CHAPTER 4

Generalized ν-Birnbaum Saunders Distribution

4.1 Introduction

Motivated by problems with vibration in commercial aircraft causing fatigue in the materials, the two-parameter BS distribution, also known as the fatigue life distribution, was proposed by Birnbaum and Saunders (1969a). The model was developed based on the impression that failure is due to the development and growth of a dominant crack. The BS distribution is now a natural model in many instances where the accumulation of a specific factor forces a quantifiable characteristic to exceed a critical threshold. A few examples of instances in which this distribution can be used are (i) heat-induced migration of metallic flaws in nano-circuits; (ii) ingestion of toxic chemicals from industrial waste by humans; (iii) pollution in the atmosphere as a result of an accumulation of pollutants over time; (iv) accumulation of deleterious substances in the lungs from air pollution; (v) events such as earthquakes and tsunamis occurring naturally, and so on. The BS distribution has two parameters modifying its shape and scale: a failure rate with an upside-down bathtub shape and a close relation to the normal distribution; see Leiva (2015).

The CDF of a two-parameter BS random variable T can be written as

$$F_T(t;\alpha,\beta) = \begin{cases} \Phi\left[\dfrac{1}{\alpha}\left(\left(\dfrac{t}{\beta}\right)^{\dfrac{1}{2}} - \left(\dfrac{\beta}{t}\right)^{\dfrac{1}{2}}\right)\right] & \text{if } t > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{4.1.1}$$

with $\alpha > 0$ and $\beta > 0$ being respectively the shape and scale parameters and $\Phi(.)$ is the standard normal CDF. The corresponding PDF of the BS model can be expressed in terms of the PDF of the standard normal distribution and is given by

$$f_T(t;\alpha,\beta) = \begin{cases} \dfrac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\dfrac{\beta}{t}\right)^{\dfrac{1}{2}} + \left(\dfrac{\beta}{t}\right)^{\dfrac{3}{2}}\right] e^{-\dfrac{1}{2\alpha^2}\left(\dfrac{t}{\beta}+\dfrac{\beta}{t}-2\right)} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.2}$$

It is known that the density function of the BS distribution is unimodal, and although the hazard rate is not an increasing function of t, the average hazard rate is nearly a non-decreasing function of t (Mann et al., (1974)).

Often, it is very likely to observe a three-phase behavior of HF in the case of studying the life cycle of an industrial product or the entire life cycle of a biological entity. For example, non-monotone hazard rates involving a U-shaped (bathtub-shaped) pattern are exhibited in the case of the age-specific death rate in human life tables. The core motivation behind developing a more flexible distribution is its capability to model the underlying monotonic and non-monotonic failure rate behavior of the observed data.

In this chapter, a distribution called the ν-Birnbaum Saunders (BS) distribution is discussed, which generalizes the BS model. It is noted here that the BS distribution only has a decreasing or upside-down bathtub shape for its hazard function. It is important to note that the shape of the distribution always depends on the power of the random variable, thus facilitating the development of more flexible models. Chacko et al. (2015) considered a generalization of the BS distribution, incorporating a new shape parameter exhibiting both monotonic and non-monotonic failure rate behaviors, but statistical inference has not been given. Since the estimation of parameters is essential for using any distribution, this chapter provides some structural properties of the distribution and the method of estimation. A discussion on maximum likelihood estimation of the parameters is given and derived the observed information matrix. The use of the distribution is justified by three real-life data sets: the industrial devices data set reported by Aarset (1987), exceedances of flood peaks data given in Choulakian and Stephens (2001), and the insurance data reported in Andrews and Herzberg (2012).

Several extensions and generalizations of the BS distribution are studied by many researchers, including its bivariate and multivariate extensions. The rest of the chapter is organized as follows: In Section 4.2, the $\nu - BS$ distribution, its structural properties, moments, quantiles, and order statistics are given. Also, the estimation procedure is given using the method of maximum likelihood. In addition to this, an extensive simulation study is carried out along with two real-life applications. Section 4.3 is devoted to the bivariate $\nu - BS$ distribution. In section 4.4, the multivariate $\nu - BS$ distribution is defined. The summary is given in the final section.

4.2 Univariate ν-Birnbaum Saunders Distribution

In this section, an extension of the BS distribution is considered, motivated by the work of Chacko et al. (2015), who call this extended version of the BS distribution ν-BS distribution. The study of ν-BS distribution is motivated by three real-life data examples-industrial devices data set, exceedances of flood peaks data, and insurance data. In order to investigate the fitness of the data to the ν-BS distribution, we have to estimate the parameters. So estimation of the parameters of ν-BS distribution is considered in this chapter.

4.2.1 Cumulative Distribution Function

The CDF of a $\nu - BS$ random variable T is given by

$$F(t;\alpha,\beta,\nu) = \begin{cases} \Phi\left(\frac{1}{\alpha}\left\{\left(\frac{t}{\beta}\right)^{\nu} - \left(\frac{t}{\beta}\right)^{-\nu}\right\}\right) & \text{if } t > 0, \alpha, \beta, \nu > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2.1}$$

where $\Phi(.)$ is the standard normal CDF. Here, $\alpha > 0$ and $\beta > 0$ are respectively, the shape parameter and the scale parameter. Note that the parameters α and β in Eq. (4.2.1) are governed by the proposed shape parameter $\nu > 0$. One can obtain the BS distribution in its particular case when $\nu = \frac{1}{2}$.

4.2.2 Probability Density Function

For a random variable T with CDF defined in Eq.(4.2.1), the corresponding PDF is given by

$$f(t;\alpha,\beta,\nu) = \begin{cases} \dfrac{\nu}{\alpha\beta\sqrt{2\pi}}\, e^{-\frac{1}{2\alpha^2}\left[\left(\frac{t}{\beta}\right)^{2\nu}+\left(\frac{t}{\beta}\right)^{-2\nu}-2\right]}\left[\left(\frac{t}{\beta}\right)^{\nu-1}+\left(\frac{t}{\beta}\right)^{-(\nu+1)}\right] & \text{if } t > 0, \\ 0 & \text{otherwise,} \end{cases} \tag{4.2.2}$$

From now on, the notation $T \sim BS(\alpha,\beta,\nu)$ is used to denote a univariate $\nu-$ BS random variable T with parameters α, β, and ν. The PDF in Figure 4.1 has been plotted for different values of the parameters. From the plot, it can be seen that the PDF is unimodal in nature.

4.2.3 Hazard Function

The following section discusses the shape characteristics of the HRF of a BS random variable. With $T \sim BS(\alpha,\beta,\nu)$, the HRF of T is given by

$$h_T(t;\alpha,\beta,\nu) = \frac{f(t;\alpha,\beta,\nu)}{\bar{F}_T(t;\alpha,\beta,\nu)}$$

It is possible to choose $\beta = 1$ without loss of generality since the HRF's form does not depend on the scale parameter β.

$$h_T(t;\alpha,1,\nu) = \frac{\frac{1}{\alpha\sqrt{2\pi}}\ \epsilon'_\nu(t)\ e^{-\frac{1}{2\alpha^2}\epsilon_\nu^2(t)}}{\Phi(-\frac{\epsilon_\nu(t)}{\alpha})} \tag{4.2.3}$$

where $\epsilon_\nu(t) = (t)^\nu - (t)^{-\nu}$, $\epsilon'_\nu(t) = \frac{\nu}{t}\Big((t)^\nu - (t)^{-\nu}\Big)$ and $\epsilon''_\nu(t) = \frac{\nu}{t^2}\Big((\nu-1)t^\nu - (\nu+1)t^{-\nu}\Big)$.

Kundu et al. (2008) then showed that the HRF in Eq. (4.2.3) is always

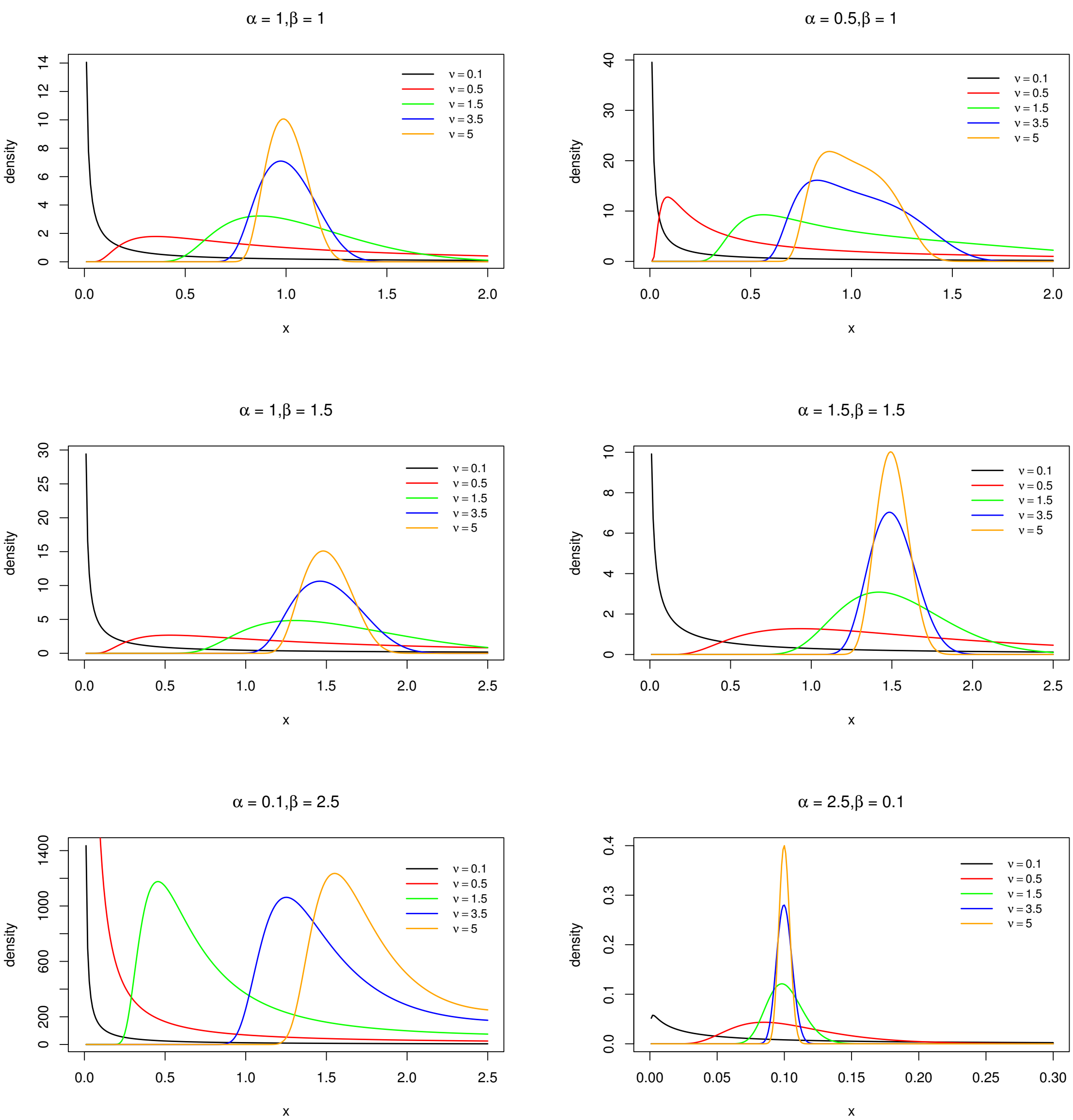


Figure 4.1: Probability density function plots

unimodal. The plots of the HF of $BS(\alpha, \beta, \nu)$ in Eq.(4.2.3) for different values of α and ν, are presented in Figure 4.2. Whenever $0 < \nu < 1$, from (4.2.3) it can shown that $\ln(h_T(t; \alpha, 1, \nu)) \to 1/2\alpha^2$ as $t \to \infty$.

Moments

If $T \sim BS(\alpha, \beta, \nu)$ (T has a ν-BS distribution with parameters α, β and ν), the moments of the random variable T can be obtained by making the following transformation:

$$Z = \frac{1}{\alpha}\left[\left(\frac{T}{\beta}\right)^{\nu} - \left(\frac{T}{\beta}\right)^{-\nu}\right]$$

or

$$T = \frac{\beta}{2^{1/\nu}}\left[\alpha Z + \sqrt{4 + (\alpha Z)^2}\right]^{1/\nu} = \beta\left[\frac{W}{\beta}\right]^{1/2\nu} \tag{4.2.4}$$

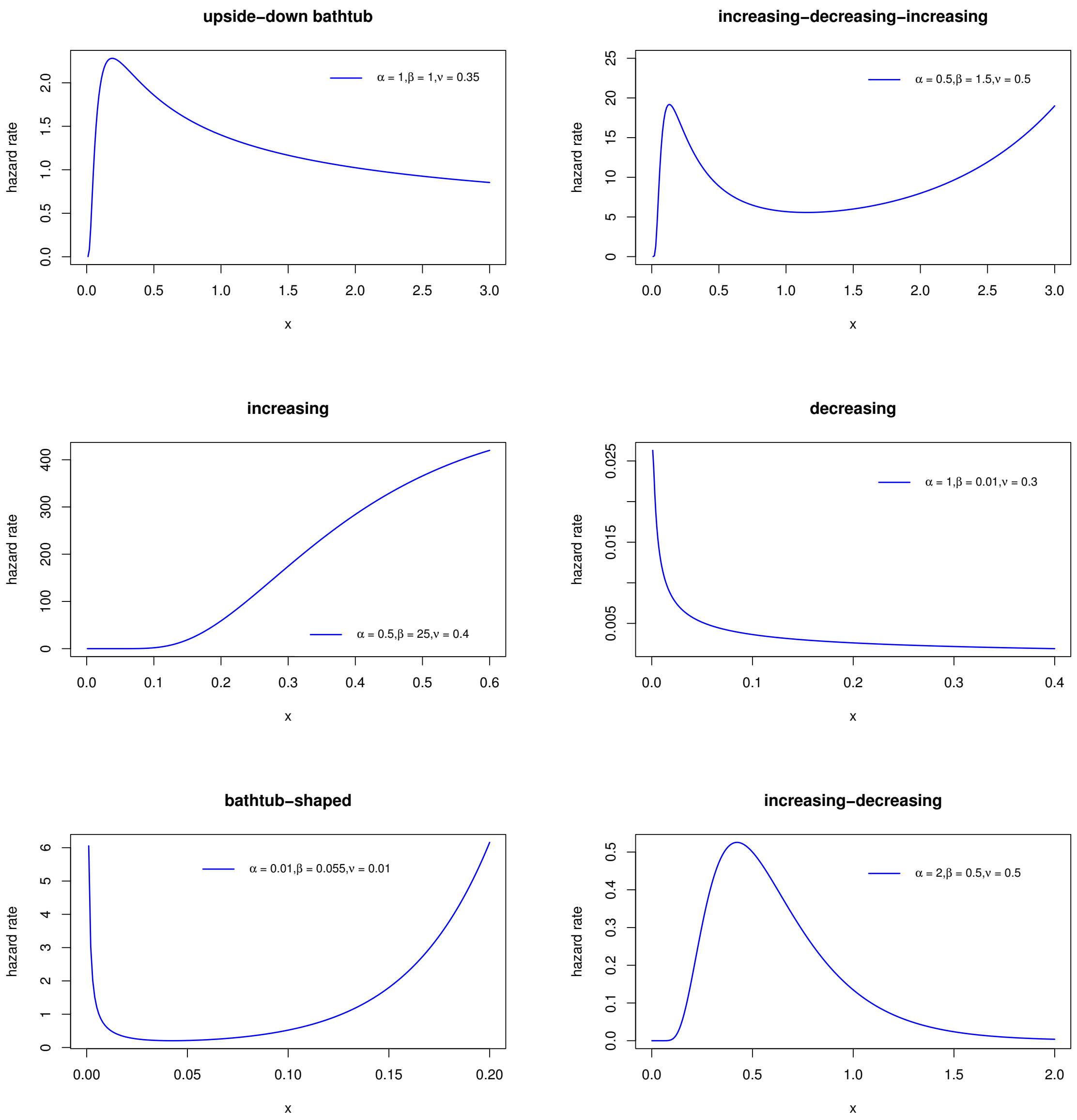


Figure 4.2: Failure rate function plots for different parameter values.

or

$$T^{2\nu} = \beta^{2\nu-1} W \tag{4.2.5}$$

where $W = \frac{\beta}{4}\Big[\alpha Z + \sqrt{1+(\alpha Z)^2}\Big]^2 \sim BS(\alpha,\beta)$ and $Z \sim N(0,1)$. Hence,

$$\begin{aligned} E(T^r) &= \beta^r E\Bigg[\left(\frac{W}{\beta}\right)^{\frac{r}{2\nu}}\Bigg] \\ &= \beta^{r-\frac{r}{2\nu}} E\Bigg[W^{\frac{r}{2\nu}}\Bigg] \end{aligned}$$

Now in case if $r/2\nu$ is an integer then

$$E(T^r) = \beta^r \sum_{j=1}^{r/\nu} \binom{r/2\nu}{2j} \sum_{i=0}^{j} \frac{(r/\nu - 2j + 2i)!}{2^{r/2\nu - j + i}(r/2\nu - j + i)!} \left(\frac{\alpha}{2}\right)^{r/\nu - 2j + 2i} \tag{4.2.6}$$

(see Leiva et al. (2009) in this regard). Rieck (1999) also obtained $E(T^r)$, for fractional values of $r/2\nu$, in terms of the Bessel function, from the MGF of $E(\ln(W))$. For $r = 2\nu$, then

$$E(T^{2\nu}) = \beta^{2\nu-1} E(W) = \frac{\beta^{2\nu}}{2}(\alpha^2 + 2). \tag{4.2.7}$$

If $T \sim BS(\alpha,\beta,\nu)$, then it can be easily shown that $T^{-1} \sim BS(\alpha,\beta^{-1},\nu^{-1})$ (T has a ν-BS distribution with parameters α, β^{-1} and ν^{-1}). Therefore, for integer r, it can be readily obtained from Eq. (4.2.6) that

$$E(T^{-r}) = \beta^{-r} \sum_{j=1}^{r\nu} \binom{r\nu/2}{2j} \sum_{i=0}^{j} \frac{(r\nu - 2j + 2i)!}{2^{r\nu/2 - j + i}(r\nu/2 - j + i)!} \left(\frac{\alpha}{2}\right)^{r\nu - 2j + 2i}. \tag{4.2.8}$$

For $r = 2\nu$, then

$$E(T^{-2\nu}) = \beta^{-2\nu+1} E(W^{-1}) = \frac{\beta^{-2\nu}}{2}(\alpha^2 + 2). \tag{4.2.9}$$

Quantiles

Quantiles can be obtained as a solution to the equation $F_T(t_q) = q$, where t_q is the qth quantile. Hence,

$$\Phi\left[\frac{1}{\alpha}\left\{\left(\frac{t_q}{\beta}\right)^{\nu} - \left(\frac{\beta}{t_q}\right)^{\nu}\right\}\right] = q.$$

Now, solving the above equation, the qth quantile $(0 < q < 1)$ can be written as

$$t_q = \frac{\beta}{2^{\frac{1}{\nu}}}\left(\alpha z_q + \sqrt{(\alpha z_q)^2 + 4}\right)^{\frac{1}{\nu}}, \tag{4.2.10}$$

where $z_q = \Phi^{-1}(q)$ is the qth quantile of a standard normal random variable. Then, using the ν- BS quantile function, that is, the inverse transform method, a generator of random numbers for the ν-BS distribution is summarized in the following Algorithm 1.

Algorithm 1 : Generator of random numbers from $\nu - BS$ distribution.

1: Generate a random number z from $Z \sim N(0, 1)$.
2: Set values for α, β and ν of $T \sim BS(\alpha, \beta, \nu)$.
3: Compute a random number t from $T \sim BS(\alpha, \beta, \nu)$ by using Eq. (4.2.10) conducting to

$$t = \frac{\beta}{2^{1/\nu}}\left[\alpha z + \sqrt{4 + (\alpha z)^2}\right]^{1/\nu}.$$

4: Repeat steps 1 to 3 until the required amount of random numbers to be completed.

Order Statistics

Order statistics make their appearance in many areas of statistical theory and practice. The density function $f_{p:n}(t)$ of the p-th order statistic $T_{p:n}$, for $p = 1, \ldots, n$, from independent identically distributed $BS(\alpha, \beta, \nu)$ random variables $T_1, \ldots, T_n$ is given by

$$f_{p:n}(t) = \frac{f(t)}{B(p, n-p+1)} F(t)^{p-1}[1 - F(t)]^{n-p}.$$

For convenience, let us consider the Eq.(4.2.1) and Eq.(4.2.2) as

$$F(t) = \Phi(\mu_t) \tag{4.2.11}$$

where $\mu_t = \frac{1}{\alpha}\epsilon_\nu(\frac{t}{\beta})$ and

$$f(t) = \phi(\mu_t)M_t \tag{4.2.12}$$

where $M_t = d\mu_t/dt$ and $\phi(.)$ is the standard normal density function. As a result of substituting Eq.(4.2.11) and Eq.(4.2.12) into the above expression,

$$f_{p:n}(t) = \frac{\phi(\mu_t)M_t}{B(p, n-p+1)}[\Phi(\mu_t)]^{p-1}[1-\Phi(\mu_t)]^{n-p}.$$

The above PDF can be expressed in terms of the binomial expansion as

$$f_{p:n}(t) = \frac{\phi(\mu_t)M_t}{B(p, n-p+1)}\sum_{k=0}^{n-p}(-1)^k\binom{n-p}{k}[\Phi(\mu_t)]^{p+k-1}.$$

Thus, this PDF of the $BS(\alpha, \beta, \nu)$ order statistics can be reduced to

$$f_{p:n}(t) = \sum_{k=0}^{n-p} m_k f(t),\ t > 0 \tag{4.2.13}$$

where $m_{i+1} = \frac{(-1)^k\binom{n-p}{k}[\Phi(\mu_t)]^{p+k-1}}{B(p,n-p+1)}$ and $f(t)$ is in Eq.(4.2.12). As a result, the PDF Eq.(4.2.13) of $BS(\alpha, \beta, \nu)$ order statistics can be viewed as a linear combination of the $BS(\alpha, \beta, \nu)$ density functions. In this way, many mathematical properties of $BS(\alpha, \beta, \nu)$ order statistics, such as moments and the generating function, can be determined from the $BS(\alpha, \beta, \nu)$ distribution.

4.2.4 Estimation and Testing of Hypothesis

In this Section, the estimation methodologies for the unknown parameters in the case of the ν-BS distribution are first discussed. The likelihood ratio (LR) test is then discussed in this setup.

Point Estimation

The point estimation of the parameters of the ν-BS distribution by the method of maximum likelihood is considered.

1. Complete data case: Let $\boldsymbol{T} = \{T_1, T_2, ..., T_n\}$ be a random sample of size n

and $\boldsymbol{\theta} = (\alpha, \beta, \nu)$ be the unknown parameter vector. Based on the random sample, $\widehat{\boldsymbol{\theta}}$, the MLE of $\boldsymbol{\theta}$, can be obtained by maximizing the log-likelihood function. The associated likelihood and the log-likelihood function are respectively given by

$$L(\boldsymbol{\theta}|\boldsymbol{t}) = \left(\frac{\nu}{\alpha\beta\sqrt{2\pi}}\right)^{n} e^{-\frac{1}{2\alpha^2}\sum_{i=1}^{n}\left[\left(\frac{t_i}{\beta}\right)^{2\nu}+\left(\frac{t_i}{\beta}\right)^{-2\nu}-2\right]} \prod_{i=1}^{n}\left[\left(\frac{t_i}{\beta}\right)^{\nu-1} + \left(\frac{t_i}{\beta}\right)^{-(\nu+1)}\right], \tag{4.2.14}$$

and

$$l(\boldsymbol{\theta}|\boldsymbol{t}) = n\ln\nu - n\ln\alpha - n\ln\beta - \frac{n}{2}\ln(2\pi) - \frac{1}{2\alpha^2}\sum_{i=1}^{n}\left[\left(\frac{t_i}{\beta}\right)^{2\nu} + \left(\frac{t_i}{\beta}\right)^{-2\nu} - 2\right]$$

$$+\sum_{i=1}^{n}\log\left[\left(\frac{t_i}{\beta}\right)^{\nu-1} + \left(\frac{t_i}{\beta}\right)^{-(\nu+1)}\right], \tag{4.2.15}$$

where $\boldsymbol{t} = \{t_1,\ t_2,...,t_n\}$ is the observed sample. The components of the score vector $\boldsymbol{U(\theta)} = (U_\alpha, U_\beta, U_\nu)^T$ are

$$U_\alpha = \frac{-n}{\alpha} + \frac{1}{\alpha^3}\sum_{i=1}^{n}\left[\left(\frac{t_i}{\beta}\right)^{2\nu} + \left(\frac{\beta}{t_i}\right)^{2\nu} - 2\right]$$

$$U_\beta = \frac{n}{\beta} + \frac{\nu}{\alpha^2}\sum_{i=1}^{n}\left[\frac{\beta^{2\nu-1}}{t_i^{2\nu}} - \frac{t_i^{2\nu}}{\beta^{2\nu+1}}\right] + (\nu+1)\sum_{i=1}^{n}\frac{\frac{\beta^\nu}{t_i^{\nu+1}} - \frac{t_i^{\nu-1}}{\beta^\nu}}{(\frac{t_i}{\beta})^{\nu-1} + (\frac{\beta}{t_i})^{\nu+1}}$$

$$U_\nu = \tfrac{n}{\nu} - \tfrac{1}{\alpha^2}\textstyle\sum_{i=1}^{n}\left[\left(\frac{t_i}{\beta}\right)^{2\nu}\log\left(\frac{t_i}{\beta}\right) + \left(\frac{\beta}{t_i}\right)^{2\nu}\log\left(\frac{\beta}{t_i}\right)\right] + \sum_{i=1}^{n}\frac{(\frac{t_i}{\beta})^{\nu-1}\log(\frac{t_i}{\beta}) + (\frac{\beta}{t_i})^{\nu+1}\log(\frac{\beta}{t_i})}{(\frac{t_i}{\beta})^{\nu-1} + (\frac{\beta}{t_i})^{\nu+1}}$$

Setting these equations to zero, $\boldsymbol{U(\theta)} = 0$, and solving them simultaneously yields $\widehat{\boldsymbol{\theta}}$ of the three parameters. From the score equation $U_\alpha = 0$, it can be written as

$$\widehat{\alpha} = \widehat{\alpha}(\beta, \nu) = \left\{\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{t_i}{\beta}\right)^{2\nu} + \left(\frac{\beta}{t_i}\right)^{2\nu} - 2\right]\right\}^{\frac{1}{2}}. \tag{4.2.16}$$

Plugging in $\widehat{\alpha}$ replacing α in the log-likelihood function $l(\boldsymbol{\theta}|\boldsymbol{t})$, the profile log-likelihood function of β and ν is obtained first and then maximized using some numerical routine to obtain $\widehat{\beta}$ and $\widehat{\nu}$. Finally, $\widehat{\alpha} = \widehat{\alpha}(\widehat{\beta}, \widehat{\nu})$ is obtained.

2. Multicensored data case: More often, censored data occur in lifetime data analysis. Some basic mechanisms of censoring are well known in the literature as, for example, Type-I and Type-II censoring. The survival function of the ν-BS distribution has a simple convenient form and hence this distribution can be employed in analyzing censored data. In this context, the general case of multicensored data is considered. Suppose there are $n = n_0 + n_1 + n_2$ units of which n_0 are known to have failed at the times $t_1, \ldots, t_{n_0}$; n_1 are known to have failed in the interval $[s_{i-1}, s_i]$ for $i = 1, \ldots, n_1$; and n_2 units have survived at least till a time r_i $(i = 1, ..., n_2)$ but not observed any longer. It is to note here that Type-I and Type-II censoring are contained as particular cases of multicensoring. The log-likelihood function of $\boldsymbol{\theta} = (\alpha, \beta, \nu)$ for this multicensored data takes the following form:

$$
\begin{aligned}
l(\boldsymbol{\theta}|\boldsymbol{t}) \propto\ & n_0 \ln \nu - n_0 \ln(\alpha\beta) - \frac{1}{2\alpha^2} \sum_{i=1}^{n_0} \left[\left(\frac{t_i}{\beta}\right)^{2\nu} + \left(\frac{t_i}{\beta}\right)^{-2\nu} - 2 \right] \\
& + \sum_{i=1}^{n_0} \log \left[\left(\frac{t_i}{\beta}\right)^{\nu-1} + \left(\frac{t_i}{\beta}\right)^{-(\nu+1)} \right] \\
& + \sum_{i=1}^{n_2} \log \left[1 - \Phi\left(\frac{1}{\alpha} \left\{ \left(\frac{r_i}{\beta}\right)^{\nu} - \left(\frac{r_i}{\beta}\right)^{-\nu} \right\} \right) \right] \\
& + \sum_{i=1}^{n_1} \log \left[\Phi\left(\frac{1}{\alpha} \left\{ \left(\frac{s_i}{\beta}\right)^{\nu} - \left(\frac{s_i}{\beta}\right)^{-\nu} \right\} \right) \right] - \sum_{i=1}^{n_1} \log \left[\Phi\left(\frac{1}{\alpha} \left\{ \left(\frac{s_{i-1}}{\beta}\right)^{\nu} - \left(\frac{s_{i-1}}{\beta}\right)^{-\nu} \right\} \right) \right].
\end{aligned}
\tag{4.2.17}
$$

The MLEs are obtained by maximizing the above log-likelihood function with respect to unknown parameters. It is not possible to obtain any of the MLEs as a function of one or others. One requires either carrying out a three-dimensional maximization of the objective function $l(\boldsymbol{\theta}|\boldsymbol{t})$ in Eq. (4.2.17) or obtaining the score vector and solving them to obtain $\boldsymbol{\theta}$.

Interval Estimation

Assuming the asymptotic normality of the MLEs, the CIs for $\boldsymbol{\theta}$ are computed using the observed information matrix $I = \left(\left(\dfrac{\partial l(\boldsymbol{\theta}|\boldsymbol{t})}{\partial\theta_i\partial\theta_j}\right)\right)$, $i, j = 1, 2, 3$, where $l(\boldsymbol{\theta}|\boldsymbol{t})$ is the log-likelihood function as defined in Eq.(4.2.15). The $100(1-\gamma)\%$ asymptotic CIs for $\boldsymbol{\theta}$ are respectively given by $\widehat{\alpha} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{11}}$, $\widehat{\beta} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{22}}$, $\widehat{\nu} \pm z_{1-\frac{\gamma}{2}}\sqrt{V_{33}}$ where V_{ij} is the $(i, j)-$th element of the inverse of the observed Fisher information matrix I. This interval estimation method is quite useful for its computational ease and provides coverage probabilities close to the nominal value.

Testing of Hypothesis

In this context, it is worthwhile to mention that the LR statistic often turns out to be useful for testing the goodness-of-fit of the ν-BS model and for comparing it with the usual BS model. One can easily check if the fit using the ν-BS model is statistically "superior" to a fit using the BS model for a given data set by computing

$$w = 2\{l(\widehat{\alpha}, \widehat{\beta}, \widehat{\nu}|\boldsymbol{t}) - l(\widetilde{\alpha}, \widetilde{\beta}, 0.5|\boldsymbol{t})\},$$

where $\widehat{\alpha}, \widehat{\beta}, \widehat{\nu}$ are the unrestricted MLEs and $\widetilde{\alpha}, \widetilde{\beta}$ are the restricted estimates. Also, the LR statistic is asymptotically distributed under the null model as χ^2 distribution with 1 degree of freedom. Further, the LR test rejects the null hypothesis if $w > \eta_n$, where η_n denotes the upper $100\eta\%$ point of the χ^2 distribution with 1 degree of freedom.

4.2.5 Simulation Study

In this section, a simulation study is performed with various sample sizes and parameter values to assess the effectiveness of the proposed estimation methodology. For illustration purposes, different sample sizes are considered $(n = 40, 60, 80, 100, 120)$ and the parameter values are taken as $\alpha = 2, \beta = 1, \nu = 1.5$. Based on the likelihood principle, the average estimates (AEs), MSEs, and biases for each unknown model parameter are computed. When it comes to the interval estimation problem, it is noted that the exact distribution of the MLEs is not possible to compute. Hence, interval estimates are computed in terms of asymptotic CIs. All the results are based on 5000 replications and are available in Table 4.1.

Some of the observations are quite evident from the results obtained in Table

4.1. As the sample size increases, the AEs approach the true values of the model parameters in all cases, and the corresponding MSEs decrease. In the case of the results associated with the interval estimates, the performance of asymptotic CIs is quite satisfactory in terms of coverage probabilities (CPs). With the increase in sample sizes, the average lengths (ALs) of all the model parameters decrease, which is quite expected.

Table 4.1: MLEs, MSEs, Biases, CPs and ALs for $\nu - BS$ model with $\alpha = 2, \beta = 1$ and $\nu = 1.5$

n	MLEs	MSE	Bias	CP	AL
40	$\hat{\alpha}$= 2.3355	2.0985	0.6355	0.9875	4.5109
	$\hat{\beta}$=1.0004	0.0053	0.0004	0.9154	0.2310
	$\hat{\nu}$=1.7261	0.3938	0.2261	0.9028	1.9249
60	$\hat{\alpha}$=2.2167	1.1183	0.4167	0.9930	3.9053
	$\hat{\beta}$=1.0047	0.0039	0.0047	0.9321	0.2139
	$\hat{\nu}$=1.6515	0.2552	0.1515	0.9261	1.7610
80	$\hat{\alpha}$=2.1666	0.7898	0.3666	0.9883	3.2447
	$\hat{\beta}$=1.0016	0.0026	0.0016	0.9201	0.1855
	$\hat{\nu}$=1.6483	0.1769	0.1483	0.9298	1.4733
100	$\hat{\alpha}$=2.1259	0.5568	0.2259	0.9710	2.8059
	$\hat{\beta}$=1.0025	0.0019	0.0025	0.9171	0.1666
	$\hat{\nu}$=1.5887	0.1248	0.0887	0.9411	1.3035
120	$\hat{\alpha}$= 2.1930	0.4671	0.1930	0.9710	2.4835
	$\hat{\beta}$=1.0015	0.0016	0.0015	0.9271	0.1537
	$\hat{\nu}$=1.5749	0.1001	0.0749	0.9461	1.1789

4.2.6 Real Life Applications

In the following, applications of the ν-BS distribution to real data are presented for illustrative purposes. In order to show how well the ν-BS distribution can be applied to real-life phenomena, three real-life data sets are used- industrial devices data given by Aarset (1987), exceedances of flood peaks data given in Choulakian and Stephens (2001), and insurance data reported in Andrews and Herzberg (2012).

Industrial devices data

At first, industrial devices' real-life data set are considered (see Aarset (1987) in this respect) which is given in Table 4.2. This data set represents the lifetimes of 50 industrial devices put on life tests at time zero. In real data applications, several authors studied this data set for different statistical models since it presents a bathtub-shaped failure rate, see for example, Ahmed (2014) and Kayal et al. (2017). A detailed summary of these data is provided in Table 4.3.

Table 4.2: Industrial devices data

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

The MLEs of all the model parameters are computed based on the principle of maximum likelihood. Despite our inability to theoretically verify the unimodality of the profile log-likelihood function of β and ν, the contour plot in Figure 4.3(a) indicates that the function is indeed unimodal. The K-S distance is also reported along with the p-value for the goodness of fit. It is observed that both the BS distribution and $\nu-$BS distribution fit the data well. However, based on the Maximum log-likelihood (MLL) value, K-S distance, and AIC value, it can be seen

Table 4.3: Descriptive statistics: Industrial devices data

Mean	Median	Variance	Skewness	Kurtosis	Minimum	Maximum
35.8800	34.0000	861.6100	-0.1400	1.4100	0.1000	83.0000

that the proposed $\nu-$BS distribution outperforms the BS distribution. All the associated results are listed in Table 4.4. The LR statistic to test the hypothesis H_0 : BS against H_1 : $\nu-$BS is 52.6200 (p-value < 0.01). Thus, using any usual significance level, the null hypothesis is rejected in favor of the $\nu-$BS distribution, i.e., the $\nu-$BS distribution is significantly better than the BS distribution.

Table 4.4: MLEs (standard errors in parentheses), K-S distance, p-values, MLL values, and AIC values: industrial devices Aarset data set

Distribution	Estimates			K-S distance	p-value	MLL	AIC
BS(α, β, ν)	31.9352	3.8157	1.2286	0.1543	0.8356	-227.1600	460.3200
	(18.9847)	(0.4530)	(0.1769)				
BS(α, β)	2.7455	7.1877		0.1783	0.7798	-253.4700	510.9400
	(0.2982)	(1.5499)					

Exceedances of flood peaks data

For our second real-life illustration, a data set corresponding to the exceedances of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada is considered. The data consist of 72 exceedances for the years 1958–1984, rounded to one decimal place (see Choulakian and Stephens (2001) in this respect) and are given in Table 4.5. Table 4.6 gives a descriptive summary of these data.

The MLEs of all the model parameters are computed based on the principle of maximum likelihood. Despite our inability to theoretically verify the unimodality of the profile log-likelihood function of β and ν, the contour plot in Figure 4.3(b) indicates that the function is indeed unimodal. The K-S distance is also reported

Table 4.5: Exceedances of flood peaks data

1.7	2.2	14.4	1.1	0.4	20.6	5.3	0.7	1.9	13.0	12.0	9.3
1.4	18.7	8.5	25.5	11.6	14.1	22.1	1.1	2.5	14.4	1.7	37.6
0.6	2.2	39.0	0.3	15.0	11.0	7.3	22.9	1.7	0.1	1.1	0.6
9.0	1.7	7.0	20.1	0.4	2.8	14.1	9.9	10.4	10.7	30.0	3.6
5.6	30.8	13.3	4.2	25.5	3.4	11.9	21.5	27.6	36.4	2.7	64.0
1.5	2.5	27.4	1.0	27.1	20.2	16.8	5.3	9.7	27.5	2.5	27.0

Table 4.6: Descriptive statistics: exceedances of flood peaks data

Mean	**Median**	**Variance**	**Skewness**	**Kurtosis**	**Minimum**	**Maximum**
12.2000	9.5000	151.2200	1.4700	5.8900	0.1000	64.0000

along with the p-value for the goodness of fit. It is observed that both the BS distribution and $\nu-$ BS distribution fit the data well. However, based on the MLL value, K-S distance, and AIC value, it can be seen that the proposed $\nu-$ BS distribution outperforms the BS distribution. All the associated results are listed in Table 4.10. The LR statistic to test the hypothesis H_0 : BS against H_1 : $\nu-$BS is 50.3400 (p-value < 0.01). Thus, the null hypothesis is rejected in favor of the $\nu-$BS distribution using any usual significance level. Therefore, the $\nu-$BS distribution is significantly better than the BS distribution based on the LR statistic.

Insurance data

Finally, the data representing Swedish third-party motor insurance for 1977 for one of several geographical zones are considered. The data were compiled by a Swedish committee on the analysis of risk premiums in motor insurance. The data points

Table 4.7: MLEs (standard errors in parentheses), K-S distance, p-values, MLL values, and AIC values: exceedances of flood peaks data set

Distribution	Estimates			K-S distance	p-value	MLL	AIC
BS(α, β, ν)	1.0897	5.1582	0.3481	0.1404	0.5996	-230.8600	467.7200
	(0.9356)	(1.4117)	(0.2447)				
BS(α, β)	1.7583	4.4179		0.1457	0.5470	-256.0300	516.2300
	(0.1477)	(0.6497)					

are the aggregate payments by the insurer in thousand Skr (Swedish currency). The data set was originally reported in Andrews and Herzberg (2012) and is as provided in Table 4.8. Table 4.9 gives a descriptive summary of these data.

Table 4.8: Insurance data

5014	5855	6486	6540	6656	6656
7212	7541	7558	7797	8546	9345
11762	12478	13624	14451	14940	14963
15092	16203	16229	16730	18027	18343
19365	21782	24248	29069	34267	38993

Table 4.9: Descriptive statistics: Insurance data

Mean	Median	Variance	Skewness	Kurtosis	Minimum	Maximum
14525.7300	14037.5000	69927726	1.3016	1.6004	5014	38993

MLEs of all the model parameters are computed based on the principle of maximum likelihood. Despite our inability to theoretically verify the unimodality of the profile log-likelihood function of β and ν, the contour plot in Figure 4.3(c) indicates that the function is indeed unimodal. The K-S distance is also reported along with the p-value for the goodness of fit. It is observed that both the BS distribution and $\nu-$ BS distribution fit the data well. However, based on the MLL value, K-S distance, and AIC value, it can be seen that the proposed $\nu-$ BS distribution outperforms the BS distribution. All the associated results are listed in Table 4.10. The LR statistic to test the hypothesis H_0 : BS against H_1 : $\nu-$BS is 7.1230 (p-value = $0.0076 < 0.01$). Thus, the null hypothesis is rejected in favor of the $\nu-$BS distribution using any usual significance level. Therefore, the $\nu-$BS distribution is significantly better than the BS distribution based on the LR statistic.

Table 4.10: MLEs (standard errors in parentheses), K-S distance, p-values, MLL values, and AIC values: insurance data set

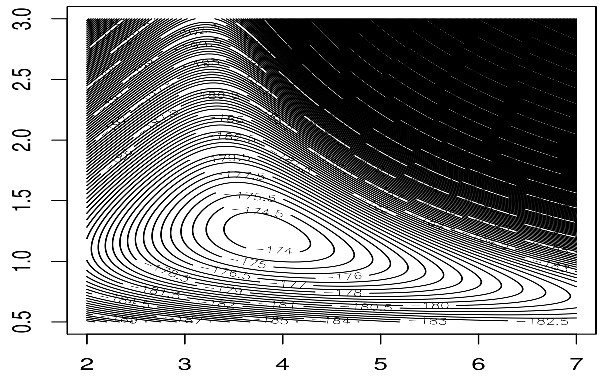
Distribution	**Estimates**			**K-S distance**	**p-value**	**MLL**	**AIC**
BS(α, β, ν)	2.4285	1.3219	1.6654	0.1305	0.7052	-16.7831	39.5662
	(1.4121)	(0.1069)	(0.5985)				
BS(α, β)	0.5595	1.2559		0.1385	0.6130	-20.3446	44.6892
	(0.0722)	(0.1233)					

4.3 Bivariate ν- Birnbaum Saunders Distribution

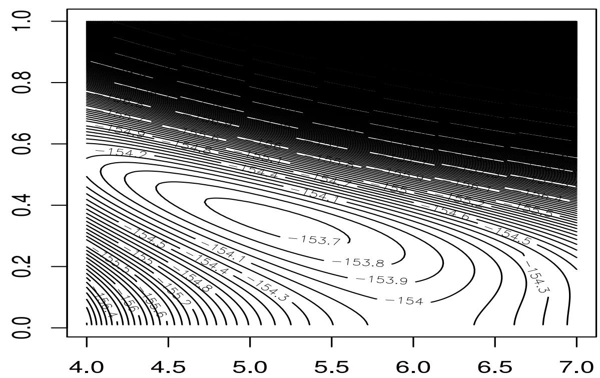
In this section, a new generalized form of BVBS distribtion is proposed and call it a ν-BVBS distribution.

4.3.1 CDF, PDF, and HRF of ν- BVBS Distribution

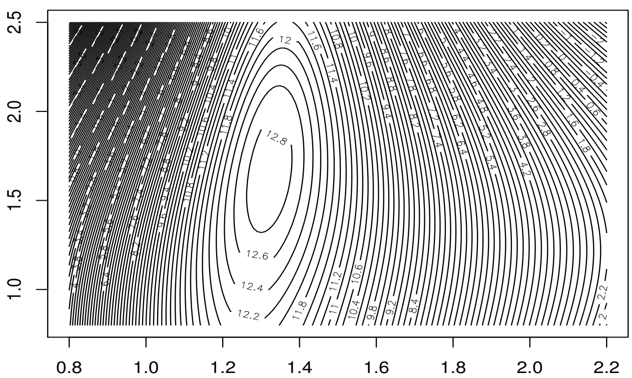
The joint CDF of a ν-BVBS random vector (T_1, T_2) with parameters $\alpha_1, \beta_1, \nu_1, \alpha_2, \beta_2, \nu_2$, and ρ can be written as



(a)



(b)



(c)

Figure 4.3: Contour plot of β and ν in (a)industrial devices data, (b)exceedances of flood peaks data and (c)insurance data using $\nu-$ Birnbaum Saunders distribution

$$F(t_1,t_2)=\Phi_2\left[\frac{1}{\alpha_1}\left(\left(\frac{t_1}{\beta_1}\right)^{\nu_1}-\left(\frac{\beta_1}{t_1}\right)^{\nu_1}\right),\frac{1}{\alpha_2}\left(\left(\frac{t_2}{\beta_2}\right)^{\nu_2}-\left(\frac{\beta_2}{t_2}\right)^{\nu_2}\right);\rho\right];t_1>0,t_2>0 \tag{4.3.1}$$

Here $\alpha_1 > 0, \beta_1 > 0, \alpha_2 > 0, \beta_2 > 0, -1 < \rho < 1$ and $\Phi_2(.;\rho)$ is CDF of standard BV normal vector (z_1, z_2) with correlation coefficient ρ. One can obtain the BVBS distribution in its particular case when $\nu_1 = \nu_2 = \frac{1}{2}$. For a BV random vector (T_1, T_2) with CDF as in Eq. (4.3.1), the corresponding joint PDF is given by

$$f_{T_1,T_2}(t_1,t_2)=\frac{\nu_1\nu_2}{2\pi\alpha_1\alpha_2\beta_1\beta_2\sqrt{1-\rho^2}}\left[\left(\frac{t_1}{\beta_1}\right)^{\nu_1-1}+\left(\frac{\beta_1}{t_1}\right)^{\nu_1+1}\right]\left[\left(\frac{t_2}{\beta_2}\right)^{\nu_2-1}+\left(\frac{\beta_2}{t_2}\right)^{\nu_2+1}\right]$$

$$exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{1}{\alpha_1^2}\left(\left(\frac{t_1}{\beta_1}\right)_1^{\nu}-\left(\frac{\beta_1}{t_1}\right)_1^{\nu}\right)^2+\frac{1}{\alpha_2^2}\left(\left(\frac{t_2}{\beta_2}\right)_2^{\nu}-\left(\frac{\beta_2}{t_2}\right)_2^{\nu}\right)^2\right.\right.$$

$$\left.\left.-\frac{2\rho}{\alpha_1\alpha_2}\left[\left(\frac{t_1}{\beta_1}\right)_1^{\nu}-\left(\frac{\beta_1}{t_1}\right)_1^{\nu}\right]\left[\left(\frac{t_2}{\beta_2}\right)_2^{\nu}-\left(\frac{\beta_2}{t_2}\right)_2^{\nu}\right]\right]\right\}$$

4.3.2 Properties of ν-BVBS Distribution

1. If $(T_1, T_2) \sim$ BVBS $(\alpha_1, \beta_1, \nu_1, \alpha_2, \beta_2, \nu_2, \rho)$ then it can be easily shown that its marginals, T_i, $\sim \nu - BS(\alpha_i, \beta_i, \nu_i)$

2. If $(T_1, T_2) \sim \nu - BS(\alpha_1, \beta_1, \nu_1, \alpha_2, \beta_2, \nu_2, \rho)$ then

   - $(T_1^{-1}, T_2^{-1}) \sim \nu - BS(\alpha_1, \frac{1}{\beta_1}, \nu_1, \alpha_2, \frac{1}{\beta_2}, \nu_2, \rho)$
   - $(T_1^{-1}, T_2) \sim \nu - BS(\alpha_1, \frac{1}{\beta_1}, \nu_1, \alpha_2, \beta_2, \nu_2, \rho)$
   - $(T_1, T_2^{-1}) \sim \nu - BS(\alpha_1, \beta_1, \nu_1, \alpha_2, \frac{1}{\beta_2}, \nu_2, \rho)$

4.4 Multivariate ν- Birnbaum Saunders Distribution

Along the same lines as the univariate and bivariate $\nu-$BS distribution, the multivariate $\nu-$BS distribution can be defined. First, let us recall the definition of the multivariate BS distribution [see Eq. 1.2.4].

Then the multivariate $\nu-$BS distribution is as follows:

Definition 4.4.1. *Let $\underline{\alpha}, \underline{\beta} \in \mathbb{R}^m$, where $\underline{\alpha} = (\alpha_1, \cdots, \alpha_m)^T$ and $\underline{\beta} = (\beta_1, \cdots, \beta_m)^T$, with $\alpha_1 > 0, \beta_i > 0$ for $i = 1, 2, \cdots, m$. Let $\mathbf{\Gamma}$ be a $m \times m$ positive definite correlation matrix. Then, the random vector $\underline{T} = (T_1, \cdots, T_m)^T$ is said to have a m-variate BS distribution with parameters $(\underline{\alpha}, \underline{\beta}, \mathbf{\Gamma}, \nu)$ if it has the joint CDF as*

$$\begin{aligned} P(\underline{T} \leq \underline{t}) &= P(T_1 \leq t_1, \cdots, T_m \leq t_m) \\ &= \Phi_m\left[\frac{1}{\alpha_1}\left(\left(\frac{t_1}{\beta_1}\right)^{\nu} - \left(\frac{\beta_1}{t_1}\right)^{\nu}\right), \cdots, \frac{1}{\alpha_m}\left(\left(\frac{t_m}{\beta_m}\right)^{\nu} - \left(\frac{\beta_m}{t_m}\right)^{\nu}\right); \Gamma\right] \end{aligned}$$

for $t_1 > 0, \cdots, t_m > 0$ and $0 < \nu < 1$. Here, for $\underline{u} = (u_1, \cdots, u_m)^T$, $\Phi_m(\underline{u}; \mathbf{\Gamma})$ denotes the joint CDF of a standard normal vector $\underline{Z} = (Z_1, \cdots, Z_m)^T$ with correlation matrix $\mathbf{\Gamma}$.

4.5 Summary

This chapter considers the univariate, bivariate, and multivariate $\nu-$ Birnbaum Saunders distributions and mainly focuses on the univariate case. Several interesting and useful properties are studied in detail. The point estimates of the model parameters of the univariate $\nu-$ Birnbaum Saunders distribution are obtained by employing the maximum likelihood principle. In order to obtain interval estimates, asymptotic CIs are computed using the observed information matrix. In an extensive simulation study, both estimation methodologies were thoroughly explored. Applications of the $\nu-$BS distribution to three real data sets are given to show that the $\nu-$ Birnbaum Saunders distribution provides consistently better modeling than the BS distribution. This extension is intended to attract a broad range of applications to the literature on fatigue life distributions.