CHAPTER 6

A Simple Step-Stress Analysis of Type II Gumbel Distribution

6.1 Introduction

Technology in the modern world evolves faster than ever before. As it gets better, every industry gains. Ultimately, we gain from their results because they lead to better products and services. Since its inception, the market has been and always will be competitive. As a result, producers compete to offer their clients the highest quality products possible. Failure time within a specific timeframe under normal operating conditions cannot be estimated because product quality is constantly advancing. Early failures using ALT methodologies are encouraged in this instance. Using this method, we put more stress than usual on promoting early failures. It lowers the price and enhances the quality of the product.

A type of ALT called step-stress life testing allows the experimenter to gradually increase the stress levels at predetermined intervals throughout the test. 'n' identical units are placed on a life-testing experiment at a starting stress level in a set-up for a multiple-step stress model. The stress level then continued to rise at pre-defined intervals. If there are only two degrees of stress, the model is known as the simple step-stress model.

A model that links the distributions under various stress levels is needed to analyze failure time data from any SSALT experiment. The cumulative exposure model (Sedyakin (1966)), its generalizations (Bagdonavičius (1978)), the proportional hazard model (Cox (1992)), tampered random variable model (Goel (1972)), tampered failure rate model (Bhattacharyya and Soejoeti (1989)), and Khamis-Higgins model (Khamis and Higgins (1998)) are the most frequently used models in the literature.

Here, a failure rate-based model with pre-fixed but arbitrarily chosen failure rates at various stress levels is used (see Kateri and Kamps (2015, 2017)). It is assumed that the HRF of the distribution for the step stress approach is as follows, where S_1 and S_2 denote the stress levels and T denotes the time at which the stress changes.

$$h(x) = \begin{cases} h_1(x) & \text{if } 0 < x \leq T \\ \\ h_2(x) & \text{if } T < x < \infty \quad . \end{cases} \tag{6.1.1}$$

The corresponding CDF is,

$$F(x) = \begin{cases} F_1(x) & \text{if } 0 < x \leq T \\ \\ 1 - \dfrac{1 - F_1(T)}{1 - F_2(T)}(1 - F_2(x)) & \text{if } T < x < \infty \quad . \end{cases} \tag{6.1.2}$$

SSALT setups with type II Gumbel lifetime distributions are rarely examined with regard to inference procedures. Dutta et al. (2023) used Gumbel type II distribution for the simple step-stress life test based on a tampered random variable model under type-II censoring.

This chapter discusses the estimation problem for the Type-II Gumbel distribution utilizing Type-II censoring in the failure rate-based SSALT model. The SSALT model with Type II Gumbel distribution under Type II censoring has been developed to comprehensively assess the reliability and failure characteristics of products or systems exposed to stress testing, which is seldom explored. A type II Gumbel distribution is selected as it is capable of modeling rare but catastrophic failures.

In addition to exploring how increasing stress levels affect the failure rate of a product, the step-stress approach also assists in understanding how the product performs under various environmental conditions. The use of type II censoring, with its periodic inspections and data collection, is an effective and efficient way to conduct long-term tests. As a result of this combination of techniques, organizations can make informed decisions regarding product design, warranty policies, and maintenance strategies by gaining insight into how stress and aging can affect failure behavior and the life expectancy of products.
The baseline lifetime X is distributed according to the Type II Gumbel distribution, whose PDF and CDF are, respectively,

$$f_i^*(x;\beta,\theta) = \begin{cases} \beta_i\theta_i x^{-(\beta_i+1)} e^{-(\theta_i x^{-\beta_i})} & \text{if } x > 0,\ \beta > 0,\ \theta > 0 \\ \\ 0 & \text{otherwise} \end{cases}, \tag{6.1.3}$$

and

$$F_i^*(x;\beta,\theta) = \begin{cases} e^{-(\theta_i x^{-\beta_i})} & \text{if } x > 0,\ \beta > 0,\ \theta > 0 \\ \\ 0 & \text{otherwise} \end{cases}, \tag{6.1.4}$$

where β, θ are the shape and scale parameters, respectively. The HRF is given by

$$h_i^*(x;\beta,\theta) = \begin{cases} \dfrac{\beta_i\theta_i x^{-(\beta_i+1)} e^{-(\theta_i x^{-\beta_i})}}{1 - e^{-(\theta_i x^{(-\beta_i)})}} & \text{if } x > 0,\ \beta > 0, \theta > 0 \\ \\ 0 & \text{otherwise} \end{cases}. \tag{6.1.5}$$

Depending on the parameter values, the Type-II Gumbel distribution's HRF decreases or takes the shape of a UBFR. The Type-II Gumbel distribution is highly adaptable to represent meteorological occurrences, reliability analysis, and life testing, as well as in medical and epidemiological applications because of these shapes of HRF.

6.2 Model Description

Under a Type-II censoring scheme, a simple SSALT model with two stress levels, S_1, and S_2, is analyzed. In the life testing experiment, n identical units are first placed at the stress level S_1. At the pre-determined time T $(0 < T < \infty)$, the stress level is increased to a higher level S_2, and the experiment ends when the rth failure occurs (r is a pre-determined integer $\leq n$).

Let n_i be the number of units that fail at $S_i (i = 1, 2)$. The following ordered failure time data given below are observed using this notation.

$$\Im = \{x_{1:n} < ... < x_{n_1:n} < T < x_{n_1+1:n} < ... < x_{r:n}\}, \tag{6.2.1}$$

where $r = n_1 + n_2$.

Assume that the lifetime distributions of the experimental units at stress levels S_1 and S_2 are Type-II Gumbel distributions, with differences in both the shape and scale parameters. To relate the CDFs of lifetime distributions at two successive stress levels to the CDFs of the lifetime under the used conditions, the assumptions from the SSALT model based on failure rate are used.

To peruse the failure time data, the HRF h(t), the CDF G(t), and the associated PDF g(t) of the lifetime of an experimental unit under the assumption of the failure rate-based SSALT model are respectively given by

$$h(x) = \begin{cases} \dfrac{\beta_1\theta_1 x^{-(\beta_1+1)} e^{-(\theta_1 x^{-\beta_1})}}{1 - e^{-(\theta_1 x^{(-\beta_1)})}} & \text{if } 0 < x \leq T \\[2ex] \dfrac{\beta_2\theta_2 x^{-(\beta_2+1)} e^{-(\theta_2 x^{-\beta_2})}}{1 - e^{-(\theta_2 x^{(-\beta_2)})}} & \text{if } T < x < \infty, \end{cases} \tag{6.2.2}$$

$$G(x) = \begin{cases} e^{-(\theta_1 x^{-\beta_1})} & \text{if } 0 < x \leq T \\[2ex] 1 - \dfrac{e^{-\theta_1 T^{-\beta_1}}}{e^{-\theta_2 T^{-\beta_2}}} e^{-(\theta_2 x^{-\beta_2})} & \text{if } T < x < \infty, \end{cases} \tag{6.2.3}$$

$$g(x) = \begin{cases} \beta_1\theta_1 x^{-(\beta_1+1)} e^{-(\theta_1 x^{-\beta_1})} & \text{if } 0 < x \leq T \\ \dfrac{\beta_2\theta_2 e^{-\theta_1 T^{-\beta_1}}}{e^{-\theta_2 T^{-\beta_2}}} x^{-(\beta_2+1)} e^{-(\theta_2 x^{-\beta_2})} & \text{if } T < x < \infty. \end{cases} \tag{6.2.4}$$

6.3 Maximum Likelihood Estimation

The MLEs of the unknown parameters $\beta_1, \theta_1, \beta_2$, and θ_2 are determined here using the likelihood function based on the observed type-II censored data in Eq.(6.2.1).

If $X_{1:n} < \cdots < X_{r:n}$ denotes the ordered Type-II censored sample from any continuous CDF $F^*(.)$, PDF $f^*(.)$, then the likelihood function of this censored sample can be stated as follows:

$$L(\boldsymbol{\theta}) = \frac{n!}{(n-r)!} \left\{ \prod_{k=1}^{n} f_X(x_{k:n}) \right\} \{1 - F_X(x_{r:n})\}^{n-r}, 0 < x_{1:n} < \ldots < x_{r:n} < \infty,$$

where $\boldsymbol{\theta}$ is the vector representing model's parameters.

Let $\boldsymbol{\theta} = (\beta_1, \theta_1, \beta_2, \theta_2)$ denotes the set of unknown parameters. Using the type-II censored data in Eq.(6.2.1) of failure time from the Type II Gumbel distribution with differences in the shape and scale parameters at each of the two stress levels and assuming a failure rate based simple SSALT model, the likelihood function is obtained as

$$L(\boldsymbol{\theta}|\Im) = \frac{n!}{(n-r)!} \beta_1^{n_1}\theta_1^{n_1}\beta_2^{n_2}\theta_2^{n_2} \prod_{k=1}^{n_1} x_{k:n}^{-(\beta_1+1)} \prod_{k=n_1+1}^{r} x_{k:n}^{-(\beta_2+1)} \prod_{k=1}^{n_1} e^{-\theta_1 x_{k:n}^{-\beta_1}}$$

$$\prod_{k=n_1+1}^{r} e^{-\theta_2 x_{k:n}^{-\beta_2}} \left(\frac{e^{\theta_1 T^{-\beta_1}}}{e^{-\theta_2 T^{-\beta_2}}} \right)^{n-n_1} (e^{-\theta_1 x_{r:n}^{-\beta_2}})^{n-r}. \tag{6.3.1}$$

The associated log-likelihood function $\ell(\boldsymbol{\theta})$ of the observed data is given by

$$\ell(\boldsymbol{\theta}) = \psi_1(\beta_1, \theta_1) + \psi_2(\beta_2, \theta_2), \tag{6.3.2}$$

where

$$\psi_1(\beta_1,\theta_1) = \ln n! + \ln(n-r)! + n_1 \ln\beta_1 + n_1 \ln\theta_1 - (\beta_1+1)\sum_{k=1}^{n_1} \ln x_{k:n}$$

$$- \theta_1 \sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} + (n-n_1)\ln[1-e^{-\theta_1 T^{-\beta_1}}], \quad (6.3.3)$$

and

$$\psi_2(\beta_2,\theta_2) = n_2 \ln\beta_2 + n_2 \ln\theta_2 - (\beta_2+1)\sum_{k=n_1+1}^{r} \ln x - \theta_2 \sum_{k=n_1+1}^{r} x_{k:n}^{-\beta_2}$$

$$-(n-n_1)\ln[1-e^{-\theta_2 T^{-\beta_2}}] + (n-r)\ln[1-e^{-\theta_2 x_{r:n}^{-\beta_2}}]. \quad (6.3.4)$$

Hence, $\hat{\boldsymbol{\theta}}$ can be obtained by maximizing the log-likelihood function Eq.(6.3.2) over the region Θ. The Eq.(6.3.2) can be written as the sum of two equations Eq.(6.3.3) and Eq.(6.3.4). Differentiating Eq.(6.3.2) with respect to $\beta_1, \theta_1, \beta_2, \theta_2$ respectively and equating them to zero, the normal equations are obtained as

$$\frac{\partial \ell}{\partial \beta_1} = \frac{n_1}{\beta_1} - \sum_{k=1}^{n_1} \ln x_{k:n} + \theta_1 \sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} \ln x_{k:n} + (n-n_1)\frac{\theta_1 T^{-\beta_1} \ln T e^{-\theta_1 T^{-\beta_1}}}{1-e^{-\theta_1 T^{-\beta_1}}}, \quad (6.3.5)$$

$$\frac{\partial \ell}{\partial \theta_1} = \frac{n_1}{\theta_1} - \sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} + (n-n_1)\frac{T^{-\beta_1} e^{-\theta_1 T^{-\beta}}}{1-e^{-\theta_1 T^{-\beta_1}}}, \quad (6.3.6)$$

$$\frac{\partial \ell}{\partial \beta_2} = \frac{n_2}{\beta_2} - \sum_{k=n_1+1}^{r} \ln x_{k:n} - \theta_2 \Sigma_{k=n_1+1}^{r} x_{k:n}^{-\beta_2} \ln x_{k:n} - (n-n_1)\frac{\theta_2 \ln T e^{-\theta_2 T^{-\beta_2}}}{1-e^{-\theta_2 T^{-\beta_2}}}$$

$$+ (n-r)\frac{\theta_2 \ln x_{r:n} e^{-\theta_2 x_{r:n}^{-\beta_2}}}{1-e^{-\theta_2 x_{r:n}^{-\beta_2}}}, \quad (6.3.7)$$

and

$$\frac{\partial \ell}{\partial \theta_2} = \frac{n_2}{\theta_2} - \sum_{k=n_1+1}^{r} x_{k:n}^{-\beta_2} - (n-n_1)\frac{T^{-\beta_2} e^{-\theta_2 T^{-\beta_2}}}{1-e^{-\theta_2 T^{-\beta_2}}} + (n-r)\frac{x_{r:n}^{-\beta_2} e^{-\theta_2 x_{r:n}^{-\beta_2}}}{1-e^{-\theta_2 x_{r:n}^{-\beta_2}}}. \quad (6.3.8)$$

Multiplying Eq.(6.3.6) with $\theta_1 \ln T$, we get

$$n_1 \ln T - n_1 \ln T \sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} + (n - n_1) \frac{\theta_1 \ln T\, T^{-\beta_1} e^{-\theta_1 T^{-\beta}}}{1 - e^{-\theta_1 T^{-\beta_1}}} = 0 \tag{6.3.9}$$

Substracting Eq.(6.3.9) from Eq.(6.3.5) and simplifying, we get

$$\theta_1 = \frac{\frac{n_1}{\beta_1} - n_1 \ln T + n_1 \ln T \sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} - \sum_{k=1}^{n_1} \ln x_{k:n}}{\sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} \ln x_{k:n}} \tag{6.3.10}$$

6.4 Interval Estimation

In this section, a method for constructing CIs for the unknown parameters $\beta_1, \theta_1, \beta_2$, and θ_2 are presented. The exact CIs of the unknown parameters cannot be obtained because the closed forms of the MLEs do not exist. The asymptotic CIs are provided, assuming the MLEs are asymptotically normal.

6.4.1 Asymptotic Confidence Intervals

Using the observed Fisher information matrix, a method is presented that assumes asymptotic normality of the MLEs to obtain the CIs for $\beta_1, \theta_1, \beta_2$, and θ_2. For large sample sizes, this method is useful due to its simplicity in computation.

To begin with, we need to obtain explicit expressions for the elements of the Fisher information matrix $I(\boldsymbol{\theta})$. The elements of $I(\boldsymbol{\theta})$ are

$$\frac{\partial^2 \ell}{\partial \beta_1^2} = -\frac{n_1}{\beta_1^2} - \theta_1 \sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} (\ln x_{k:n})^2 - \frac{(n-n_1)\theta_1 (\ln T)^2 T^{-\beta_1} [(1 - \theta_1 T^{-\beta_1}) e^{\theta_1 T^{-\beta_1}} - 1]}{(e^{\theta_1 T^{-\beta_1}} - 1)^2}$$

$$\frac{\partial^2 \ell}{\partial \theta_1 \partial \beta_1} = \sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} \ln x_{k:n} - \frac{(n-n_1) \ln T\, T^{-\beta_1} [(\theta_1 T^{-\beta_1} - 1) e^{\theta_1 T^{-\beta_1}} + 1]}{(e^{\theta_1 T^{-\beta_1}} - 1)^2}$$

$$\frac{\partial^2 \ell}{\partial \theta_1^2} = -\frac{n_1}{\theta_1^2} - (n-n_1) T^{-2\beta_1} \frac{e^{\theta_1 T^{-\beta_1}}}{(e^{\theta_1 T^{-\beta_1}} - 1)^2}$$

$$\frac{\partial^2 \ell}{\partial \beta_1 \partial \theta_1} = -\sum_{k=1}^{n_1} x_{k:n}^{-\beta_1} \ln x_{k:n} + (n-n_1) \ln T \frac{T^{-\beta_1} [(1 - \theta_1 T^{-\beta_1}) e^{\theta_1 T^{-\beta_1}} - 1]}{(e^{\theta_1 T^{-\beta_1}} - 1)^2}$$

$$\frac{\partial^2 \ell}{\partial \beta_2^2} = -\frac{n_2}{\beta_2^2} + \theta_2 \sum_{k=n_1+1}^{r} x_{k:n}^{-\beta_2} (\ln x_{k:n})^2 - (n-n_1)\, \theta_2^2\, (\ln T)^2 \frac{T^{-\beta_2} e^{\theta_2 T^{-\beta_2}}}{(e^{\theta_2 T^{-\beta_2}}-1)^2} + (n-r)\theta_2^2\, (\ln x_{r:n})^2\, \frac{e^{\theta_2 x_{r:n}^{-\beta_2}}}{x_{r:n}^{\beta_2}(e^{\theta_2 x_{r:n}^{-\beta_2}}-1)^2}$$

$$\frac{\partial^2 \ell}{\partial \theta_2 \partial \beta_2} = -\sum_{k=n_1+1}^{r} x_{k:n}^{-\beta_2} \ln x_{k:n} + \frac{(n-n_1)\ln T\, [(\theta_2 T^{-\beta_2}-1)e^{\theta_2 T^{-\beta_2}}+1]}{(e^{\theta_2 T^{-\beta_2}}-1)^2} - (n-r)\frac{\ln x_{r:n}\, [(\theta_2 x_{r:n}^{-\beta_2}-1)e^{\theta_2 x_{r:n}^{-\beta_2}}+1]}{(e^{\theta_2 x_{r:n}^{-\beta_2}}-1)^2}$$

$$\frac{\partial^2 \ell}{\partial \theta_2^2} = -\frac{n_2}{\theta_2^2} + (n-n_1)\frac{T^{-2\beta_2} e^{\theta_2 T^{-\beta_2}}}{(e^{\theta_2 T^{-\beta_2}}-1)^2} - (n-r)\frac{x_{r:n}^{-2\beta_2} e^{\theta_2 x_{r:n}^{-\beta_2}}}{(e^{\theta_2 x_{r:n}^{-\beta_2}}-1)^2}$$

$$\frac{\partial^2 \ell}{\partial \beta_2 \partial \theta_2} = \sum_{k=n_1+1}^{r} x_{k:n}^{-\beta_2} \ln x_{k:n} - (n-n_1)\theta_2^2 \frac{(\ln T)^2\, T^{-\beta_2} e^{\theta_2 T^{-\beta_2}}}{(e^{\theta_2 T^{-\beta_2}}-1)^2} + (n-r)\theta_2^2 \frac{(\ln x_{r:n})^2\, x_{r:n}^{-\beta_2} e^{\theta_2 x_{r:n}^{-\beta_2}}}{(e^{\theta_2 x_{r:n}^{-\beta_2}}-1)^2}.$$

Then, the $100(1-\alpha)\%$ asymptotic CIs for $\beta_1, \theta_1, \beta_2$, and θ_2 are, respectively

$(\hat{\beta}_1 \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{11}})$, $(\hat{\theta}_1 \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{22}})$, $(\hat{\beta}_2 \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{33}})$, and $(\hat{\theta}_2 \pm z_{1-\frac{\alpha}{2}}\sqrt{V_{44}})$,

where V_{ij} represents the (i,j)th element in the inverse of the Fisher information matrix I and z_p is the p-th upper percentile of a standard normal distribution.

6.5 Summary

This study introduces a simple step stress life testing model with type-II Gumbel lifetime distribution. A flexible failure-rate based SSALT model is considered based on type-II censoring. The point estimate of parameters using the maximum likelihood method is described under the notion of a failure rate-based model.