^2 \log L}{\partial \theta \partial \lambda} & \frac{\partial^2 \log L}{\partial \theta^2} \end{pmatrix}$$

where

$$\frac{\partial^2 \log L}{\partial \lambda^2} = -\frac{n}{\lambda^2} - \lambda\sum_{i=1}^{n} x_i e^{-\lambda x_i} - (\theta-1)\lambda\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}((e^{\lambda x_i} - 1)e^{1-(\lambda x_i) - e^{-\lambda x_i}} - 1)}{(e^{1-e^{-\lambda x_i}} - 1)^2},$$

$$\frac{\partial^2 \log L}{\partial \theta \partial \lambda} = \frac{\partial^2 \log L}{\partial \lambda \partial \theta} = \sum_{i=1}^{n} x_i \frac{e^{1-\lambda x_i - e^{-\lambda x_i}}}{(e^{1-e^{-\lambda x_i}} - 1)},$$

and

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{n}{\theta^2}.$$

For λ and θ, the $100(1-\gamma)\%$ asymptotic CIs are as follows: $\hat{\lambda} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{11}}$ and $\hat{\theta} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{22}}$, where V_{ij} represents the (i,j)th element in the inverse of the Fisher information matrix I. The computational efficiency of this interval estimation method makes it particularly useful.

2.2.3 Simulation Study

To illustrate the accuracy of the maximum likelihood estimation procedure for *PGDUSE* distribution, a Monte Carlo simulation study is carried out using the inversion method. Samples of sizes 50, 75, 100, 500, and 1000 for the parameter combinations (0.5, 0.5), (0.5, 1.5), (1, 1.5), and (1.5, 2.5) corresponding to (λ, θ) are generated. The performance of the estimation procedure is studied by calculating the bias and mean square error (MSE) of the MLEs. It can be seen from Table 2.1, 2.2, 2.3, and 2.4 that, as the sample size increases, the bias and MSEs of the estimates decrease.

Table 2.1: Estimate, Biases and MSEs for PGDUSE model at $\lambda = 0.5$ and $\theta = 0.5$

n	Estimated value of Parameters	Bias	MSE
50	$\hat{\lambda}$=0.5248	0.0248	0.0126
	$\hat{\theta}$=0.5223	0.0223	0.0087
75	$\hat{\lambda}$=0.5162	0.0162	0.0086
	$\hat{\theta}$=0.5137	0.0137	0.0055
100	$\hat{\lambda}$=0.5114	0.0114	0.0044
	$\hat{\theta}$=0.5104	0.0104	0.0035
500	$\hat{\lambda}$=0.5101	0.0101	0.0010
	$\hat{\theta}$=0.5066	0.0066	0.0007
1000	$\hat{\lambda}$=0.5019	0.0019	0.0004
	$\hat{\theta}$=0.5042	0.0042	0.0003

2.2.4 Data Analysis

Real data analysis is given to assess how well the proposed distribution works have been performed. The data given in Lawless (1982) that contains the number of million revolutions before the failure of 23 ball bearings put on life test is considered. See Table 2.5.

Further, the proposed distribution has been compared with the generalized DUS exponential (GDUSE) by Maurya et al. (2017b), DUS exponential (DUSE), exponential (ED), and Kavya-Manoharan exponential (KME) by Kavya and Manoharan (2021) distributions. AIC, BIC, the value of KS statistic, p-value,

Table 2.2: Estimate, Biases and MSEs for PGDUSE model at $\lambda = 0.5$ and $\theta = 1.5$

n	Estimated value of Parameters	Bias	MSE
50	$\hat{\lambda}$=0.5192	0.0192	0.0066
	$\hat{\theta}$=1.6242	0.1242	0.1442
75	$\hat{\lambda}$=0.5165	0.0165	0.0042
	$\hat{\theta}$=1.5711	0.0790	0.0758
100	$\hat{\lambda}$=0.5158	0.0158	0.0031
	$\hat{\theta}$=1.5719	0.0719	0.0607
500	$\hat{\lambda}$=0.5025	0.0025	0.0005
	$\hat{\theta}$=1.5122	0.0122	0.0083
1000	$\hat{\lambda}$=0.5009	0.0009	0.0003
	$\hat{\theta}$=1.4709	-0.0291	0.0045

Table 2.3: Estimate, Biases and MSEs for PGDUSE model at $\lambda = 1$ and $\theta = 1.5$

n	Estimated value of Parameters	Bias	MSE
50	$\hat{\lambda}$=1.0236	0.0236	0.0255
	$\hat{\theta}$=1.5655	0.0655	0.1267
75	$\hat{\lambda}$=1.0190	0.0190	0.0161
	$\hat{\theta}$=1.5484	0.0484	0.0793
100	$\hat{\lambda}$=1.0116	0.0116	0.0113
	$\hat{\theta}$=1.5062	0.0062	0.0434
500	$\hat{\lambda}$=1.0091	0.0091	0.0023
	$\hat{\theta}$=1.5178	0.0178	0.0098
1000	$\hat{\lambda}$=0.9889	-0.0111	0.0010
	$\hat{\theta}$=1.4805	-0.0195	0.0039

and log-likelihood value have been used for model selection.

Table 2.6 elucidates that the proposed distribution gives the lowest AIC, BIC, and KS values, the greatest log-likelihood, and the p-value. Thus, it can be concluded that the $PGDUSE(\lambda, \theta)$ distribution provides a better fit for the given data set when compared with other competing distributions. The empirical cumulative density function (ECDF) plot is depicted in Figure 2.3.

Table 2.4: Estimate, Biases and MSEs for PGDUSE model at $\lambda = 1.5$ and $\theta = 2.5$

n	Estimated value of Parameters	Bias	MSE
50	$\hat{\lambda}$=1.5536	0.0536	0.0453
	$\hat{\theta}$=2.7200	0.2200	0.4536
75	$\hat{\lambda}$=1.5363	0.0363	0.0290
	$\hat{\theta}$=2.6169	0.1169	0.2836
100	$\hat{\lambda}$=1.5229	0.0229	0.0210
	$\hat{\theta}$=2.6154	0.1154	0.2005
500	$\hat{\lambda}$=1.5052	0.0052	0.0040
	$\hat{\theta}$=2.5271	0.0271	0.0314
1000	$\hat{\lambda}$=1.4897	-0.0103	0.0020
	$\hat{\theta}$=2.4774	-0.0226	0.0144

Table 2.5: Ball bearings dataset

17.88	28.92	33.00	41.52	42.12	45.60
48.80	51.84	51.96	54.12	55.56	67.80
68.64	68.64	68.88	84.12	93.12	98.64
105.12	105.84	127.92	128.04	173.40	

Table 2.6: MLEs of the parameters, Log-likelihoods, AIC, BIC, KS Statistics and p-values of the fitted models

Model	MLEs	log **L**	AIC	BIC	KS	p-value
PGDUSE	$\hat{\lambda} = 0.0336$ $\hat{\theta} = 3.8066$	-113.0030	230.0060	232.2770	0.1103	0.9425
GDUSE	$\hat{\alpha} = 4.7391$ $\hat{\beta} = 0.0355$	-113.0466	230.0931	232.3641	0.1179	0.9064
DUSE	$\hat{a} = 0.0182$	-127.4622	256.9244	261.1954	0.2774	0.0580
KME	$\hat{\theta} = 0.0095$	-123.1065	248.2129	252.4839	0.3110	0.0234
ED	$\hat{\theta} = 0.0138$	-121.4393	244.8786	246.0141	0.30673	0.0264

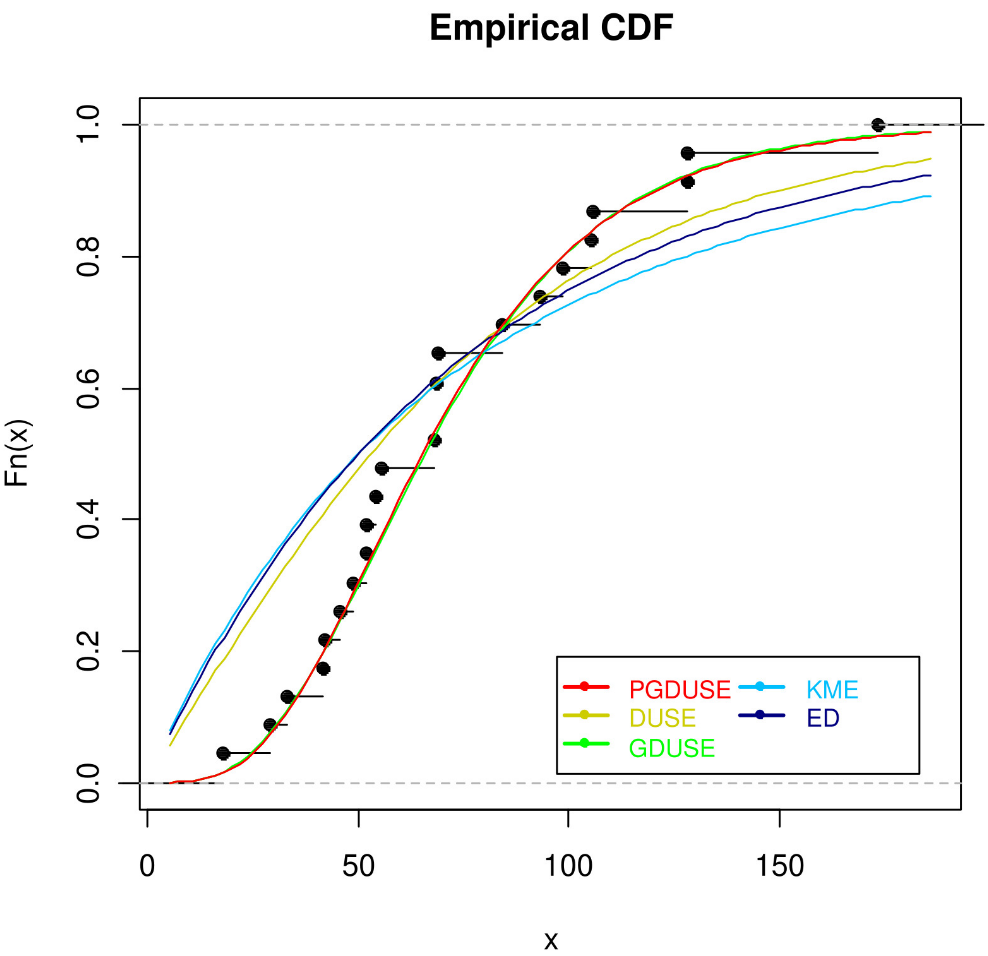

Figure 2.3: The empirical CDFs of the models.

2.3 PGDUS Weibull Distribution

Weibull distribution is used as the baseline distribution for PGDUS transformation and investigated the distributional properties. The CDF of Weibull distribution with parameters α and β is

$$G(x) = 1 - e^{-(x\beta)^{\alpha}}, \alpha, \beta > 0, x > 0. \tag{2.3.1}$$

and corresponding PDF is

$$g(x) = \alpha\beta(x\beta)^{\alpha-1}e^{-(x\beta)^{\alpha}}, \alpha, \beta > 0, x > 0 \tag{2.3.2}$$

Using Eq.(2.3.1) in Eq.(2.1.1), the CDF of PGDUS transformation of Weibull

(PGDUSW) distribution is as follows:

$$F(x) = \left(\frac{e^{1-e^{-(x\beta)^{\alpha}}} - 1}{e-1} \right)^{\theta}, \alpha, \beta > 0, \theta > 0, x > 0. \tag{2.3.3}$$

and the corresponding PDF is

$$f(x) = \frac{\theta \alpha \beta^{\alpha}}{(e-1)^{\theta}} x^{\alpha-1} (e^{1-e^{-(x\beta)^{\alpha}}} - 1)^{\theta-1} e^{1-(x\beta)^{\alpha}-e^{-(x\beta)^{\alpha}}}, \alpha, \beta, \theta > 0, x > 0. \tag{2.3.4}$$

In relation to Eq.(2.3.3) and Eq.(2.3.4), the HRF is,

$$h(x) = \frac{\theta \alpha \beta^{\alpha} x^{\alpha-1} (e^{1-e^{-(x\beta)^{\alpha}}} - 1)^{\theta-1} e^{1-(x\beta)^{\alpha}-e^{-(x\beta)^{\alpha}}}}{(e-1)^{\theta} - (e^{1-e^{-(x\beta)^{\alpha}}} - 1)^{\theta}}, \alpha, \beta, \theta > 0, x > 0. \tag{2.3.5}$$

The distribution with CDF Eq.(2.3.3) and PDF Eq.(2.3.4) is referred to as PGDUSW distribution with parameters α, β and θ and is denoted as $PGDUSW(\alpha, \beta, \theta)$. Figures 2.4 and 2.5 provide the graphical representation of the pdf and HRF respectively for various parameter values.

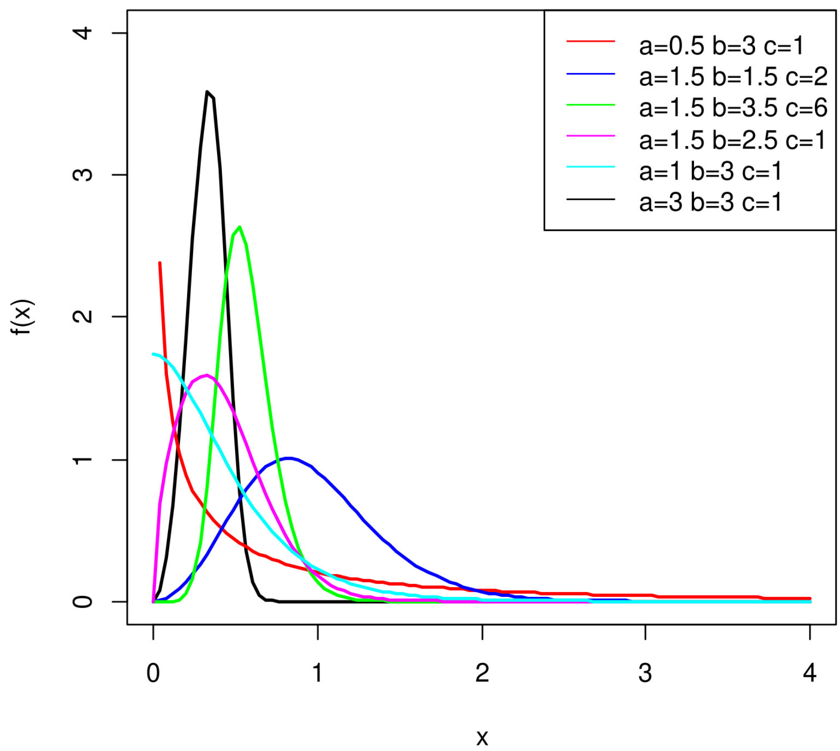

Figure 2.4: Density plot for PGDUSW

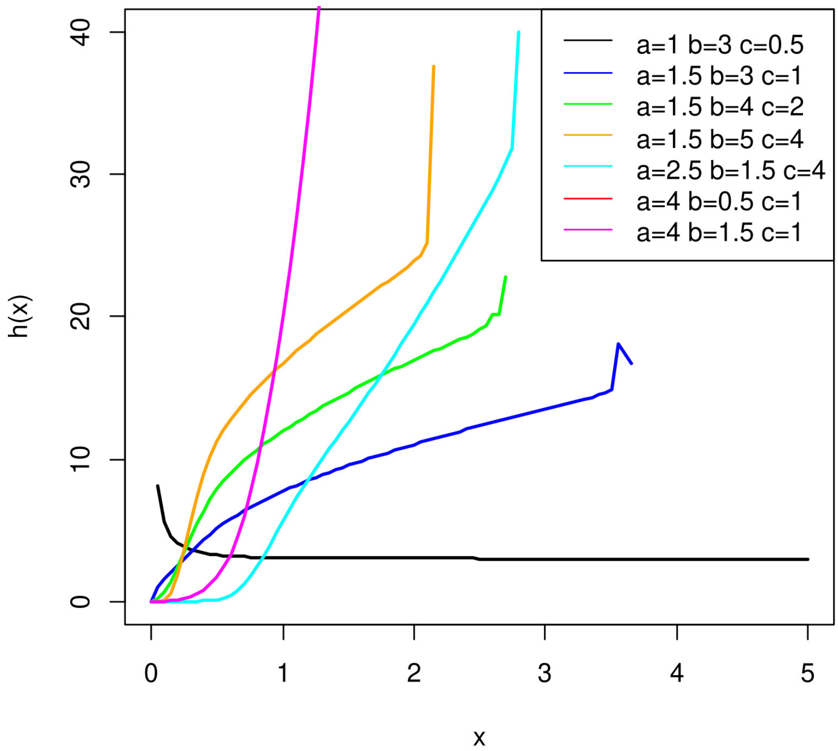

Figure 2.5: Failure rate plot for PGDUSW

2.3.1 Statistical Properties of PGDUSW Distribution

Moments, MGF, CF, CGF, QF, distribution of order statistics, and Rényi entropy of the proposed $PGDUSW(\alpha, \beta, \theta)$ distribution are derived.

Moments

The rth raw moment of the $PGDUSW(\alpha, \beta, \theta)$ distribution is given by

$$\mu_r' = \frac{\theta \beta^{-r} e}{(e-1)^{\theta}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+k}}{m!} e^{\theta-k-1} \binom{\theta-1}{k} (\theta-k)^m \frac{\Gamma(\frac{r}{\alpha}+1)}{(1+m)^{\frac{r}{\alpha}+1}}.$$

Moment Generating Function

The MGF of $PGDUSW(\alpha, \beta, \theta)$ distribution is

$$M_X(t) = \frac{\theta \alpha}{(e-1)^{\theta}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{k+m+n}}{m!n!} \binom{\theta-1}{k} e^{\theta-k} (\theta-k)^m (1+m)^n \beta^{\alpha+\alpha n} \frac{\Gamma(\alpha+\alpha n)}{t^{\alpha+\alpha n}}.$$

Characteristic Function and Cumulant Generating Function

The CF of $PGDUSW(\alpha,\beta,\theta)$ is given by

$$\phi_X(t)=\frac{\theta\alpha}{(e-1)^\theta}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+m+n}}{m!n!}\binom{\theta-1}{k}e^{\theta-k}(\theta-k)^m(1+m)^n\beta^{\alpha+\alpha n}\frac{\Gamma(\alpha+\alpha n)}{(it)^{\alpha+\alpha n}},$$

and the CGF of $PGDUSW(\alpha,\beta,\theta)$ is given by

$$\begin{aligned}K_X(t)&=\log\phi_X(t)\\&=\log\left[\frac{\theta\alpha}{(e-1)^\theta}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+m+n}}{m!n!}\binom{\theta-1}{k}e^{\theta-k}(\theta-k)^m(1+m)^n\beta^{\alpha+\alpha n}\frac{\Gamma(\alpha+\alpha n)}{(it)^{\alpha+\alpha n}}\right]\end{aligned}$$

where $i=\sqrt{-1}$ is the unit imaginary number.

Quantile Function

The pth quantile $Q(p)$ of the $PGDUSW(\alpha,\beta,\theta)$ is the real solution of the following equation

$$((e^{1-e^{-(\beta Q(p))^\alpha}}-1)/(e-1))^\theta=p$$

where $p\sim Uniform(0,1)$. Solving the above equation for $Q(p)$, it is obtained that

$$Q(p)=\frac{-1}{\beta^\alpha}\log[1-\log{(e-1)p^{\frac{1}{\theta}}}+1]^{\frac{1}{\alpha}}. \tag{2.3.6}$$

Setting $p=0.5$ in the Eq.(2.3.6) yields the median. Thus,

$$Median=\frac{-1}{\beta^\alpha}\log[1-\log{(e-1)0.5^{\frac{1}{\theta}}}+1]^{\frac{1}{\alpha}}.$$

Similarly, the quartiles Q_1 and Q_3 are obtained respectively by setting $p=\frac{1}{4}$ and $p=\frac{3}{4}$ in Eq.(2.3.6).

Distribution of Order Statistic

Let $X_1,X_2,\ldots,X_m$ be m independent random variables from the $PGDUSW(\alpha,\beta,\theta)$ distribution with CDF Eq.(2.3.3) and PDF Eq.(2.3.4). Then the PDF of rth order statistics $X_{(r)}$ of the $PGDUSW(\alpha,\beta,\theta)$ distribution is given by

$$f_{X_{(r)}} = \frac{m!}{(r-1)!(m-r)!}\frac{\theta\alpha\beta^{\alpha}x^{\alpha-1}}{(e-1)^{\theta m}}\left(e^{1-e^{-(x\beta)^{\alpha}}}-1\right)^{\theta r-1} e^{1-(x\beta)^{\alpha}-e^{-(x\beta)^{\alpha}}} \\ \left[(e-1)^{\theta}-(e^{1-e^{-(x\beta)^{\alpha}}})^{\theta}\right]^{m-r}, r=1,2,\ldots,m. \quad (2.3.7)$$

Then, the PDF of $X_{(1)}$ and $X_{(m)}$ are obtained by setting $r=1$ and $r=m$ respectively in Eq.(2.3.7). This can be used in reliability analysis of series and parallel system.

Rényi Entropy

Rényi entropy introduced by Rényi (1961) is defined as

$$\beth_R(\nu) = \frac{1}{1-\nu}\log\left(\int f^{\nu}(x)dx\right)$$

where $\nu > 0$ and $\nu \neq 1$.

$$\int_0^{\infty} f^{\nu}(x)dx = \frac{(\theta\alpha)^{\nu}}{(e-1)^{\theta\nu}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!}\binom{\nu\theta-\nu}{k}(\nu\theta-k)^m e^{\nu\theta-k}\frac{\Gamma(\nu-\frac{\nu}{\alpha}+1)}{(\nu+m)^{\nu-\frac{\nu}{\alpha}+1}\beta^{\alpha-\nu}}$$

Then the Rényi entropy of the $PGDUSW(\alpha,\beta,\theta)$ becomes

$$\beth_R(\nu) = \frac{1}{1-\nu}\log\left[\frac{(\theta\alpha)^{\nu}}{(e-1)^{\theta\nu}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!}\binom{\nu\theta-\nu}{k}(\nu\theta-k)^m e^{\nu\theta-k}\frac{\Gamma(\nu-\frac{\nu}{\alpha}+1)}{(\nu+m)^{\nu-\frac{\nu}{\alpha}+1}\beta^{\alpha-\nu}}\right]$$

2.3.2 Estimation of PGDUSW Distribution

To estimate the unknown parameters of the $PGDUSW(\alpha,\beta,\theta)$, the maximum likelihood estimation method is utilized. For this, a random sample of size n from the $PGDUSW(\alpha,\beta,\theta)$ distribution was chosen. Therefore, the likelihood function is given by,

$$L(x) = \prod_{i=1}^{n} f(x) = \prod_{i=1}^{n}\frac{\theta\alpha\beta^{\alpha}}{(e-1)^{\theta}}x^{\alpha-1}e^{1-(x_i\beta)^{\alpha}-e^{-(x_i\beta)^{\alpha}}}(e^{1-e^{-(x_i\beta)^{\alpha}}}-1)^{\theta-1} \quad (2.3.8)$$

Applying the natural logarithm to Eq.(2.3.8), the log-likelihood function becomes

$$\log L = n\log(\theta) + n\log(\alpha) + \alpha n\log(\beta) - \theta n\log(e-1) + n + \sum_{i=0}^{n}(\alpha-1)\log(x_i)$$

$$-\sum_{i=0}^{n}(x_i\beta)^{\alpha} - \sum_{i=0}^{n} e^{-(x_i\beta)^{\alpha}} + (\theta-1)\sum_{i=0}^{n}\log(e^{1-e^{-(x_i\beta)^{\alpha}}}-1).$$

Computing the first order partial derivatives,

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=0}^{n}(x_i\beta)^{\alpha}\log(x_i\beta) + \sum_{i=0}^{n}\log(x_i) + \sum_{i=0}^{n}(x_i\beta)^{\alpha}e^{-(x_i\beta)^{\alpha}}\log(x_i\beta)$$
$$+n\log(\beta) + \frac{(\theta-1)(x_i\beta)^{\alpha}}{(e^{1-e^{-(x_i\beta)^{\alpha}}}-1)}\log(x_i\beta)e^{1-(x_i\beta)^{\alpha}-e^{-(x_i\beta)^{\alpha}}}, \tag{2.3.9}$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n\alpha}{\beta} - \sum_{i=0}^{n}\frac{\alpha(x_i\beta)^{\alpha}}{\beta} + \sum_{i=0}^{n}\frac{\alpha(x_i\beta)^{\alpha}}{\beta}e^{-(x_i\beta)^{\alpha}}$$
$$+(\theta-1)\frac{\alpha}{\beta}\sum_{i=0}^{n}(x_i\beta)^{\alpha}\frac{e^{1-(x_i\beta)^{\alpha}-e^{-(x_i\beta)^{\alpha}}}}{(e^{1-e^{-(x_i\beta)^{\alpha}}}-1)}, \tag{2.3.10}$$

and

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - n\log(e-1) + \sum_{i=0}^{n}\log(e^{1-e^{-(x_i\beta)^{\alpha}}}-1). \tag{2.3.11}$$

Equations (2.3.9), (2.3.10) and (2.3.11) are not in closed form. The solution to these explicit equations can be obtained analytically and can be solved numerically using R software by taking arbitrary initial values. In the case of asymptotic normal MLEs, the confidence interval(CI)s for α, β, and θ are calculated by computing the observed information matrix given by

$$I = \begin{pmatrix} \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \beta} & \frac{\partial^2 \log L}{\partial \alpha \partial \theta} \\ \\ \frac{\partial^2 \log L}{\partial \beta \partial \alpha} & \frac{\partial^2 \log L}{\partial \beta^2} & \frac{\partial^2 \log L}{\partial \beta \partial \theta} \\ \\ \frac{\partial^2 \log L}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L}{\partial \theta \partial \beta} & \frac{\partial^2 \log L}{\partial \theta^2} \end{pmatrix}$$

where

$$\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{n}{\alpha} - \sum_{i=1}^{n} \log^2(x_i\beta)(x_i\beta)^\alpha((x_i\beta)^\alpha - 1)e^{-(x_i\beta)^\alpha}$$

$$+(\theta-1)\textstyle\sum_{i=1}^{n} \frac{(x_i\beta)^\alpha \log^2(x_i\beta)e^{1-(x_i\beta)^\alpha - e^{-(x_i\beta)^\alpha}}((((x_i\beta)^\alpha - 1)e^{1-e^{-(x_i\beta)^\alpha}} - (x_i\beta)^\alpha e^{-(x_i\beta)^\alpha}) + (1-(x_i\beta)^\alpha))}{(e^{1-e^{-(x_i\beta)^\alpha}} - 1)^2},$$

$$\frac{\partial^2 \log L}{\partial \alpha \partial \beta} = -\sum_{i=1}^{n} (x_i\beta)^\alpha e^{-(x_i\beta)^\alpha}[\alpha((x_i\beta)^\alpha - 1)\ln(x_i\beta) - 1]$$

$$+\frac{n}{\beta} - (\theta-1)\sum_{i=1}^{n} \frac{(x_i\beta)^\alpha([\alpha[(x_i\beta)^\alpha - 1]e^{(x_i\beta)^\alpha} - \alpha(x_i\beta)^\alpha]e^{1-(x_i\beta)^\alpha - e^{-(x_i\beta)^\alpha}}}{\beta(e^{1-e^{-(x_i\beta)^\alpha}} - 1)^2}$$

$$-\sum_{i=1}^{n} x_i^\alpha \beta^{\alpha-1}(\alpha\ln(x_i\beta) + 1) - (\theta-1)\sum_{i=1}^{n} \frac{\alpha(1-(x_i\beta)^\alpha)\ln(x_i\beta) - e^{1-e^{1-(x_i\beta)^\alpha}} + 1)}{\beta(e^{1-e^{-(x_i\beta)^\alpha}} - 1)^2},$$

$$\frac{\partial^2 \log L}{\partial \alpha \partial \theta} = \sum_{i=1}^{n} \frac{(x_i\beta)^\alpha \log(x_i\beta)e^{1-(x_i\beta)^\alpha - e^{-(x_i\beta)^\alpha}}}{(e^{1-e^{-(x_i\beta)^\alpha}} - 1)},$$

$$\frac{\partial^2 \log L}{\partial \beta^2} = \frac{-\alpha n}{\beta^2} - \frac{\alpha(\alpha-1)}{\beta^2}\sum_{i=1}^{n}(x_i\beta)^{\alpha} - \frac{\alpha}{\beta^2}\sum_{i=1}^{n}(x_i\beta)^{\alpha}(\alpha(x_i\beta)^{\alpha} - \alpha + 1)e^{-(x_i\beta)^{\alpha}}$$

$$+(\theta-1)\tfrac{\alpha}{\beta^2}\textstyle\sum_{i=1}^{n}(x_i\beta)^{\alpha}\frac{(((\alpha(x_i\beta)^{\alpha}-\alpha+1)e^{(x_i\beta)^{\alpha}}-\alpha(x_i\beta)^{\alpha})e^{1-(x_i\beta)^{\alpha}-e^{-(x_i\beta)^{\alpha}}}(-\alpha(x_i\beta)^{\alpha}+\alpha-1))}{(e^{1-e^{-(x_i\beta)^{\alpha}}}-1)^2}$$

$$\frac{\partial^2 \log L}{\partial \beta \partial \theta} = \frac{\alpha}{\beta}\sum_{i=1}^{n}\frac{(x_i\beta)^{\alpha}e^{1-(x_i\beta)^{\alpha}-e^{-(x_i\beta)^{\alpha}}}}{(e^{1-e^{-(x_i\beta)^{\alpha}}}-1)},$$

and

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{n}{\theta^2}.$$

For α, β, and θ, the $100(1-\gamma)\%$ asymptotic CIs are as follows: $\hat{\alpha} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{11}}$, $\hat{\beta} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{22}}$, and $\hat{\theta} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{33}}$, where V_{ij} represents the (i,j)th element in the inverse of the Fisher information matrix I.

2.3.3 Simulation Study

To illustrate the performance of the maximum likelihood method for $PGDUSW(\alpha, \beta, \theta)$ distribution, the inverse transformation method is used. For different values of α, β and θ, samples of sizes $n = 100, 250, 500, 750$ and 1000 are generated from the proposed model. For 1000 repetitions, the bias and mean square error (MSE) of the estimated parameters are computed. The selected parameter values are $\alpha = 0.5, \beta = 0.5$ and $\theta = 0.5$, $\alpha = 0.5, \beta = 1$ and $\theta = 0.5$ and $\alpha = 1, \beta = 1$ and $\theta = 0.5$. From the Tables 2.7, 2.8 and 2.9, it is noted that bias and MSE decrease for the selected parameter values as sample size increases.

2.3.4 Data Analysis

A real data analysis is carried out to determine the performance of the proposed model. For this, the data on the number of million revolutions before the failure of 23 ball bearings put on test is considered (Lawless (1982)), see Table 2.5.

Table 2.7: Estimate, Biases and MSEs for PGDUSW model at $\alpha = 0.5, \beta = 0.5$ and $\theta = 0.5$

n	Estimated Parameter values	Bias	MSE
100	$\hat{\alpha}$=0.5668	0.0668	0.0473
	$\hat{\beta}$=0.7541	0.2541	1.0617
	$\hat{\theta}$=0.5021	0.0031	0.0413
250	$\hat{\alpha}$=0.5251	0.0251	0.0118
	$\hat{\beta}$=0.5831	0.0831	0.1488
	$\hat{\theta}$=0.5032	0.0022	0.0165
500	$\hat{\alpha}$=0.5297	0.0189	0.0057
	$\hat{\beta}$=0.4929	0.0177	0.0318
	$\hat{\theta}$=0.4922	0.0007	0.0068
750	$\hat{\alpha}$=0.5188	0.0188	0.0034
	$\hat{\beta}$=0.4935	-0.0065	0.0223
	$\hat{\theta}$=0.5026	0.0003	0.0050
1000	$\hat{\alpha}$=0.5165	0.0165	0.0025
	$\hat{\beta}$=0.4795	-0.0205	0.0159
	$\hat{\theta}$=0.4922	-0.0078	0.0035

Different distributions namely, Inverse Weibull (IW) distribution, DUS Exponential (DUSE) distribution by Kumar et al. (2015), and Kavya-Manoharan Weibull (KMW) by Kavya and Manoharan (2021) distribution are used to compare the performance with the proposed $PGDUSW(\alpha, \beta, \theta)$ distribution.

To check the acceptability of the $PGDUSW(\alpha, \beta, \theta)$ distribution for the given data set AIC, Corrected Akaike Information Criterion (AICc), log-likelihood value,

Table 2.8: Estimate, Biases and MSEs for PGDUSW model at $\alpha = 0.5, \beta = 1$ and $\theta = 0.5$

n	Estimated Parameter values	Bias	MSE
100	$\hat{\alpha}$=0.5729	0.0729	0.0460
	$\hat{\beta}$=1.4827	0.4827	3.7354
	$\hat{\theta}$=0.51341	0.0434	0.0485
250	$\hat{\alpha}$=0.5019	0.0019	0.0083
	$\hat{\beta}$=1.2852	0.2852	0.6372
	$\hat{\theta}$=0.5333	0.0393	0.0169
500	$\hat{\alpha}$=0.4943	-0.0057	0.0041
	$\hat{\beta}$=1.2236	0.2236	0.2915
	$\hat{\theta}$=0.5399	0.0339	0.0102
750	$\hat{\alpha}$=0.4886	-0.0109	0.0023
	$\hat{\beta}$=1.1045	0.1814	0.1353
	$\hat{\theta}$=0.5244	0.0244	0.0050
1000	$\hat{\alpha}$=0.4822	-0.0178	0.0022
	$\hat{\beta}$=1.1814	0.1045	0.1195
	$\hat{\theta}$=0.5207	0.0207	0.0042

and KS goodness of fit test statistic with the p-value are used and the computed values are provided in Table 2.10. It is worth noting that in the goodness of fit test, the purpose is to determine whether the sets of data with the distribution function $F(y)$ and the hypothesised distribution $F_{PGDUSW}(y)$ are compatible. This problem can be formulated as $H_0 : F(y) = F_{PGDUSW}(y)$ versus the alternative $H_1 : F(y) \neq F_{PGDUSW}(y)$.

Table 2.9: Estimate, Biases and MSEs for PGDUSW model at $\alpha = 1, \beta = 1$ and $\theta = 0.5$

n	Estimated Parameter values	Bias	MSE
100	$\hat{\alpha}$=1.1273	0.1273	0.1628
	$\hat{\beta}$=1.1460	0.1460	0.8851
	$\hat{\theta}$=0.5223	0.0223	0.0545
250	$\hat{\alpha}$=1.0184	0.0184	0.0449
	$\hat{\beta}$=1.0889	0.0889	0.1068
	$\hat{\theta}$=0.5205	0.0205	0.0177
500	$\hat{\alpha}$=1.0109	0.0109	0.0185
	$\hat{\beta}$=1.0490	0.0490	0.0447
	$\hat{\theta}$=0.5151	0.0151	0.0085
750	$\hat{\alpha}$=1.0056	0.0056	0.0107
	$\hat{\beta}$=1.0381	0.0381	0.0260
	$\hat{\theta}$=0.5095	0.0095	0.0049
1000	$\hat{\alpha}$=0.9851	-0.0149	0.0074
	$\hat{\beta}$=1.0239	0.0239	0.0167
	$\hat{\theta}$=1.0012	0.0012	0.0035

From Table 2.10, it is noted that the $PGDUSW(\alpha, \beta, \theta)$ distribution fits well for the given data set. To facilitate a better understanding of the results, the plot of the ECDF is shown in the Figure 2.6 along with the plot of fitted densities in the Figure 2.7 of the distributions for the ball bearings dataset. Furthermore, our proposed distribution is found to fit better than those of the other distributions.

Table 2.10: Findings for PGDUSW Distribution

Model	MLEs	$\log \mathbf{L}$	AIC	AICc	KS	p-value
IW	$\hat{\lambda} = 1.8341$ $\hat{\theta} = 0.0206$	-115.7887	235.5774	236.1774	0.1328	0.8118
DUSE	$\hat{a} = 0.0182$	-127.4622	256.9244	257.1149	0.2774	0.0580
KMW	$\hat{\lambda} = 2.3169$ $\hat{\kappa} = 0.0107$	-113.4076	230.8152	231.4152	0.1421	0.7419
PGDUSW	$\hat{\alpha} = 0.9362$ $\hat{\beta} = 0.0383$ $\hat{\theta} = 4.4478$	-113.0114	230.0228	230.6228	0.10921	0.9467

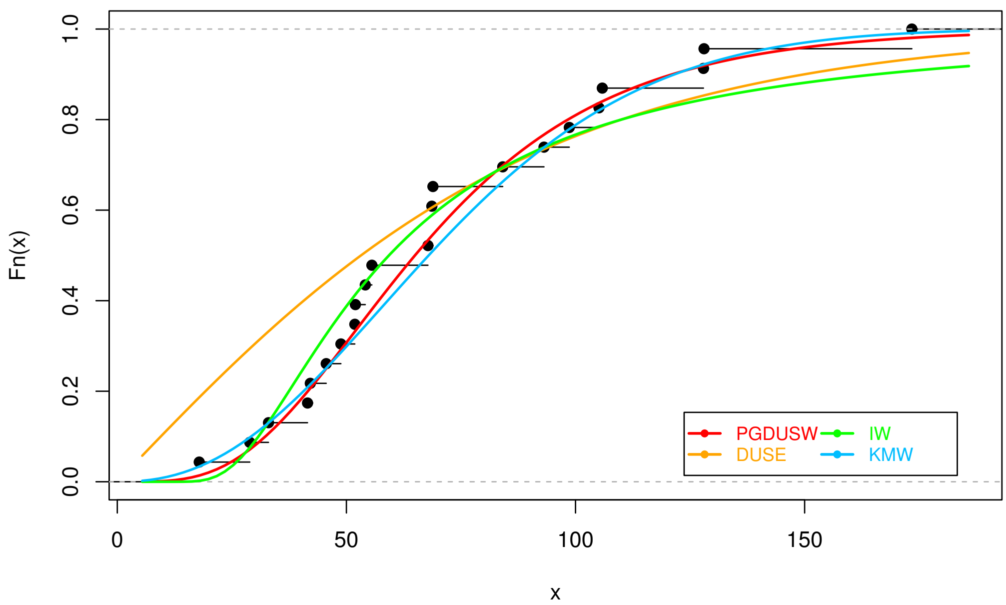

Figure 2.6: ECDF plot for various distributions.

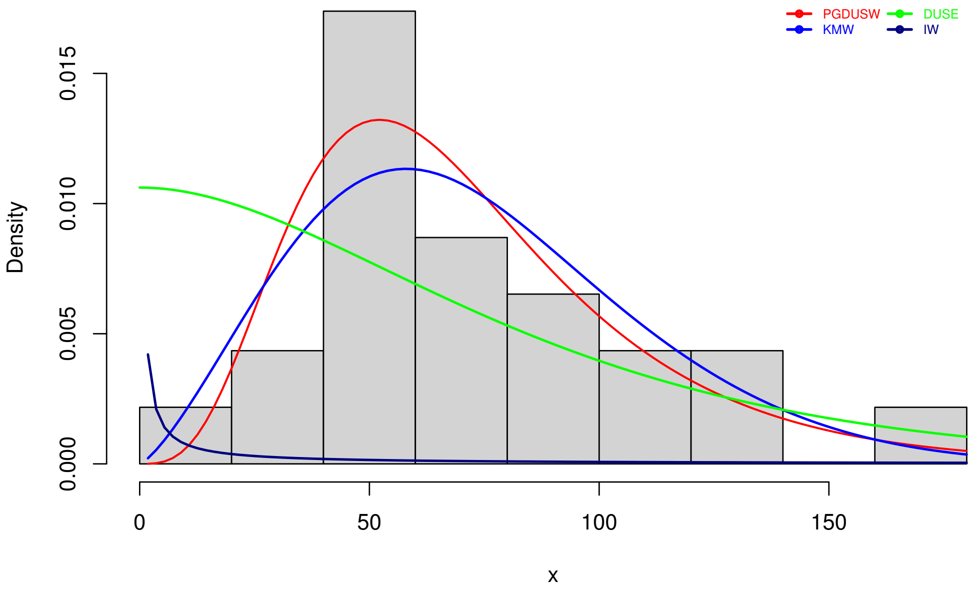

Figure 2.7: Fitted Density plot for various distributions.

2.4 PGDUS Lomax Distribution

Power Generalized DUS Lomax (PGDUSL) Distribution, denoted as $PGDUSL(\alpha, \beta, \theta)$, is obtained using PGDUS transformation with Lomax distribution as baseline distribution. Then the CDF of the $PGDUSL(\alpha, \beta, \theta)$ distribution using Eq.(2.1.1) is given by

$$F(x) = \left(\frac{e^{1-(1+x\beta)^{-\alpha}} - 1}{e - 1}\right)^{\theta}, \alpha, \beta > 0, \theta > 0, x > 0. \tag{2.4.1}$$

Then the PDF is

$$f(x) = \frac{\theta\alpha\beta}{(e-1)^{\theta}}(e^{1-(1+x\beta)^{-\alpha}} - 1)^{\theta-1}e^{1-(1+x\beta)^{-\alpha}}(1 + x\beta)^{-(\alpha+1)}. \tag{2.4.2}$$

The HRF is

$$h(x) = \frac{\theta\alpha\beta(e^{1-(1+x\beta)^{-\alpha}} - 1)^{\theta-1}e^{1-(1+x\beta)^{-\alpha}}(1 + x\beta)^{-(\alpha+1)}}{(e-1)^{\theta} - (e^{1-(1+x\beta)^{-\alpha}} - 1)^{\theta}}$$

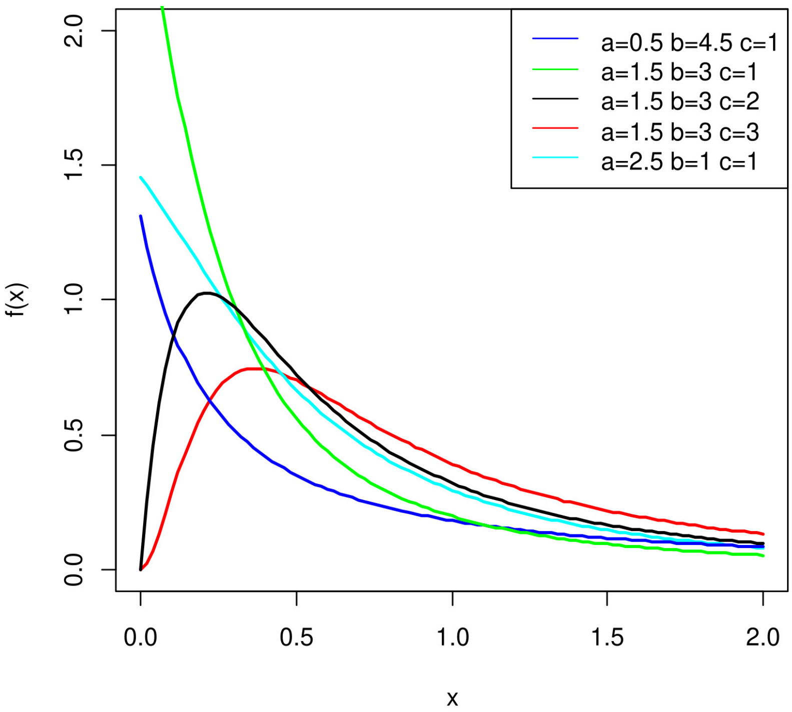

Figure 2.8: PGDUSL distribution density plot for various parameter values.

2.4.1 Properties of PGDUSL Distribution

Here, a few properties of the PGDUSL distribution are explored.

Moments

The r^{th} raw moments of $PGDUSL(\alpha, \beta, \theta)$ is

$$\mu_r' = \frac{\theta\alpha}{(e-1)^\theta} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{k+m+n}}{n!} \binom{\alpha+k}{k} \binom{\theta-1}{m} \beta^{k+1}\, e^{\theta-m}\, (\theta-m)^n$$
$$B(r+k+1, \alpha n - r - k - 1).$$

Moment Generating Function

The MGF of $PGDUSL(\alpha, \beta, \theta)$ is

$$M_X(t) = \frac{\theta\alpha}{(e-1)^\theta} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{k+m+n}}{n!\, l!\, \beta^l} \binom{\alpha-k}{k} \binom{\theta-1}{m} e^{\theta-m}\, (\theta-m)^n\, t^l$$

$$B(k+l+1, \alpha\, n - k - l - 1).$$

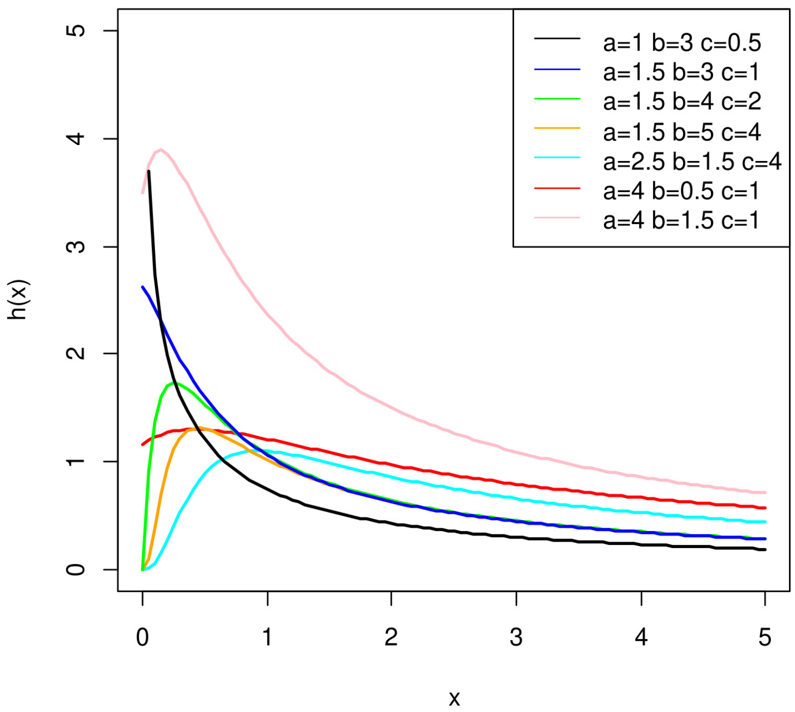

Figure 2.9: PGDUSL distribution HRF plot for various parameter values.

Characteristic Function and Cumulant Generating Function

The CF of the proposed distribution is given by

$$\phi_X(t) = \frac{\theta\alpha}{(e-1)^\theta} \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{k+m+n}}{n!\, l!\, \beta^l} \binom{\alpha-k}{k}\binom{\theta-1}{m} e^{\theta-m}\,(\theta-m)^n\,(it)^l$$

$$B(k+l+1, \alpha\, n-k-l-1).$$

The CGF of the proposed distribution is given by

$$K_X(t) = \log\left\{ \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{k+m+n}}{n!\, l!\, \beta^l} \binom{\alpha-k}{k}\binom{\theta-1}{m} e^{\theta-m}\,(\theta-m)^n\,(it)^l \right.$$

$$\left. B(k+l+1, \alpha\, n-k-l-1) \right\} + \log\left(\frac{\theta\alpha}{(e-1)^\theta}\right).$$

Quantile Function

The pth quantile $Q(p)$ of the $PGDUSL(\alpha, \beta, \theta)$ is the real solution of the following equation

$$((e^{1-1+(\beta Q(p))^{\alpha}} - 1)/(e-1))^{\theta} = p,$$

where $p \sim Uniform(0,1)$. Solving the above equation for $Q(p)$, it can be obtained as

$$Q(p) = \frac{1}{\beta}\left\{ \left[1 - \log\left[p^{\frac{1}{\theta}}(e-1)+1\right]\right]^{\frac{-1}{\alpha}} - 1\right\}.$$

Setting $p = 0.5$ in the above equation yields median. Thus,

$$Median = \frac{1}{\beta}\left\{ \left[1 - \log\left[0.5^{\frac{1}{\theta}}(e-1)+1\right]\right]^{\frac{-1}{\alpha}} - 1\right\}.$$

2.4.2 Estimation of PGDUSL Distribution

Method of Maximum likelihood estimation is used to estimate the unknown parameters of $PGDUSL(\alpha, \beta, \theta)$. For this, a random sample of size n from $PGDUSL(\alpha, \beta, \theta)$ distribution was chosen. Then the likelihood function is given by,

$$L(x) = \prod_{i=1}^{n} f(x) = \frac{(\theta\alpha\beta)^n}{(e-1)^{\theta n}} \prod_{i=1}^{n} (e^{1-(1+x_i\beta)^{-\alpha}} - 1)^{\theta-1} e^{1-(1+x_i\beta)^{-\alpha}} (1+x_i\beta)^{-\alpha+1} \quad (2.4.3)$$

The log-likelihood function becomes

$$\log L = n\log(\theta) + n\log(\alpha) + n\log(\beta) - \theta n \log(e-1) + n - \sum_{i=1}^{n}(1+x_i\beta)^{-\alpha}$$

$$-(\alpha+1)\sum_{i=1}^{n}\log(1+x_i\beta) + (\theta-1)\sum_{i=1}^{n}\log(e^{1-(1+x_i\beta)^{-\alpha}} - 1). \quad (2.4.4)$$

Computing the first order partial derivatives of Eq.(2.4.4),

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log(1+x_i\beta)(1+x_i\beta)^{-\alpha} - \sum_{i=1}^{n}\log(1+x_i\beta)$$

$$+\sum_{i=1}^{n}\frac{(\theta-1)\log(1+x_i\beta)e^{1-(1+x_i\beta)^{-\alpha}}(1+x_i\beta)^{-\alpha}}{(e^{1-(1+x_i\beta)^{-\alpha}} - 1)}. \quad (2.4.5)$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \alpha x_i (1 + x_i\beta)^{-(\alpha+1)} - (\alpha + 1) \sum_{i=1}^{n} \frac{x_i}{1 + x_i\beta}$$

$$- \sum_{i=1}^{n} \frac{\alpha x_i (\theta - 1)(1 + x_i\beta)^{-(\alpha+1)}}{(e^{1-(1+x_i\beta)^{-\alpha}} - 1)}, \tag{2.4.6}$$

and

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - n \log(e - 1) + \sum_{i=1}^{n} \log(e^{1-(1+x_i\beta)^{-\alpha}} - 1). \tag{2.4.7}$$

Equations (2.4.5), (2.4.6) and (2.4.7) are not in closed form. The solution of these explicit equations can be obtained analytically and can be solved numerically using R software by taking arbitrary initial values.

In the case of asymptotic normal MLEs, the confidence interval(CI)s for α, β, and θ are calculated by computing the observed information matrix given by

$$I = \begin{pmatrix} \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \beta} & \frac{\partial^2 \log L}{\partial \alpha \partial \theta} \\ \frac{\partial^2 \log L}{\partial \beta \partial \alpha} & \frac{\partial^2 \log L}{\partial \beta^2} & \frac{\partial^2 \log L}{\partial \beta \partial \theta} \\ \frac{\partial^2 \log L}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L}{\partial \theta \partial \beta} & \frac{\partial^2 \log L}{\partial \theta^2} \end{pmatrix}$$

where

$$\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \sum_{i=1}^{n} (1 + \beta x_i)^{-\alpha} \log^2(1 + \beta x_i)$$

$$+ (\theta - 1) \sum_{i=1}^{n} \frac{\log^2(1 + x_i\beta) e^{1-(1+x_i\beta)^{-\alpha}} (1 + x_i\beta)^{-\alpha} [1 - (1 + x_i\beta)^{-\alpha} - e^{1-(1+x_i\beta)^{-\alpha}}]}{(e^{1-(1+x_i\beta)^{-\alpha}} - 1)^2},$$

$$\frac{\partial^2 \log L}{\partial\alpha\partial\beta} = -\sum_{i=1}^{n} x_i A^{-1} - (\theta - 1)\sum_{i=1}^{n} A^{-(\alpha+1)} e^{1-A^{-\alpha}} \frac{[x_i(e^{1-A^{-\alpha}} - 1 - \alpha x_i A^{-(\alpha+1)})]}{(e^{1-A^{-\alpha}} - 1)^2}$$

$$+ (\theta - 1)\sum_{i=1}^{n} A^{-(\alpha+1)} e^{1-A^{-\alpha}} \frac{[\alpha \log(A) e^{1-A^{-\alpha}} (A^{-\alpha}\log(A) - 1 + e^{-(1-A^{-\alpha})} - A^{-\alpha} e^{-(1-A^{-\alpha})})]}{(e^{1-A^{-\alpha}} - 1)^2}$$

$$-\sum_{i=1}^{n} x_i A^{-(\alpha+1)}[\alpha \log(A) - 1]$$

where $A = (1 + x_i\beta)$

$$\frac{\partial^2 \log L}{\partial\alpha\partial\theta} = \frac{\partial^2 \log L}{\partial\theta\partial\alpha} = \sum_{i=1}^{n} \frac{\log(1+\beta x_i) e^{1-(1+\beta x_i)^{-\alpha}} (1+\beta x_i)^{-\alpha}}{(e^{1-(1+\beta x_i)^{-\alpha}} - 1)},$$

$$\frac{\partial^2 \log L}{\partial\beta^2} = -\frac{n}{\beta^2} - \sum_{i=1}^{n} \alpha(\alpha+1) x_i^2 (1+\beta x_i)^{-(\alpha+2)} + (\alpha+1)\sum_{i=1}^{n} x_i^2 (1+\beta x_i)^{-2} +$$

$$\alpha(\theta - 1)\textstyle\sum_{i=1}^{n} x_i^2 \frac{(\alpha+1)(1+\beta x_i)^{-(\alpha+2)}(e^{1-(1+\beta x_i)^{-\alpha}} - 1) - \alpha(1+\beta x_i)^{-2(\alpha+1)} e^{1-(1+\beta x_i)^{-\alpha}}}{(e^{1-(1+\beta x_i)^{-\alpha}} - 1)^2},$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\theta} = -\sum_{i=1}^{n} \frac{\alpha x_i (1+\beta x_i)^{-(\alpha+1)}}{(e^{1-(1+\beta x_i)^{-\alpha}} - 1)},$$

and

$$\frac{\partial^2 \log L}{\partial\theta^2} = -\frac{n}{\theta^2}.$$

For α, β, and θ, the $100(1-\gamma)\%$ asymptotic CIs are as follows: $\hat{\alpha} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{11}}$, $\hat{\beta} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{22}}$, and $\hat{\theta} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{33}}$, where V_{ij} represents the (i, j)th element in the inverse of the Fisher information matrix I.

2.4.3 Simulation Study

In order to demonstrate the performance of the maximum likelihood method for the proposed $PGDUSL(\alpha, \beta, \theta)$ distribution, the inverse transformation method is used.

For different combinations of values of α, β, and θ, samples of sizes n = 250, 500, 750, and 1000 are generated from the $PGDUSL(\alpha, \beta, \theta)$ model. The bias and mean square error (MSE) of the estimated parameters are calculated for 1000 iterations. The selected parameter values are $\alpha = 0.5, \beta = 0.5$ and $\theta = 0.5$, $\alpha = 1, \beta = 1.5$ and $\theta = 0.5$ and $\alpha = 1, \beta = 1.5$ and $\theta = 1$. From Tables 2.11, 2.12, and 2.13, it is observed that bias and MSE decreases for the selected parameter values as the sample size increases.

Table 2.11: Estimate, Biases and MSEs for PGDUSL model at $\alpha = 0.5, \beta = 0.5$ and $\theta = 0.5$

n	Estimated value of Parameters	Bias	MSE
250	$\hat{\alpha}$=0.5100	0.0100	0.0031
	$\hat{\beta}$=0.5520	0.0720	0.0665
	$\hat{\theta}$=0.5218	0.0218	0.0049
500	$\hat{\alpha}$=0.4921	-0.0039	0.0016
	$\hat{\beta}$=0.5926	0.0526	0.0422
	$\hat{\theta}$=0.5197	0.0197	0.0023
750	$\hat{\alpha}$=0.4960	-0.0079	0.0010
	$\hat{\beta}$=0.5313	0.0343	0.0181
	$\hat{\theta}$=0.5088	0.0088	0.0013
1000	$\hat{\alpha}$=0.4889	-0.0111	0.0008
	$\hat{\beta}$=0.5343	0.0313	0.0134
	$\hat{\theta}$=0.5046	0.0046	0.0009

2.4.4 Real Data Application

Real data analysis is used to determine the applicability of the PGDUSL model. The data set shown in Table 2.14 is uncensored. Among 128 patients with bladder cancer in a random sample, it corresponds to the number of months they experienced

Table 2.12: Estimate, Biases and MSEs for PGDUSL model at $\alpha = 1, \beta = 1.5$ and $\theta = 0.5$

n	Estimated value of Parameters	Bias	MSE
250	$\hat{\alpha}$=1.0268	0.0268	0.0314
	$\hat{\beta}$=1.6452	0.1800	0.4484
	$\hat{\theta}$=0.5217	0.0217	0.0037
500	$\hat{\alpha}$=1.0140	0.0140	0.0131
	$\hat{\beta}$=1.6800	0.1452	0.2215
	$\hat{\theta}$=0.5187	0.0187	0.0017
750	$\hat{\alpha}$=0.9838	-0.0070	0.0080
	$\hat{\beta}$=1.6374	0.1374	0.1404
	$\hat{\theta}$=0.5040	0.0050	0.0008
1000	$\hat{\alpha}$=0.9930	-0.0162	0.0059
	$\hat{\beta}$=1.6070	0.1070	0.0906
	$\hat{\theta}$=0.5050	0.0040	0.0006

remission, as reported by Lee and Wang (2003). Different distributions, namely the Lomax distribution (LD) by Lomax (1954), the DUSE distribution by Kumar et al. (2015), and the DUS Lomax (DUSL) distribution by Deepthi and Chacko (2020), are used to compare the performance with the proposed $PGDUSL(\alpha, \beta, \theta)$ distribution.

To check the acceptability of the $PGDUSL(\alpha, \beta, \theta)$ distribution for the given data set AIC, Corrected AIC (AICc), log-likelihood value, KS value and p-value are used and the computed values are provided in Table 2.15. From Table 2.15, it is clear that $PGDUSL(\alpha, \beta, \theta)$ distribution fits well for the given data set. To facilitate a better understanding of the results, the plot of the ECDF is shown in the Figure 2.10 along with fitted density plot in the Figure 2.11 of the distributions

Table 2.13: Estimate, Biases and MSEs for PGDUSL model at $\alpha = 1, \beta = 1.5$ and $\theta = 1$

n	Estimated value of Parameters	Bias	MSE
250	$\hat{\alpha}$=1.0284	0.0284	0.0194
	$\hat{\beta}$=1.69386	0.19386	0.71071
	$\hat{\theta}$=1.05298	0.05297	0.03426
500	$\hat{\alpha}$=1.0179	0.0179	0.0082
	$\hat{\beta}$=1.5999	0.0999	0.1999
	$\hat{\theta}$=1.0472	0.0472	0.0144
750	$\hat{\alpha}$=0.9917	-0.0083	0.0049
	$\hat{\beta}$=1.5596	0.0596	0.1101
	$\hat{\theta}$=1.0145	0.0145	0.0068
1000	$\hat{\alpha}$=0.9836	-0.0164	0.0033
	$\hat{\beta}$=1.5187	0.0187	0.0755
	$\hat{\theta}$=0.9967	-0.0033	0.0051

for the blood cancer patients dataset. Furthermore, our proposed distribution is found to fit better than those of the other distributions.

2.5 Summary

In this chapter, a new class of distribution generalizing the DUS transformation, called the PGDUS transformation, is introduced. A new lifetime distribution called the PGDUSE distribution with exponential as the baseline distribution is proposed. The generalized form provides greater flexibility in modeling real datasets. When a parallel system is considered, if the components are distributed as DUS transformations of some baseline models, PGDUS transformation is the only solution. Different statistical properties such as moments, MGF, CF, quantile function, CGF, order statistic, and entropy of the PGDUSE distribution are derived. The parameter

Table 2.14: Blood Cancer Patients Dataset

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23
0.52	4.98	6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09
0.22	13.80	25.74	0.50	2.46	3.64	5.09	7.26	9.47	14.24
0.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81
0.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32
0.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
0.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01
0.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33
0.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64	17.36
0.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46	4.40	5.85
0.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37	12.02	2.02
0.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76	12.07
0.73	2.07	3.36	6.93	8.65	12.63	22.69	5.49		

Table 2.15: Findings for PGDUSL distribution

Model	**MLEs**	$\log \mathbf{L}$	**AIC**	**AICc**	**KS**	**p-value**
LD	$\hat{\lambda} = 15.2817$ $\hat{\theta} = 0.0074$	-414.98	833.960	834.056	0.094	0.208
DUSE	$\hat{\mu} = 0.1342$	-433.139	868.278	868.309	0.081	0.366
DUSL	$\hat{\lambda} = 6.471$ $\hat{\theta} = 0.0253$	-413.077	830.153	830.249	0.075	0.463
PGDUSL	$\hat{\alpha} = 3.842$ $\hat{\beta} = 0.0605$ $\hat{\theta} = 1.3984$	-411.019	828.039	828.2324	0.035	0.998

estimation has been done using the method of maximum likelihood. Monte Carlo simulations are carried out. Real data analysis is performed to show that the

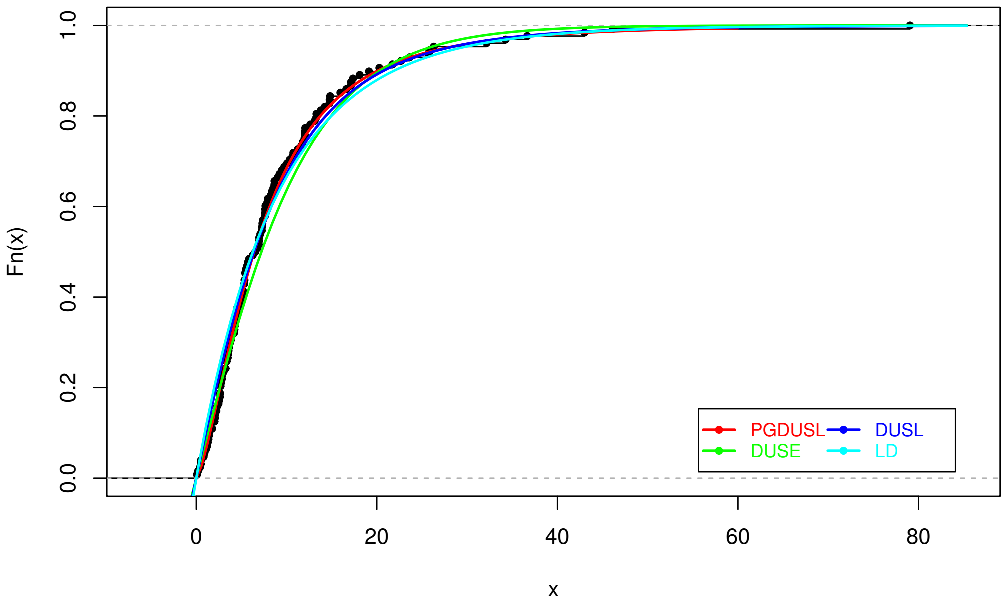

Figure 2.10: ECDF plot of the models for blood cancer patients dataset.

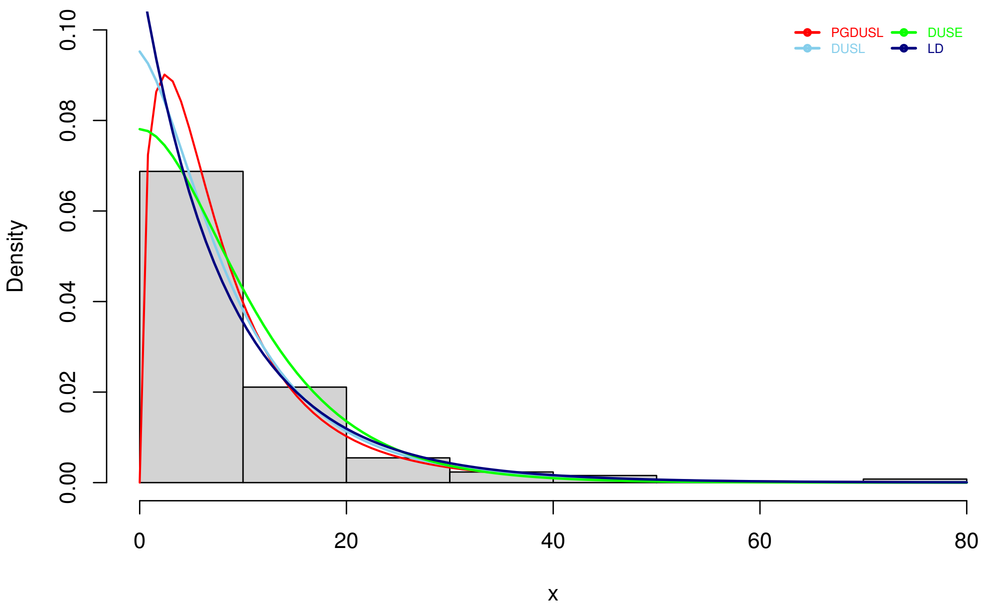

Figure 2.11: Estimated densities of the models for the blood cancer patients dataset.

proposed generalization of the DUS transformation using exponential distributions can be used effectively to provide better fits.

Similarly, the power generalized DUS transformations of Weibull and Lomax distributions have been proposed. Studies on fundamental properties like moments, MGF, CF, CGF, quantile function, distribution of order statistics, and Rényi entropy are also carried out. The parameter estimation has been given by the maximum likelihood method. By using the simulation study, it is observed that the estimates of the proposed distributions have a smaller bias and mean square error when the sample size is large. Real data applications have been performed to determine the applicability of the proposed model. Furthermore, a better fit is adjudged for the proposed model when compared with a few existing models. When conducting reliability analyses on a parallel system where each of the components has a specific DUS-transformed lifetime distribution, the PGDUS approach is highly useful.