CHAPTER 2

MHD Nanofluid Flow Between Two Vertical Porous Plates Moving In Opposite Direction *

2.1 INTRODUCTION

The theoretical analysis of MHD flow through porous channels (flow between two parallel plates) has gained great attention due to its increased advantages in the industrial, mechanical engineering, and biomedical fields. Flow in the capillaries, blood dialysis, the design of filters, and flow in blood oxygenators are some examples. Only a countable number of studies are accounted for the flow between parallel plates moving in the opposite direction.

This chapter investigates the MHD flow of water-based nanofluid between two vertical porous plates moving in the opposite direction. The perturbation technique is used to solve governing equations and the consequences of various parameters on velocity, temperature, and concentration are examined via graphs using MATLAB software. The physical quantities Nusselt number and Sherwood number are scrutinized using statistical techniques correlation, probable error, and regression

*Published in Heat Transfer (Wiley), 2021;50(5);5170–5197.

analysis.

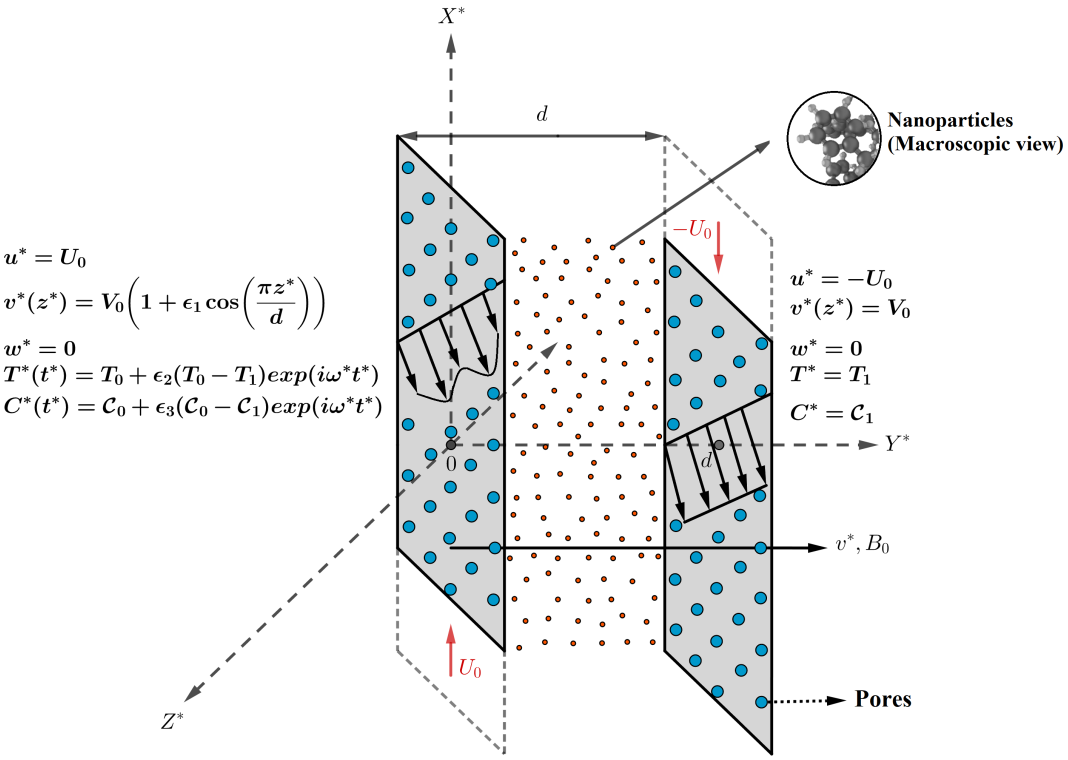

Figure 2.1: *Physical configuration*

2.2 MATHEMATICAL FORMULATION

An unsteady hydromagnetic fluctuating flow of a water based nanofluid between two insulated vertical porous plates is considered (Figure. 2.1). The problem is studied under the following assumptions:

1. Plates are moving in opposite direction with uniform velocity.

2. The upward moving plate is subjected to a transverse sinusoidal injection velocity whereas the downward moving plate is subjected to a constant suction velocity.

3. A magnetic field of uniform strength is applied perpendicular to the plane of the plate.

4. The induced magnetic field has been neglected due to the assumption of small magnetic Reynolds number.

5. The injection velocity distribution is of the form

   $v^*(z^*) = V_0(1 + \varepsilon_1 \cos(\pi z^*/d))$

6. Without loss of generality, the distance d between the plates is taken equal to the wave length of the injection velocity.

7. The temperature of the upward moving plate fluctuating with time is given as $T^*(t^*) = T_0 + \varepsilon_2(T_0 - T_1)e^{i\omega^* t^*}$

8. The concentration of the upward moving plate fluctuating with time is given as $C^*(t^*) = C_0 + \varepsilon_3(C_0 - C_1)e^{i\omega^* t^*}$

9. The temperature and concentration of the downward moving plate is at constant temperature T_1 and constant concentration C_1 respectively

Using the above assumptions and Boussinesq's approximation, the governing equations (Singh & Mathew, 2009b) takes the form:

$$\frac{\partial v^*}{\partial y^*} + \frac{\partial w^*}{\partial z^*} = 0 \tag{2.2.1}$$

$$\frac{\partial u^*}{\partial t^*} + v^*\frac{\partial u^*}{\partial y^*} + w^*\frac{\partial u^*}{\partial z^*} = -\frac{1}{\rho_{nf}}\left[\frac{\partial p^*}{\partial x^*} - \mu_{nf}\left(\frac{\partial^2 u^*}{\partial y^{*2}} + \frac{\partial^2 u^*}{\partial z^{*2}}\right) + \sigma_{nf} B_0^2 u^*\right] + g\beta_{nf}\left(T^* - T_1\right) + g\beta_{nf}\left(C^* - C_1\right) \tag{2.2.2}$$

$$\frac{\partial v^*}{\partial t^*} + v^*\frac{\partial v^*}{\partial y^*} + w^*\frac{\partial v^*}{\partial z^*} = -\frac{1}{\rho_{nf}}\left[\frac{\partial p^*}{\partial y^*} - \mu_{nf}(\frac{\partial^2 v^*}{\partial y^{*2}} + \frac{\partial^2 v^*}{\partial z^{*2}})\right] \tag{2.2.3}$$

$$\frac{\partial w^*}{\partial t^*} + v^*\frac{\partial w^*}{\partial y^*} + w^*\frac{\partial w^*}{\partial z^*} = -\frac{1}{\rho_{nf}}\left[\frac{\partial p^*}{\partial z^*} - \mu_{nf}\left(\frac{\partial^2 w^*}{\partial y^{*2}} + \frac{\partial^2 w^*}{\partial z^{*2}}\right) + \sigma_{nf} B_0^2 w^*\right] \tag{2.2.4}$$

$$\frac{\partial T^*}{\partial t^*} + v^*\frac{\partial T^*}{\partial y^*} + w^*\frac{\partial T^*}{\partial z^*} = \frac{K_{nf}}{\left(\rho C_p\right)_{nf}}\left[\frac{\partial^2 T^*}{\partial y^{*2}} + \frac{\partial^2 T^*}{\partial z^{*2}}\right] \tag{2.2.5}$$

$$\frac{\partial C^*}{\partial t^*}+v^*\frac{\partial C^*}{\partial y^*}+w^*\frac{\partial C^*}{\partial z^*}=D_B\left[\frac{\partial^2 C^*}{\partial y^{*2}}+\frac{\partial^2 C^*}{\partial z^{*2}}\right]-K_l\left(C^*-C_1\right) \tag{2.2.6}$$

The boundary conditions for the problem are:

$$\left.\begin{array}{c} y^*=0, u^*=U_0, v^*\left(z^*\right)=V_0\left(1+\varepsilon_1\cos\frac{\pi z^*}{d}\right), w^*=0, \\ T^*(t^*=T_0+\varepsilon_2\left(T_0-T_1\right)e^{i\omega^* t^*}, C^*(t^*)=C_0+\varepsilon_3\left(C_0-C_1\right)e^{i\omega^* t^*}, \\ y^*=d, u^*=-U_0, v^*(z^*)=V_0, w^*=0, T^*=T_1, C^*=C_1 \end{array}\right\} \tag{2.2.7}$$

The nanofluid model used is

Effective Dynamic Viscosity:

$$\frac{1}{A_1}=\frac{\mu_{nf}}{\mu_f}=\frac{1}{(1-\phi)^{2.5}}$$

Effective Density:

$$A_2=\frac{\rho_{nf}}{\rho_f}=(1-\phi)+\phi\left(\frac{\rho_s}{\rho_f}\right)$$

Effective Electrical Conductivity:

$$A_3=\frac{\sigma_{nf}}{\sigma_f}=1+\frac{3\left(\frac{\sigma_s}{\sigma_f}-1\right)\phi}{\left(\frac{\sigma_s}{\sigma_f}+2\right)-\left(\frac{\sigma_s}{\sigma_f}-1\right)\phi}$$

Effective Coefficient Of Thermal Expansion:

$$A_4=\frac{\beta_{nf}}{\beta_f}=(1-\phi)+\phi\left(\frac{\beta_s}{\beta_f}\right)$$

Effective Specific Heat:

$$A_5=\frac{(\rho C_p)_{nf}}{(\rho C_p)_f}=(1-\phi)+\phi\left(\frac{(\rho C_p)_s}{(\rho C_p)_f}\right)$$

Effective Thermal Conductivity:

$$A_6=\frac{K_{nf}}{K_f}=\frac{K_s+2K_f-2\phi\left(K_f-K_s\right)}{K_s+2K_f+2\phi\left(K_f-K_s\right)}$$

Introducing the following non-dimensional quantities

$$y=\frac{y^*}{d},\ z=\frac{z^*}{d},\ t=t^*\omega^*,\ u=\frac{u^*}{U_0},\ v=\frac{v^*}{V_0},\ w=\frac{w^*}{V_0},$$

$$p = \frac{p^*}{\rho_{nf} V_0^2},\ \omega = \frac{\omega^* d^2}{\vartheta},\ \theta = \frac{T^* - T_1}{T_0 - T_1},\ C = \frac{C^* - C_1}{C_0 - C_1}$$

into equations (2.2.1) to (2.2.7), excluding (2.2.2)

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.2.8}$$

$$\frac{\omega}{Re}\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \frac{1}{A_1 A_2 Re}\left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{2.2.9}$$

$$\frac{\omega}{Re}\frac{\partial w}{\partial t} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{A_1 A_2 Re}\left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) - \frac{A_3}{A_2 Re} H^2 w \tag{2.2.10}$$

$$\frac{\omega}{Re}\frac{\partial \theta}{\partial t} + v\frac{\partial \theta}{\partial y} + w\frac{\partial \theta}{\partial z} = \frac{A_6}{A_5 Pr Re}\left(\frac{\partial^2 \theta}{\partial y^2} + \frac{\partial^2 \theta}{\partial z^2}\right) \tag{2.2.11}$$

$$\frac{\omega}{Re}\frac{\partial C}{\partial t} + v\frac{\partial C}{\partial y} + w\frac{\partial C}{\partial z} = \frac{1}{ReSc}\left(\frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}\right) - \frac{K_c}{Re} C \tag{2.2.12}$$

Equation (2.2.2) reduces to the following partial differential equations based on the two cases:

Case 1: Magnetic Field is applied on the upward moving plate (at y=0)

$$\frac{\omega}{Re}\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{1}{A_1 A_2 Re}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \frac{A_3}{A_2 Re} H^2 (u-1) + A_4 Gr Re \theta + A_4 Gm Re C \tag{2.2.13}$$

Case 2: Magnetic Field is applied on the downward moving plate (at y=1)

$$\frac{\omega}{Re}\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{1}{A_1 A_2 Re}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \frac{A_3}{A_2 Re} H^2 (u+1) + A_4 Gr Re \theta + A_4 Gm Re C \tag{2.2.14}$$

The boundary conditions in the non-dimensional form are given by

$$\left.\begin{array}{c} y=0, u=1, v(z)=1+\varepsilon_1\cos\pi z, w=0, \theta=1+\varepsilon_2 e^{it}, C=1+\varepsilon_3 e^{it} \\ y=1, u=-1, v=1, w=0, \theta=0, C=0 \end{array}\right\} \tag{2.2.15}$$

2.3 METHOD OF SOLUTION

Since $\varepsilon = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is very small, assume that the solution of the problem is of the form

$$f(y,z,t) = f_0(y) + \varepsilon f_1(y,z,t) + O(\varepsilon^2) \tag{2.3.1}$$

2.3.1 Steady Flow Solution

When $\varepsilon = 0$, the current problem contracts to a steady two dimensional hydro magnetic flow between two vertical porous plates moving in opposite direction with uniform injection/suction which is governed by the following equations:

Case 1: Magnetic Field is applied on the upward moving plate (at y=0)

$$u_0'' - A_1A_2Reu_0' - A_1A_3H^2(u_0-1) + A_1A_2A_4Re^2Gr\theta_0 + A_1A_2A_4Re^2GmC_0 = 0 \tag{2.3.2}$$

Case 2: Magnetic Field is applied on the downward moving plate (at y=1)

$$u_0'' - A_1A_2Reu_0' - A_1A_3H^2(u_0+1) + A_1A_2A_4Re^2Gr\theta_0 + A_1A_2A_4Re^2GmC_0 = 0 \tag{2.3.3}$$

with $v_0 = 1, w_0 = 0, p_0 =$ a constant and

$$\theta_0'' - \frac{A_5PrRe}{A_6}\theta_0' = 0 \tag{2.3.4}$$

$$C_0'' - ScReC_0' - K_cScC_0 = 0 \tag{2.3.5}$$

where primes denote the derivative with respect to y.

The corresponding boundary condition for the above two cases reduces to:

$$\left.\begin{array}{r} y=0, u_0=1, \theta_0=1, C_0=1 \\ y=1, u_0=-1, \theta_0=0, C_0=0 \end{array}\right\} \tag{2.3.6}$$

The solutions of equations (2.3.2) to (2.3.5) subject to boundary conditions (2.3.6) are

$$\theta_0 = \frac{1}{e^a - 1}(e^a - e^{ay}) \tag{2.3.7}$$

$$C_0 = \frac{1}{e^{m_2} - e^{m_1}}\left(e^{m_2}e^{m_1 y} - e^{m_1}e^{m_2 y}\right) \tag{2.3.8}$$

Case 1: Magnetic Field is applied on the upward moving plate (at y=0)

$$u_0 = \frac{1}{e^{m_4} - e^{m_3}}\left[(\alpha_1 e^{m_4} - \beta_1)e^{m_3 y} + (\beta_1 - \alpha_1 e^{m_3})e^{m_4 y}\right] + B_1 e^{ay} + B_2 e^{m_1 y} + B_3 e^{m_2 y} + B_4 + 1 \tag{2.3.9}$$

Case 2: Magnetic Field is applied on the downward moving plate (at y=1)

$$u_0 = \frac{1}{e^{m_4} - e^{m_3}}\left[\left((\alpha_1 + 2)e^{m_4} - (\beta_1 + 2)\right)e^{m_3 y} + \left((\beta_1 + 2) - (\alpha_1 + 2)e^{m_3}\right)e^{m_4 y}\right] + B_1 e^{ay} + B_2 e^{m_1 y} + B_3 e^{m_2 y} + B_4 - 1 \tag{2.3.10}$$

2.3.2 Cross Flow Solution

When $\varepsilon \neq 0$ substituting equation (2.3.1) in equations (2.2.8) to (2.2.10) and comparing the coefficients of ε and neglecting the terms with $O(\varepsilon^2)$, obtain the following first order equations:

$$\frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0 \tag{2.3.11}$$

$$\frac{\omega}{Re}\frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial y} = -\frac{\partial p_1}{\partial y} + \frac{1}{A_1 A_2 Re}\left[\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right] \tag{2.3.12}$$

$$\frac{\omega}{Re}\frac{\partial w_1}{\partial t} + \frac{\partial w_1}{\partial y} = -\frac{\partial p_1}{\partial z} + \frac{1}{A_1 A_2 Re}\left[\frac{\partial^2 w_1}{\partial y^2} + \frac{\partial^2 w_1}{\partial z^2}\right] - \frac{A_3}{A_2 Re}H^2 w_1 \tag{2.3.13}$$

Corresponding boundary conditions are

$$\left.\begin{aligned} y = 0, v_1 = \cos \pi z, w_1 = 0 \\ y = 1, v_1 = 0, w_1 = 0 \end{aligned}\right\} \tag{2.3.14}$$

These are the linear partial differential equations describing the three dimensional cross flow which are independent of the main flow component, u_1, temperature field, θ_1 and concentration field, C_1.

Assume that the solutions for v_1, w_1, p_1 are of the following form

$$v_1(y, z, t) = v_{11}(y)e^{it} + v_{12}(y) \cos \pi z \tag{2.3.15}$$

$$w_1(y, z, t) = -(z{v_{11}}'(y)e^{it} + \frac{1}{\pi}{v_{12}}'(y) \sin \pi z) \tag{2.3.16}$$

$$p_1(y, z, t) = p_{11}(y)e^{it} + p_{12}(y) \cos \pi z \tag{2.3.17}$$

where prime denotes the derivative with respect to y. Expressions (2.3.15) and (2.3.16) have been chosen in such a way that the continuity equation (2.3.11) is precisely satisfied. Substituting these equations into equations (2.3.12) and (2.3.13) and applying the corresponding boundary conditions, the solutions of v_1, w_1, p_1 obtained as

$$v_1 = \frac{1}{D} \sum_{i=1}^{4} D_i e^{r_i y} \cos \pi z \tag{2.3.18}$$

$$w_1 = -\frac{1}{\pi D} \sum_{i=1}^{4} r_i D_i e^{r_i y} \sin \pi z \tag{2.3.19}$$

$$p_1 = -\frac{1}{A_1 A_2 Re \pi^2 D} \sum_{i=1}^{4} D_i (r_i^3 - A_1 A_2 Re r_i^2 - (A_1 A_3 H^2 + \pi^2) r_i) e^{r_i y} \cos \pi z \tag{2.3.20}$$

2.3.3 Temperature and Concentration Field

Similarly using equation (2.3.1) when $\varepsilon \neq 0$ the first order equation for temperature and concentration fields are

$$\frac{\omega}{Re}\frac{\partial \theta_1}{\partial t} + \frac{\partial \theta_1}{\partial y} = \frac{A_6}{A_5 Pr Re}\left(\frac{\partial^2 \theta_1}{\partial y^2} + \frac{\partial^2 \theta_1}{\partial z^2}\right) \tag{2.3.21}$$

$$\frac{\omega}{Re}\frac{\partial C_1}{\partial t} + \frac{\partial C_1}{\partial y} = \frac{1}{ScRe}\left(\frac{\partial^2 C_1}{\partial y^2} + \frac{\partial^2 C_1}{\partial z^2}\right) - \frac{K_c}{Re}C_1 \tag{2.3.22}$$

with corresponding boundary conditions

$$\left.\begin{array}{r} y = 0, \theta_1 = e^{it}, C_1 = e^{it} \\ y = 1, \theta_1 = 0, C_1 = 0 \end{array}\right\} \tag{2.3.23}$$

Equations (2.3.21)-(2.3.23) are solved with an assumption that the solutions are of the form:

$$\theta_1(y,z,t) = \theta_{11}e^{it} + \theta_{12}\cos \pi z \tag{2.3.24}$$

$$C_1(y,z,t) = C_{11}e^{it} + C_{12}\cos \pi z \tag{2.3.25}$$

Substituting equation (2.3.24) into (2.3.21) and equation (2.3.25) into (2.3.22), one can obtain the followig equations:

$$\theta_{11}'' - \frac{A_5 Pr Re}{A_6}\theta_{11}' - \frac{A_5 Pr \omega i}{A_6}\theta_{11} = 0 \tag{2.3.26}$$

$$\theta_{12}'' - \frac{A_5 Pr Re}{A_6}\theta_{12}' - \pi^2\theta_{12} = 0 \tag{2.3.27}$$

$$C_{11}'' - ScReC_{11}' - (Sc\omega i + K_c Sc)C_{11} = 0 \tag{2.3.28}$$

$$C_{12}'' - ScReC_{12}' - (ScK_c + \pi^2)C_{12} = 0 \tag{2.3.29}$$

with the corresponding transformed boundary conditions

$$\left.\begin{array}{l} y=0, \theta_{11}=1, \theta_{12}=0, C_{11}=1, C_{12}=0 \\ y=1, \theta_{11}=0, \theta_{12}=0, C_{11}=0, C_{12}=0 \end{array}\right\} \tag{2.3.30}$$

Solving equations (2.3.26) – (2.3.29) under the boundary condition (2.3.30), the solutions are obtained as

$$\theta_1(y,z,t) = \frac{1}{e^{n_2} - e^{n_1}}(e^{n_2}e^{n_1 y} - e^{n_1}e^{n_2 y})e^{it} \tag{2.3.31}$$

$$C_1(y,z,t) = \frac{1}{e^{m_6} - e^{m_5}}(e^{m_6}e^{m_5 y} - e^{m_5}e^{m_6 y})e^{it} \tag{2.3.32}$$

2.3.4 **Main Flow Solution**

When $\varepsilon \neq 0$, with the aid of equation (2.3.1) and comparing coefficients of ε, can deduce the first order equation for the main flow component,u_1 , as

$$\frac{\omega}{Re}\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial y} + v_1 u_0{}' = \frac{1}{A_1 A_2 Re}\left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_1}{\partial z^2}\right) - \frac{A_3}{A_2 Re}H^2 u_1 + A_4 Gr Re\theta_1 + A_4 Gm Re C_1 \tag{2.3.33}$$

The corresponding boundary conditions are

$$\left.\begin{array}{l} y=0, u_1=0 \\ y=1, u_1=0 \end{array}\right\} \tag{2.3.34}$$

To solve the above differential equation for the main flow velocity component,$u_1(y,z,t)$, assume that the solution is of the form:

$$u_1(y,z,t) = u_{11}(y)e^{it} + u_{12}(y)\cos \pi z \tag{2.3.35}$$

Substituting equation (2.3.35) in (2.3.33) and neglecting terms of $O(\varepsilon^2)$ one can obtain the following differential equations

$$u_{11}'' - A_1 A_2 Re u_{11}' - (A_1 A_2 \omega i + A_1 A_3 H^2)u_{11} = -A_1 A_2 A_4 Re^2 Gr\theta_{11} -$$

$$A_1A_2A_4Re^2GmC_{11} \quad (2.3.36)$$

$$u_{12}'' - A_1A_2Reu_{12}' - (\pi^2 + A_1A_3H^2)u_{12} = -A_1A_2Rev_{12}u_0' \quad (2.3.37)$$

where prime denotes the derivative with respect to y. The corresponding boundary conditions are given by:

$$\left. \begin{array}{c} y = 0, u_{11} = 0, u_{12} = 0 \\ y = 1, u_{11} = 0, u_{12} = 0 \end{array} \right\} \quad (2.3.38)$$

Solving equations (2.3.36) and (2.3.37) considering (2.3.38) and using (2.3.35) one can derive the solutions as:

Case 1: Magnetic Field is applied on the upward moving plate (at y=0)

$$u_1(y, z, t) = \left\{ \begin{array}{c} \frac{1}{e^{r_6} - e^{r_5}}[(\alpha_2 e^{r_6} - \beta_2)e^{r_5 y} + (\beta_2 - \alpha_2 e^{r_5})e^{r_6 y}] + \\ K_1 e^{n_1 y} + K_2 e^{n_2 y} + K_3 e^{m_5 y} + K_4 e^{m_6 y} \end{array} \right\} e^{it} +$$

$$\left\{ \begin{array}{c} \frac{1}{e^{n_4} - e^{n_3}}[(\alpha_3 e^{n_4} - \beta_3)e^{n_3 y} + (\beta_3 - \alpha_3 e^{n_3})e^{n_4 y}] + \\ \sum_{i=1}^{4} K_{i1} e^{(r_i + m_3)y} + K_{i2} e^{(r_i + m_4)y} + K_{i3} e^{(r_i + a)y} + \\ K_{i4} e^{(r_i + m_1)y} + K_{i5} e^{(r_i + m_2)y} \end{array} \right\} \cos \pi z \quad (2.3.39)$$

Case 2: Magnetic Field is applied on the downward moving plate (at y=1)

$$u_1(y, z, t) = \left\{ \begin{array}{c} \frac{1}{e^{r_6} - e^{r_5}}[(\alpha_2 e^{r_6} - \beta_2)e^{r_5 y} + (\beta_2 - \alpha_2 e^{r_5})e^{r_6 y}] + \\ K_1 e^{n_1 y} + K_2 e^{n_2 y} + K_3 e^{m_5 y} + K_4 e^{m_6 y} \end{array} \right\} e^{it} +$$

$$\left\{ \begin{array}{c} \frac{1}{e^{n_4} - e^{n_3}}[(\alpha_4 e^{n_4} - \beta_4)e^{n_3 y} + (\beta_4 - \alpha_4 e^{n_3})e^{n_4 y}] + \\ \sum_{i=1}^{4} M_{i1} e^{(r_i + m_3)y} + M_{i2} e^{(r_i + m_4)y} + \\ M_{i3} e^{(r_i + a)y} + M_{i4} e^{(r_i + m_1)y} + M_{i5} e^{(r_i + m_2)y} \end{array} \right\} \cos \pi z \quad (2.3.40)$$

2.4 PHYSICAL QUANTITIES

For practical analysis of the problem, scientists and engineers are always interested in understanding the physical quantities like drag coefficients, Nusselt number and Sherwood number as they measures the surface drag, rate of heat transfer and rate of mass transfer, respectively. The modified form of these physical quantities are given by:

$$Sh = \frac{d}{D_B(C_0 - C_1)} \left| D_B \left(\frac{dC^*}{dy^*}\right)_{y^*=d} \right| = \left| \left(\frac{dC_0}{dy}\right)_{y=1} + \varepsilon \left(\frac{dC_1}{dy}\right)_{y=1} \right| \quad (2.4.1)$$

$$Nu = \frac{d}{K_f(T_0 - T_1)} \left| K_{nf} \left(\frac{dT^*}{dy^*}\right)_{y^*=d} \right| = A_6 \left| \left(\frac{d\theta_0}{dy}\right)_{y=1} + \varepsilon \left(\frac{d\theta_1}{dy}\right)_{y=1} \right| \quad (2.4.2)$$

Case 1: Magnetic Field is applied on the upward moving plate (at y=0)

$$Cf = \frac{d}{\mu_f U_0} \left| \mu_{nf} \left(\frac{du^*}{dy^*}\right)_{y^*=d} \right| = \frac{1}{A_1} \left| \left(\frac{du_0}{dy}\right)_{y=1} + \varepsilon \left(\frac{du_1}{dy}\right)_{y=1} \right| \quad (2.4.3)$$

Case 2: Magnetic Field is applied on the downward moving plate (at y=1)

$$Cf = \frac{d}{\mu_f U_0} \left| \mu_{nf} \left(\frac{du^*}{dy^*}\right)_{y^*=d} \right| = \frac{1}{A_1} \left| \left(\frac{du_0}{dy}\right)_{y=1} + \varepsilon \left(\frac{du_1}{dy}\right)_{y=1} \right| \quad (2.4.4)$$

2.5 RESULTS AND DISCUSSION

The impact of various nanofluids, volume fraction of nanoparticle (ϕ), Grashof number (Gr), chemical reaction parameter (K_c), injection/suction parameter (Re), Schmidt number (Sc), Hartmann number (H) and modified Grashof number (Gm) on concentration (C), temperature (θ) and velocity (u) profiles are analysed through Figures 2.2 - 2.12. The analysis has been carried out for $Cu - H_2O$ nanofluid at $z = 0, t = \frac{\pi}{2}$ and $Pr = 7$. The thermophysical properties of the base fluid and nanoparticles are given in Table 2.1.

The consequence of Gr, the ratio of buoyancy force to viscous force, on u is presented in Figure 2.2. An increase in Gr causes augmentation in u for both cases (magnetic field fixed with the upward and downward moving plate). Physically, an increase in Gr paves to the increase in buoyancy force which in turn increases the velocity. Figure 2.3 confirms that u increases in both cases when Gm is increased; the physical reason being the increase in buoyancy force due to a concentration difference. The influence of H, the ratio of electromagnetic force to the viscous force, on u is graphed in Figures 2.4 and 2.5. It is evident that a rise in H drastically reduces the velocity in the case of a downward-moving plate. Physically this is due to the fact that the presence of a magnetic field induces a Lorentz force against the fluid flow which retards the velocity profile. However, in the case of upward moving plate, H expresses a mixed behavior on u. In the beginning, an increase in H reduces the velocity but after a certain distance, the behavior is reversed. Figures 2.6 and 2.7 illustrates the exponential increase in u both cases corresponding to the augmentation in Re. Figure 2.8 depicts the positive impact of ϕ on u. The impact of various nanofluids (water-based $Cu/Fe_3O_4/TiO_2$ nanofluid) on u is depicted in Figure 2.9.

The effect of ϕ on θ is elucidated in Figure 2.10. ϕ has a negative impact on the temperature profile, meaning an increase in ϕ fuels a decrease in θ. Figure 2.11 depicts the consequence of Re on temperature. It is observed that Re induces an increase in θ. The influence of Sc on C is illustrated in Figure 2.12. Concentration profile (C) experiences a slight increase when Sc values are improved.

The consequence of heat and mass transfer rate on fluctuating parameter along with the slope of linear regression via data points and the enhancement/decrement rate are assessed in table 2.2 and 2.3, respectively. Negative sign in slope and enhancement/decrement rate symbolises that the corresponding parameter has a negative impact on heat and mass transfer meaning augmentation in parameter reduces the heat and mass transfer rate while the magnitude of slope represents the quantity of change.

It is observed that t and Re have positive impact on Nu and ϕ and ω have negative impact on Nu. The respective rates (slope) are shown in Table 2.2. For Sh, when all other parameters are kept constant, Sh increases with an increase in ω at the rate 0.000322, Re at 0.104051 and Sc at the rate 0.355898. Also, when K_c and t increases Sh decreases at the rate of 0.03518 and 0.00751, respectively. Figures 2.13, 2.14, 2.15, and 2.16 describe the parallel effect of $\phi - Re$ and $H - Gr$ on Cf. From 2.13 and 2.15, it can be deduced that ϕ, Re, H and Gr promotes Cf on the upward moving plate. From 2.14 and 2.16, it can be summarised that ϕ, Re and Gr promotes Cf while H demotes Cf on the downward moving plate.

2.6 STATISTICAL ANALYSIS

2.6.1 Correlation and Probable Error

Correlation, a statistical technique, which enables the user in finding the degree of relationship between two or more variables, has gained a lot of interest in the recent times. The effects of different parameters on Nu and Sh are comprehensively carried out and established using tables. A more detailed study is carried out with the help of correlation coefficient (r) and Probable Error (PE). The nature of relationship and the intensity of the relationship is communicated through the sign and magnitude of r, respectively. Probable Error (PE) of r helps in determining the accuracy and reliability of the calculated correlation coefficient. According to (Fisher, 1950), correlation is said to be significant if $\left|\frac{r}{PE}\right| > 6$, where Probable Error, $PE = (\frac{1-r^2}{\sqrt{n}})0.6745$ and n is the number of observations

From Table 2.4, it can infer that Nu is highly negatively correlated with ϕ and ω and highly positively correlated with Re and t. From $\left|\frac{r}{PE}\right|$ values, it can conclude that all parameters are significant. Table 2.5 proposes that Sh is highly positively correlated with ω, Re and Sc and highly negatively correlated with K_c and t. As earlier, it is observed that all parameters are significant.

2.6.2 Regression Analysis

All correlations are perceived to be significant and hence further analysis can be carried out using regression. Nu and Sh are estimated using multiple linear regression models. The estimated models are of the form:

$$Nu_{est} = b_t t + b_\phi \phi + b_\omega \omega + b_{Re} Re + c$$

$$Sh_{est} = b_t t + b_\omega \omega + b_{Re} Re + b_{Sc} Sc + b_{K_c} K_c + c$$

where b_t, b_ϕ, b_ω, b_{Re}, b_{Sc}, and b_{K_c}, are the estimated regression coefficients.

Nu and Sh is estimated from 30 sets of values chosen in the range [0 ,0.04] for ϕ, [5 ,15] for ω, [1 ,1.2] for Re, [0.2 ,1] for Sc, [0.5 ,2] for K_c and [0.5 ,$\frac{\pi}{2}$] for t and the regression coefficients are found using Microsoft Excel. As the $p-$values for all the physical parameters are less than 0.05, the regression coefficients are significant. The estimated regression models are given by:

$$Nu_{est} = 0.014996t - 0.807981\phi - 0.006215\omega + 6.953733Re + 0.065176$$

$$Sh_{est} = -0.007645t + 0.000344\omega + 0.103518Re + 0.359017Sc - 0.034689K_c + 0.938922$$

Negative sign of the estimated regression coefficient implies that the corresponding parameter reduces the corresponding physical quantity. The estimated regression equation corresponds with the results achieved in Tables 2.2 and 2.3. Figures 2.17 and 2.18 illustrate the accuracy of the regression model for the chosen sample.

2.7 CONCLUSION

The main conclusions drawn from the current study are listed below:

- The main flow velocity profile is directly proportional to Grashof number (Gr), volume fraction of nanoparticle (ϕ) and modified Grashof number (Gm).
- The main flow velocity profile is greater when the magnetic field is applied on the upward moving plate as compared to the main flow velocity when magnetic field is applied on the downward moving plate.
- The injection parameter (Re) has a constructive effect on the main flow velocity.
- Injection parameter enhances the temperature profile whereas nanoparticle volume fraction diminishes the temperature profile.
- On the upward moving plate, main flow velocity profile exhibits a mixed behaviour when Hartmann number (H) is increased. Initially the velocity reduces and after travelling a certain length the velocity increases.
- On the downward moving plate, velocity profile is inversely proportional to the Hartmann number.
- Schmidt number has a constructive effect on the concentration profile.
- Nusselt number is highly positively correlated with the injection parameter and highly negatively correlated with nanoparticle volume fraction.

- The regression models are found to be faultless for the chosen range of values of the parameters.

TABLES AND GRAPHS

Table 2.1: *Thermophysical properties of base fluid and nanoparticles*

Physical Properties	H_2O	Cu	TiO_2	Fe_3O_4
ρ	997.1	8933	4250	5180
C_p	4179	385	686.2	670
β	$21 * 10^5$	$1.67 * 10^5$	$0.9 * 10^5$	$1.3 * 10^5$
σ	$5 * 10^{-2}$	$5.96 * 10^7$	$2.38 * 10^6$	25000
k	0.613	400	8.9538	9.7

Table 2.2: *Variation in Nu for differing parameter values at $y = 1$ when $\phi = 0.02, \omega = 5, Re = 1, t = \pi/2, z = 0$ and $Pr = 7$*

t	ϕ	ω	Re	Nu	**Enhancement/Decrement Rate**
0.5				6.96773	
1				6.98036	0.18%
1.5				6.99453	0.20%
Slope				**0.02681**	
	0.01			7.00444	
	0.02			6.99646	-0.11%
	0.03			6.98918	-0.10%
Slope				**-0.76297**	
		5		6.99646	
		6		6.9855	-0.16%
		7		6.97932	-0.09%
Slope				**-0.0086**	
			1	6.99646	
			1.1	7.6937	9.97%
			1.2	8.39399	9.10%
Slope				**6.98762**	

Table 2.3: *Variation in Sh for differing parameter values at$y = 1$ when $\omega = 5, Re = 1, Sc = 0.2, Kr = 1, t = \pi/2, z = 0$ and $Pr = 7$*

t	ω	Re	Sc	K_c	Sh	**Enhancement/Decrement Rate**
0.5					1.07745	
1					1.07457	-0.27%
1.5					1.06994	-0.43%
Slope					**-0.0075**	
	5				1.0692	
	6				1.06954	0.03%
	7				1.06987	0.03%
Slope					**0.00033**	
		1			1.0692	
		1.1			1.07958	0.97%
		1.2			1.09001	0.97%
Slope					**0.10405**	
			0.2		1.0692	
			0.4		1.1398	6.60%
			0.6		1.21156	6.30%
Slope					**0.3559**	
				0.5	1.08699	
				1	1.0692	-1.64%
				1.5	1.05182	-1.63%
Slope					**-0.0352**	

Table 2.4: *Correlation coefficient (r), Probable error (PE) and $\left|\frac{r}{PE}\right|$ values of Nu at $y = 1$ with respect to the parameters ϕ, ω, Re and t*

Parameter	r	PE	$\left\|\frac{r}{PE}\right\|$
ϕ	-0.998525	0.000889	1122.9909
ω	-0.961356	0.025561	37.6096
Re	0.99999	$5.02 * 10^{-7}$	199156.61
t	0.999451	0.000428	2337.49

Table 2.5: *Correlation coefficient (r), Probable error (PE) and $\left|\frac{r}{PE}\right|$ values of Sh at $y = 1$ with respect to the parameters ω, Re, Sc, Kr and t*

Parameter	r	PE	$\left\|\frac{r}{PE}\right\|$
ω	0.999963802	$2.44 * 10^{-5}$	40956.47543
Re	0.999998301	$1.02 * 10^{-6}$	975880.9088
Sc	0.999968876	$1.88 * 10^{-5}$	53256.12316
K_c	-0.99994774	$3.52 * 10^{-5}$	28369.64913
t	-0.99111797	0.006887008	143.9112518

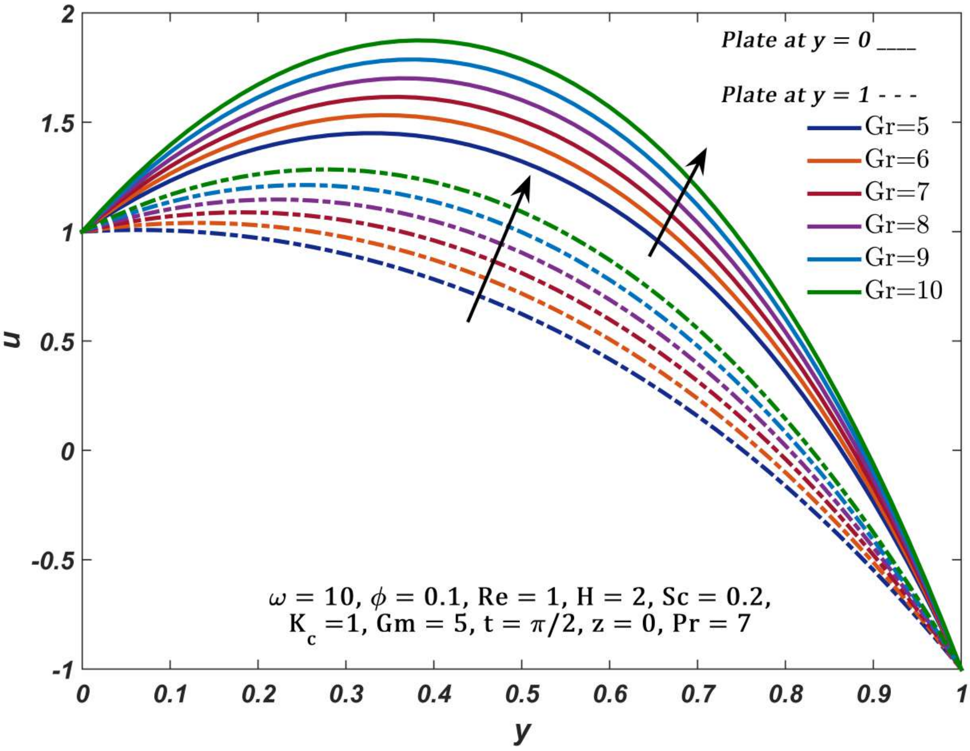

Figure 2.2: *Variation in u with Gr*

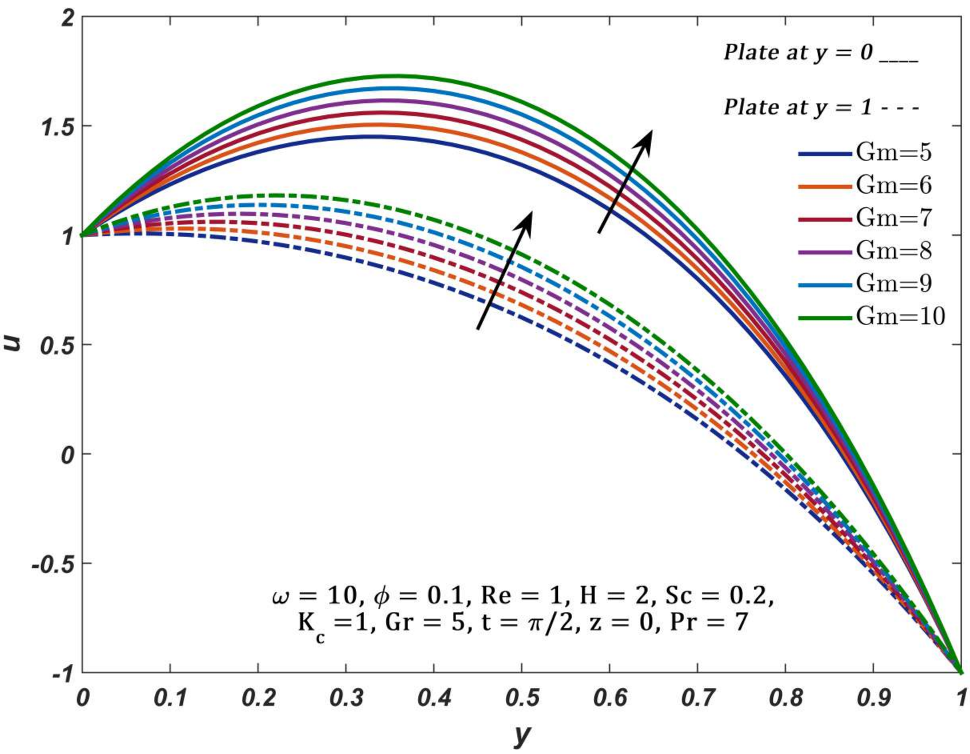

Figure 2.3: *Variation in u with Gm*

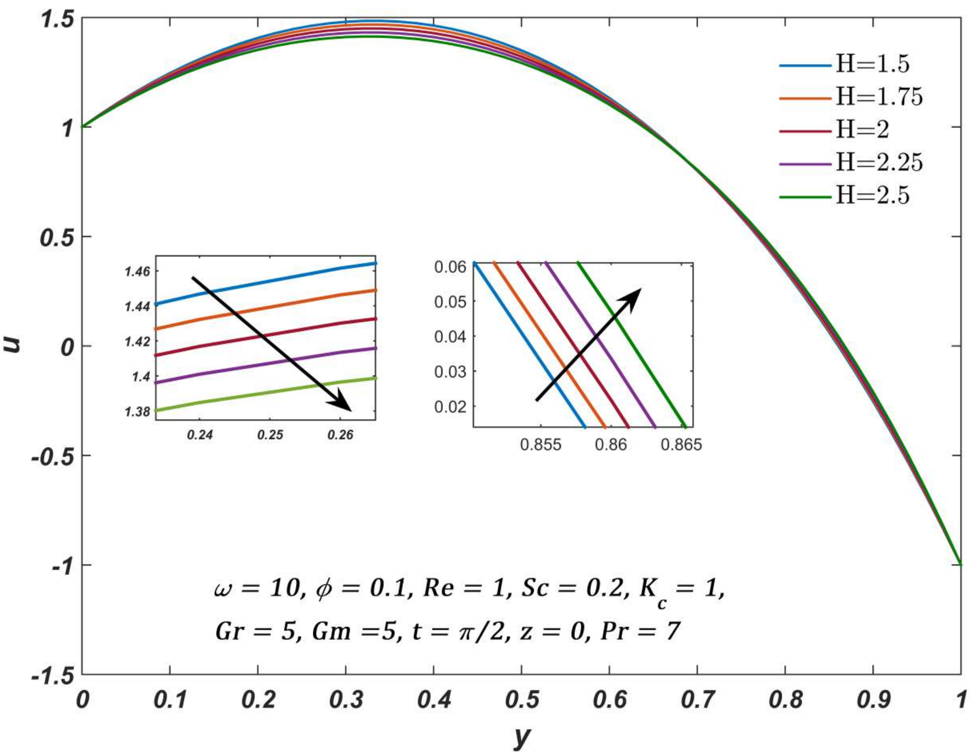

Figure 2.4: *Variation in u with H at $y = 0$*

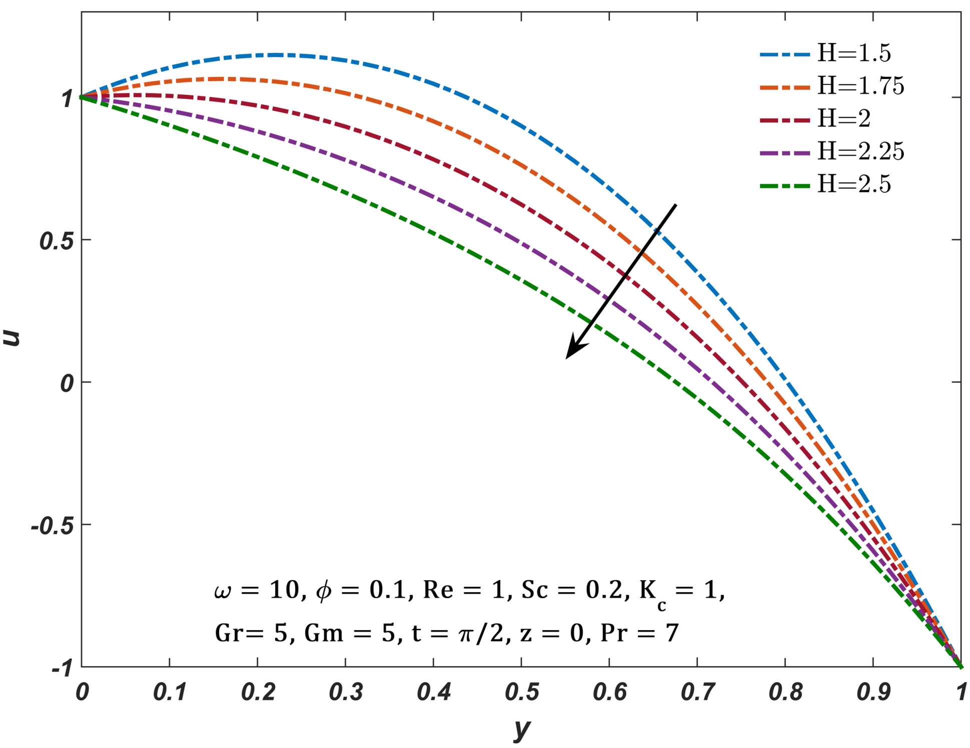

Figure 2.5: *Variation in u with H at $y = 1$*

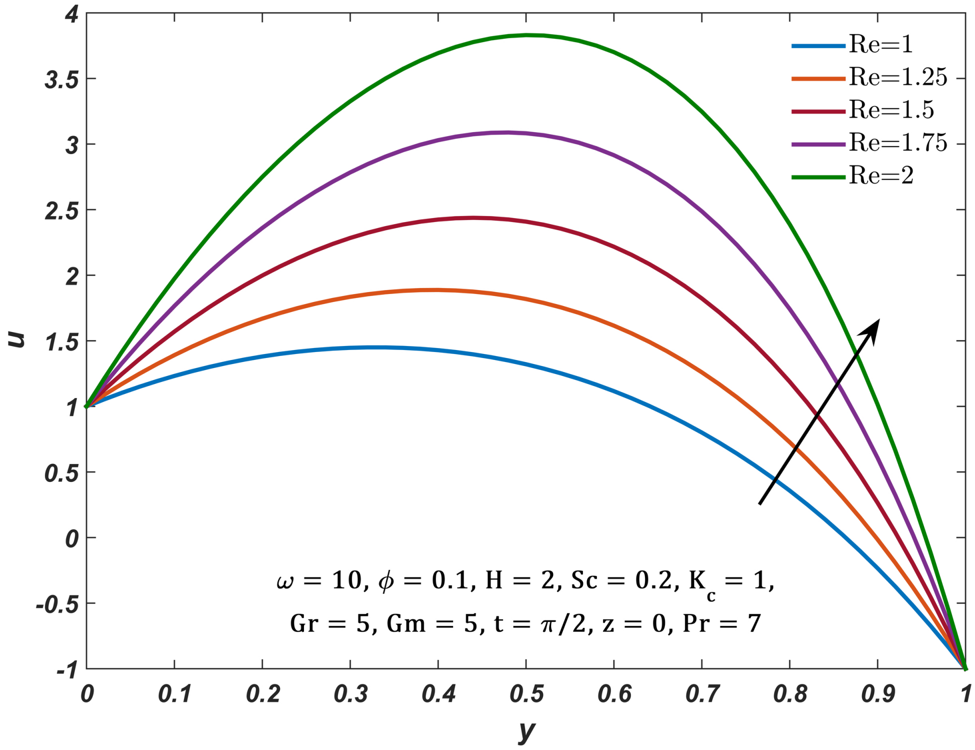

Figure 2.6: *Variation in u with Re at $y = 0$*

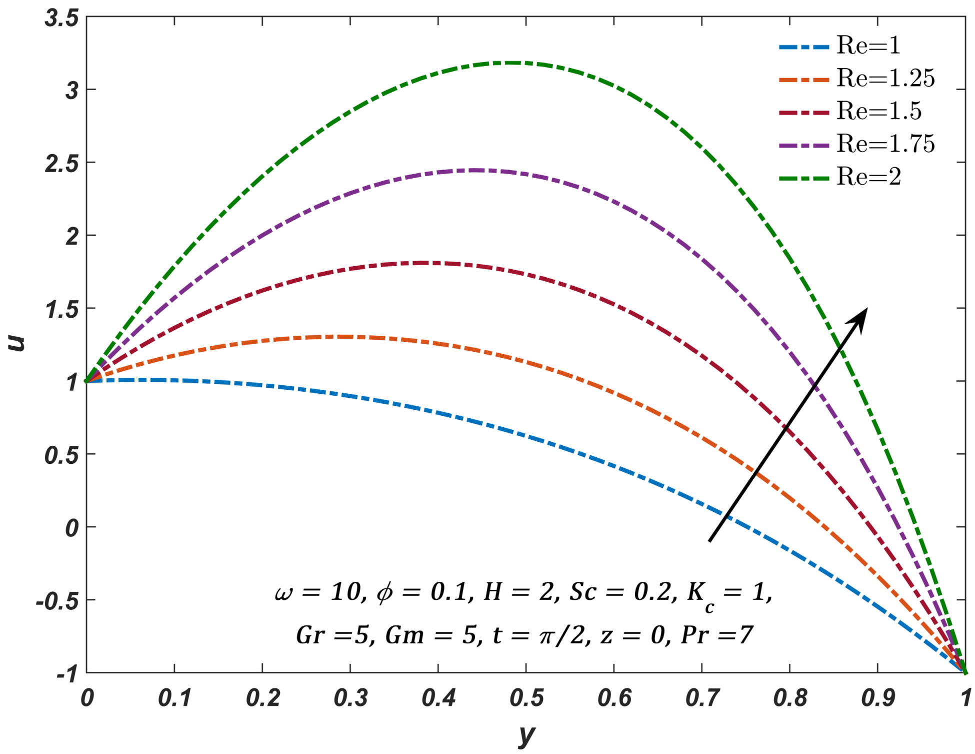

Figure 2.7: *Variation in u with Re at $y = 1$*

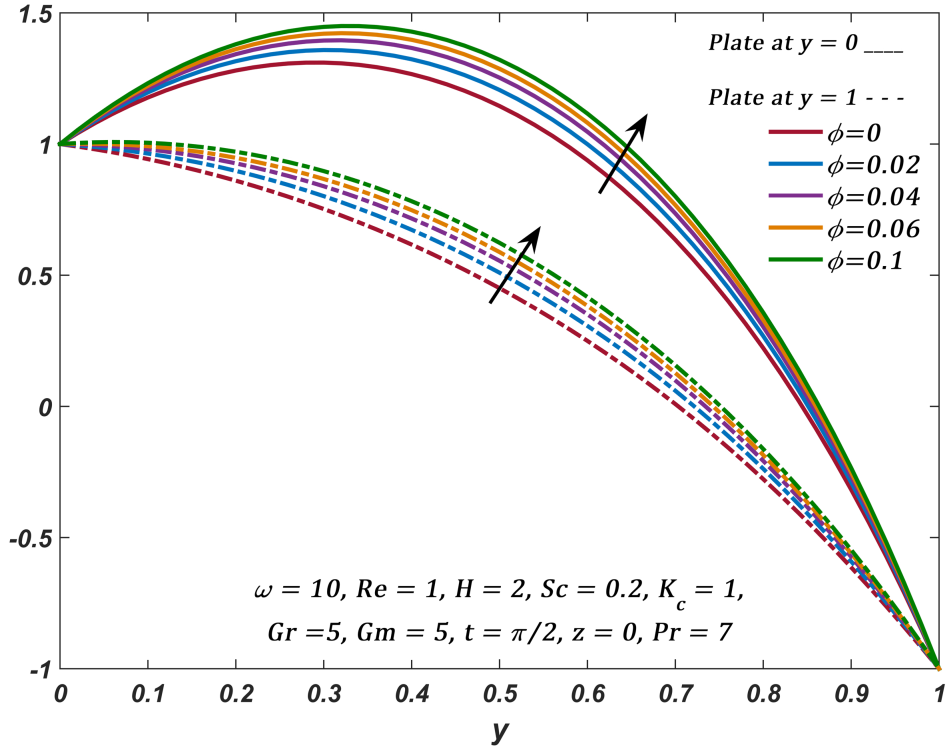

Figure 2.8: *Variation in u with ϕ*

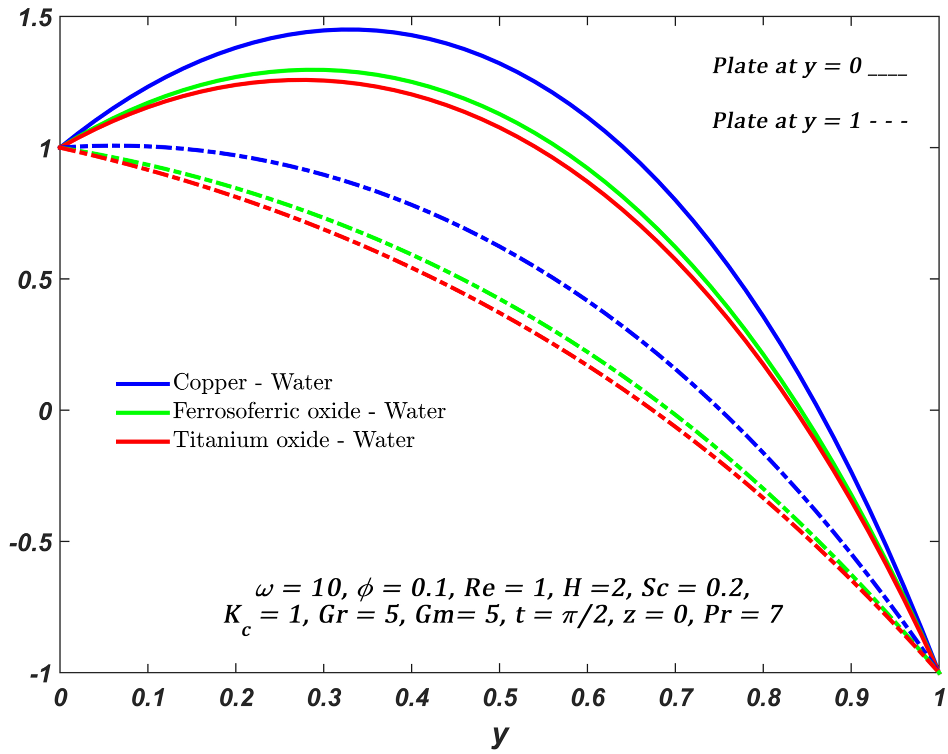

Figure 2.9: *Variation in u with different nanofluids*

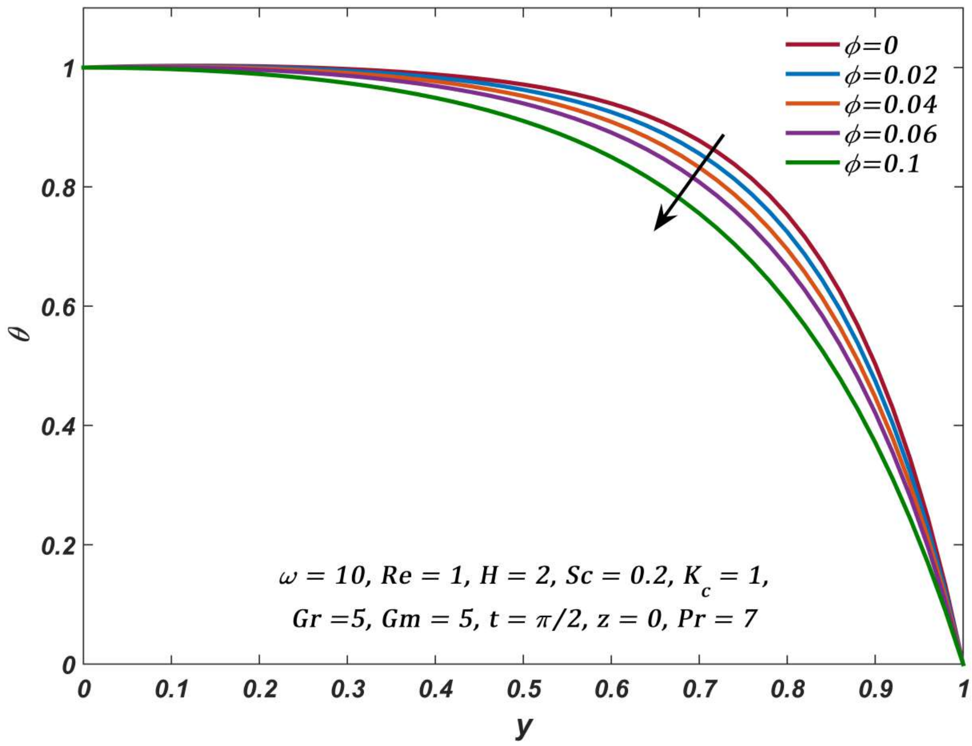

Figure 2.10: *Variation in θ with ϕ*

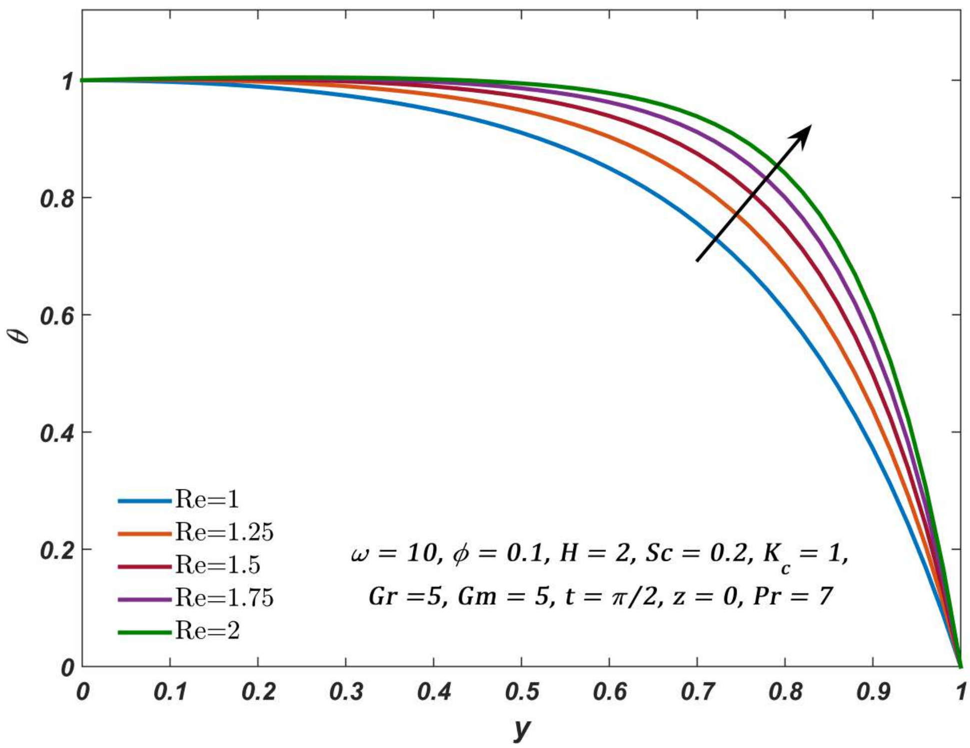

Figure 2.11: *Variation in θ with Re*

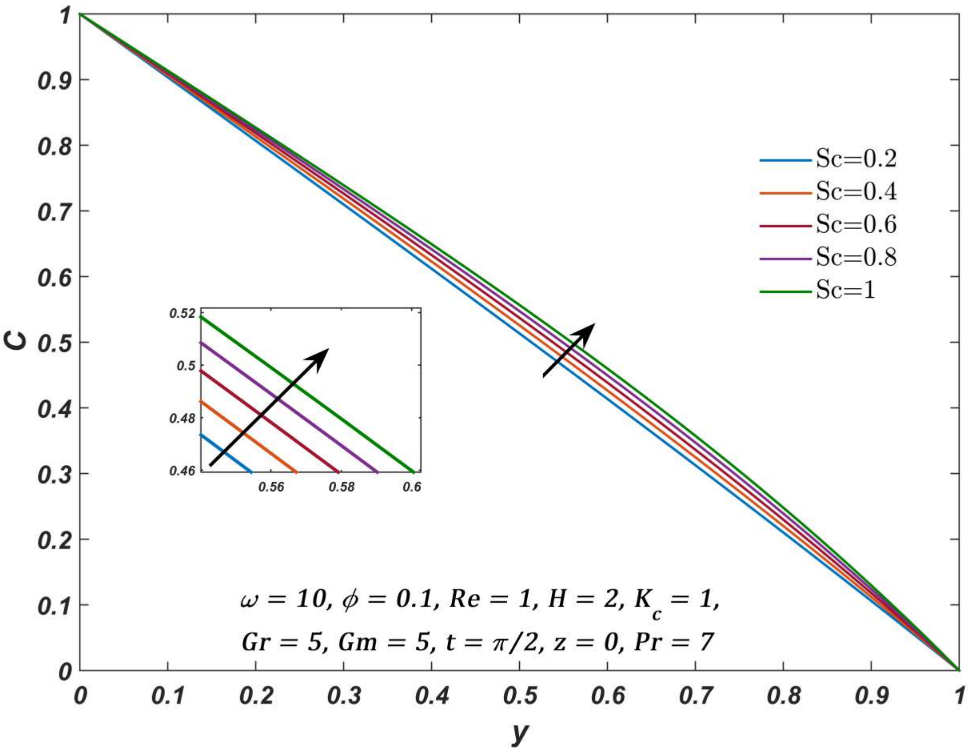

Figure 2.12: *Variation in C with Sc*

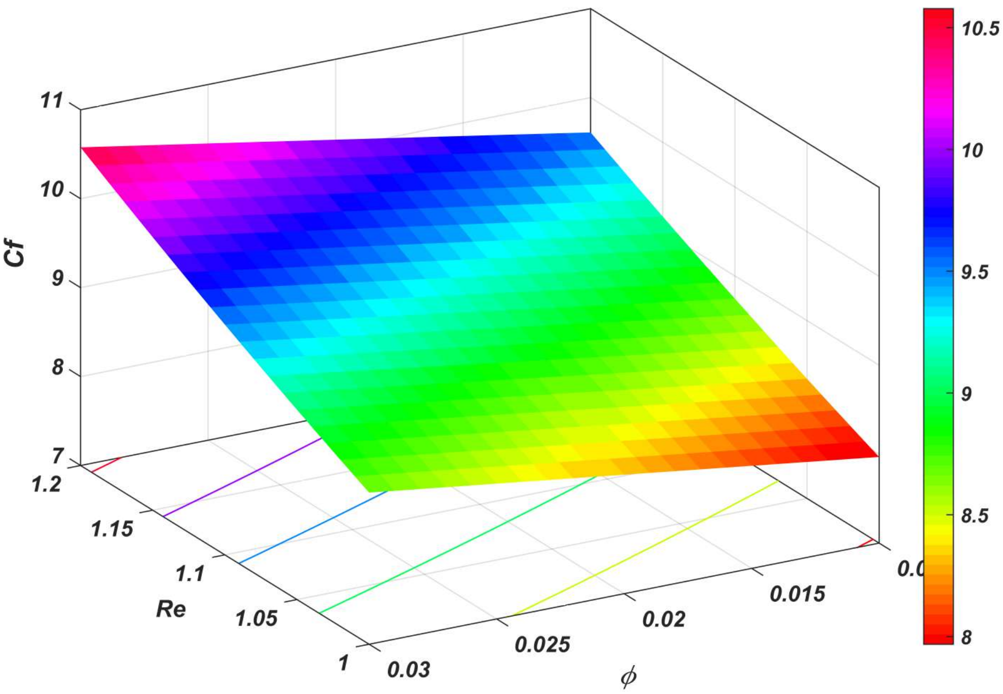

Figure 2.13: *Variation in Cf with Re and ϕ at $y = 0$*

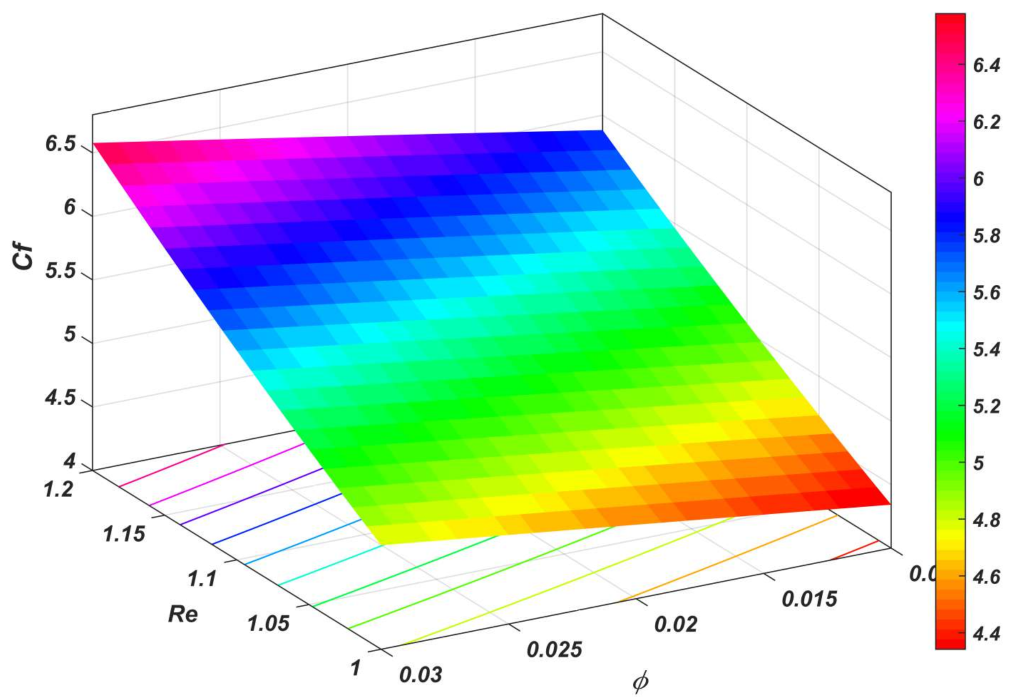

Figure 2.14: *Variation in Cf with Re and ϕ at $y = 1$*

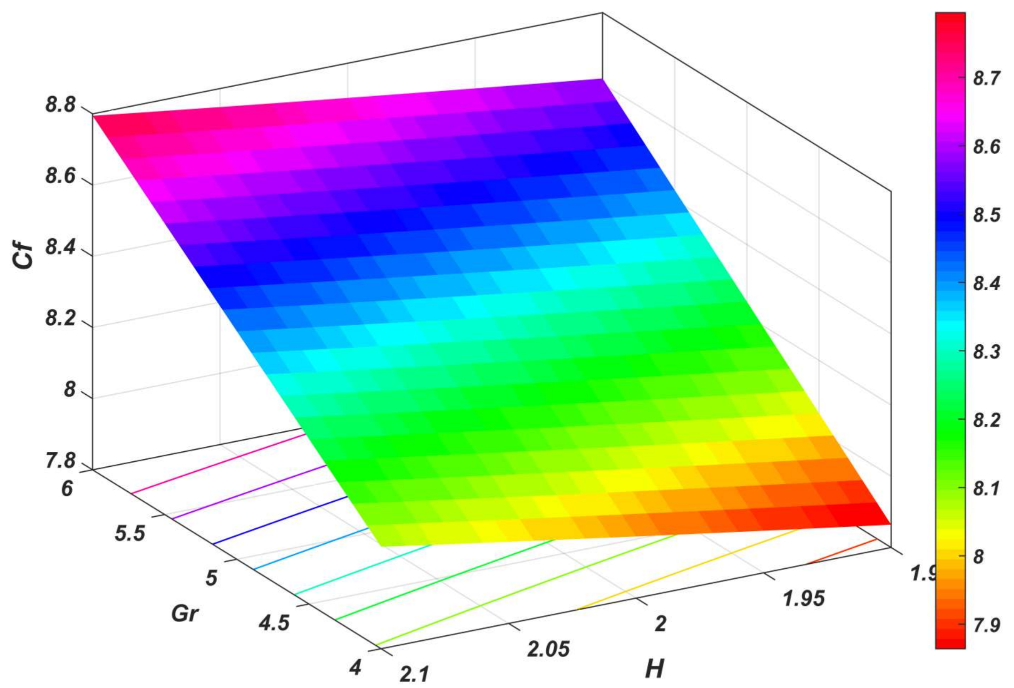

Figure 2.15: *Variation in Cf with Gr and H at $y = 0$*

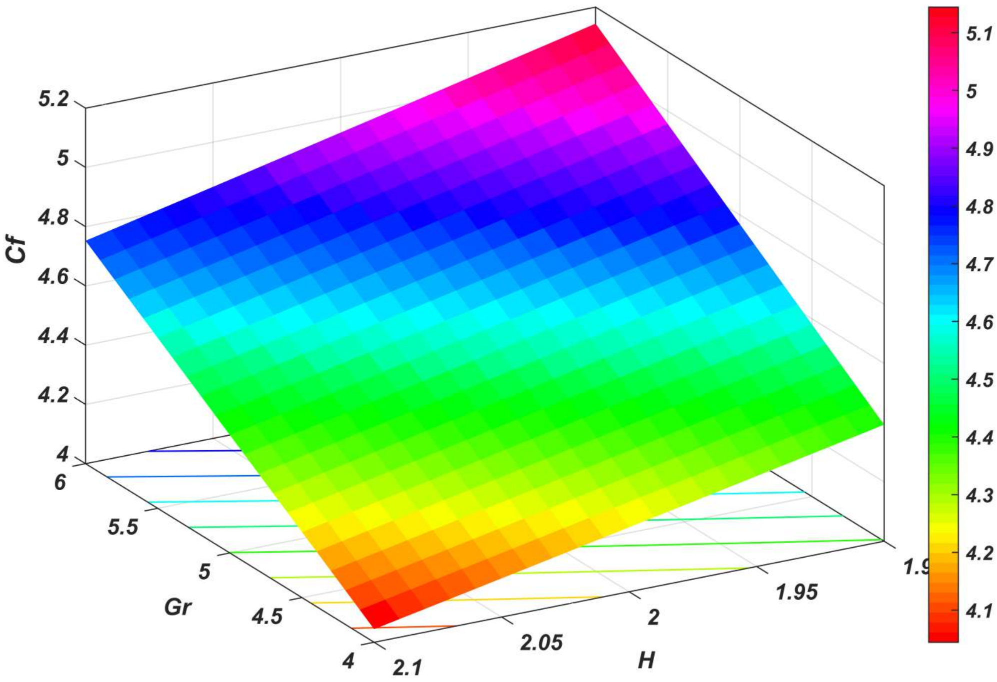

Figure 2.16: *Variation in Cf with Gr and H at $y = 1$*

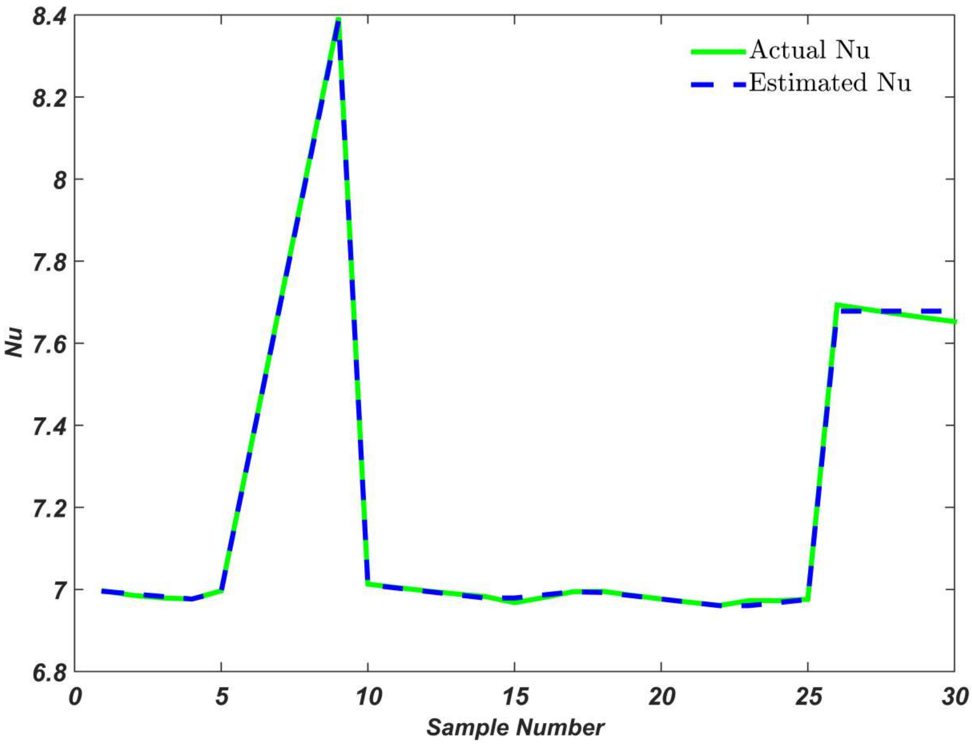

Figure 2.17: *Actual Nu versus estimated Nu*

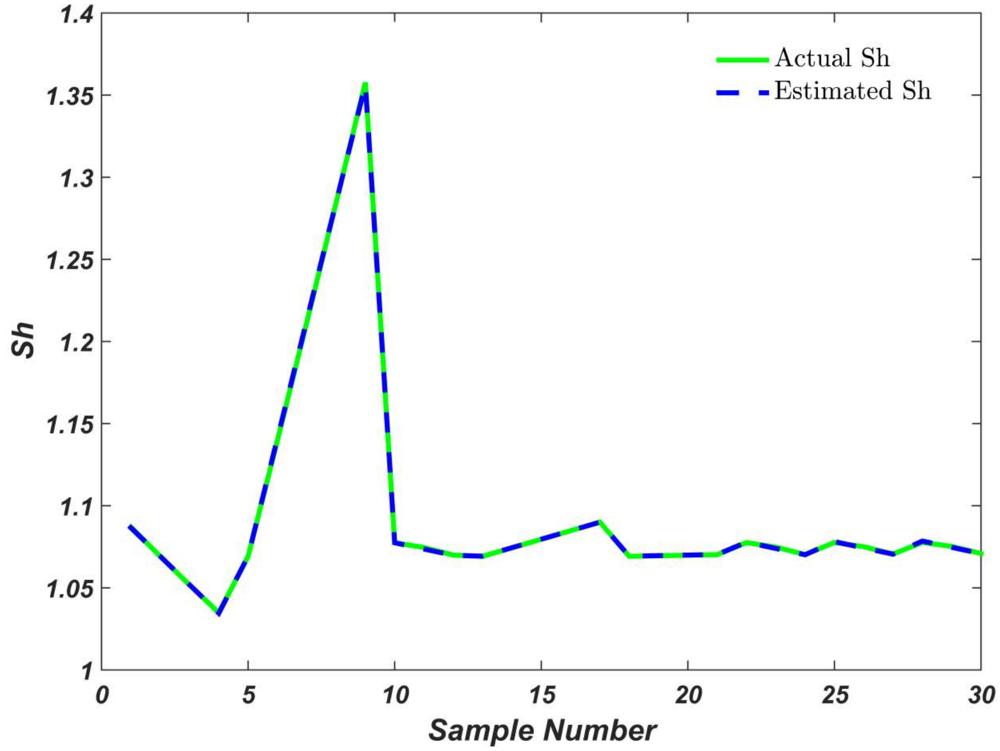

Figure 2.18: *Actual Sh versus estimated Sh*

Appendix I: Non-Dimensional Quantities

$Pr = \frac{(\mu c_p)_f}{\kappa_f}$	Prandtl Number
$Re = \frac{V_0 d}{\vartheta_f}$	Injection/ Suction Parameter
$K_c = \frac{K_l d^2}{\vartheta_f}$	Chemical reaction Parameter
$Sc = \frac{\vartheta_f}{D_B}$	Schmidth Number
$H = B_0 d\sqrt{\frac{\sigma_f}{\rho_f \vartheta_f}}$	Hartmann Number
$Gr = \frac{g\beta_f \vartheta_f (T_0 - T_1)}{U_0 V_0^2}$	Grashof Number
$Gm = \frac{g\beta_f \vartheta_f (\mathcal{C}_0 - \mathcal{C}_1)}{U_0 V_0^2}$	Modified Grashof Number

Appendix II: Nomenclature

u^*, v^*, w^*	Velocity components	T^*	Fluid temperature
C^*	Fluid concentration	g	Acceleration due to gravity
t^*	Time	D_B	Chemical molecular diffusivity
C_0	Nanoparticle concentration near the upward moving plate	C_1	Nanoparticle concentration near the downward moving plate
T_0	Temperature of the fluid near the upward moving plate	T_1	Temperature of the fluid near the downward moving plate
B_0	Strength of magnetic field	U_0	Velocity of the moving plates
V_0	Injection velocity	K_l	Chemical reaction coefficient
σ	Electrical conductivity	p^*	Pressure
μ	Dynamic viscosity	ρ	Density
ν	Kinematic viscosity	ω^*	Angular velocity
K	Thermal conductivity	s	Nanoparticle
C_p	Specific heat at constant pressure	f	Base fluid
ϕ	Volume fraction of nanoparticles	nf	Nanofluid
$\varepsilon_1, \varepsilon_2, \varepsilon_3$	Very small reference constants	d	Distance between the plates

Appendix III: Short keys

$$a = \frac{A_5 Pr Re}{A_6}$$

$$m_1 = \frac{ScRe+\sqrt{Sc^2Re^2+4ScK_c}}{2} \qquad m_2 = \frac{ScRe-\sqrt{Sc^2Re^2+4ScK_c}}{2}$$

$$m_3 = \frac{A_1A_2Re+\sqrt{A_1^2A_2^2Re^2+4A_1A_3H^2}}{2} \qquad m_4 = \frac{A_1A_2Re-\sqrt{A_1^2A_2^2Re^2+4A_1A_3H^2}}{2}$$

$$B_1 = \frac{A_1A_2A_4Re^2Gr}{(e^a-1)(a^2-A_1A_2Rea-A_1A_3H^2)} \qquad B_2 = \frac{A_1A_2A_4Re^2Gme^{m_2}}{(e^{m_1}-e^{m_2})(m_1^2-A_1A_2Rem_1-A_1A_3H^2)}$$

$$B_3 = \frac{A_1A_2A_4Re^2Gme^{m_1}}{(e^{m_2}-e^{m_1})(m_2^2-A_1A_2Rem_2-A_1A_3H^2)} \qquad B_4 = \frac{A_1A_2A_4Re^2Gre^a}{(e^a-1)(A_1A_3H^2)}$$

$$\alpha_1 = -(B_1+B_2+B_3+B_4) \qquad \beta_1 = -2-(B_1e^a+B_2e^{m_1}+B_3e^{m_2}+B_4)$$

$$r_1 = \frac{m_3+\sqrt{m_3^2+4\pi^2}}{2} \qquad r_2 = \frac{m_3-\sqrt{m_3^2+4\pi^2}}{2}$$

$$r_3 = \frac{m_4+\sqrt{m_4^2+4\pi^2}}{2} \qquad r_4 = \frac{m_4-\sqrt{m_4^2+4\pi^2}}{2}$$

$$r_5 = \frac{A_1A_2Re+\sqrt{A_1^2A_2^2Re^2+4(i\omega A_1A_2+A_1A_3H^2)}}{2} \qquad r_6 = \frac{A_1A_2Re-\sqrt{A_1^2A_2^2Re^2+4(i\omega A_1A_2+A_1A_3H^2)}}{2}$$

$$D_1 = r_3r_4(e^{r_2+r_4}-e^{r_2+r_3})+r_2r_4(e^{r_2+r_3}-e^{r_3+r_4})+r_2r_3(e^{r_3+r_4}-e^{r_2+r_4})$$

$$D_2 = r_3r_4(e^{r_1+r_3}-e^{r_1+r_4})+r_1r_4(e^{r_3+r_4}-e^{r_1+r_3})+r_1r_3(e^{r_1+r_4}-e^{r_3+r_4})$$

$$D_3 = r_2r_4(e^{r_1+r_4}-e^{r_1+r_2})+r_1r_4(e^{r_1+r_2}-e^{r_2+r_4})+r_1r_2(e^{r_2+r_4}-e^{r_1+r_4})$$

$$D_4 = r_2r_3(e^{r_1+r_2}-e^{r_1+r_3})+r_1r_3(e^{r_2+r_3}-e^{r_1+r_2})+r_1r_2(e^{r_1+r_3}-e^{r_2+r_3})$$

$$D = D_1+D_2+D_3+D_4$$

$$n_1 = \frac{\frac{A_5PrRe}{A_6}+\sqrt{\frac{A_5^2Pr^2Re^2}{A_6^2}+\frac{4A_5Pr\omega i}{A_6}}}{2} \qquad n_2 = \frac{\frac{A_5PrRe}{A_6}-\sqrt{\frac{A_5^2Pr^2Re^2}{A_6^2}+\frac{4A_5Pr\omega i}{A_6}}}{2}$$

$$m_5 = \frac{ScRe+\sqrt{Sc^2Re^2+4Sc(\omega i+K_c)}}{2} \qquad m_6 = \frac{ScRe-\sqrt{Sc^2Re^2+4Sc(\omega i+K_c)}}{2}$$

$$K_1 = \frac{A_1A_2A_4Re^2Gre^{n_2}}{(e^{n_1}-e^{n_2})(n_1^2-A_1A_2Ren_1-A_1A_2i\omega-A_1A_3H^2)} \qquad K_2 = \frac{A_1A_2A_4Re^2Gre^{n_1}}{(e^{n_2}-e^{n_1})(n_2^2-A_1A_2Ren_2-A_1A_2i\omega-A_1A_3H^2)}$$

$$K_3 = \frac{A_1A_2A_4Re^2Gme^{m_6}}{(e^{m_5}-e^{m_6})(m_5^2-A_1A_2Rem_5-A_1A_2i\omega-A_1A_3H^2)} \qquad K_4 = \frac{A_1A_2A_4Re^2Gme^{m_5}}{(e^{m_6}-e^{m_5})(m_6^2-A_1A_2Rem_6-A_1A_2i\omega-A_1A_3H^2)}$$

$$\alpha_2 = -(K_1+K_2+K_3+K_4) \qquad \beta_2 = -(K_1e^{n_1}+K_2e^{n_2}+K_3e^{m_5}+K_4e^{m_6})$$

$$n_3 = \frac{A_1A_2Re+\sqrt{A_1^2A_2^2Re^2+4(A_1A_3H^2+\pi^2)}}{2} \qquad n_4 = \frac{A_1A_2Re-\sqrt{A_1^2A_2^2Re^2+4(A_1A_3H^2+\pi^2)}}{2}$$

$$K_{i1} = \frac{m_3(\alpha_1 e^{m_4}-\beta_1)A_1A_2ReD_i}{(e^{m_4}-e^{m_3})D[(r_i+m_3)^2-A_1A_2Re(r_i+m_3)-(A_1A_3H^2+\pi^2)]}, i=1,2,3,4$$

$$K_{i2} = \frac{m_4(\beta_1-\alpha_1 e^{m_3})A_1A_2ReD_i}{(e^{m_4}-e^{m_3})D[(r_i+m_4)^2-A_1A_2Re(r_i+m_4)-(A_1A_3H^2+\pi^2)]}, i=1,2,3,4$$

$$M_{i1} = \frac{m_3((\alpha_1+2)e^{m_4}-(\beta_1+2))A_1A_2ReD_i}{(e^{m_4}-e^{m_3})D[(r_i+m_3)^2-A_1A_2Re(r_i+m_3)-(A_1A_3H^2+\pi^2)]}, i=1,2,3,4$$

$$M_{i2} = \frac{m_4((\beta_1+2)-(\alpha_1+2)e^{m_3})A_1A_2ReD_i}{(e^{m_4}-e^{m_3})D[(r_i+m_4)^2-A_1A_2Re(r_i+m_4)-(A_1A_3H^2+\pi^2)]}, i=1,2,3,4$$

$$K_{i3} = M_{i3} = \frac{aB_1A_1A_2ReD_i}{D[(r_i+a)^2-A_1A_2Re(r_i+a)-(A_1A_3H^2+\pi^2)]}, i=1,2,3,4$$

$$K_{i4} = M_{i4} = \frac{m_1B_2A_1A_2ReD_i}{D[(r_i+m_1)^2-A_1A_2Re(r_i+m_1)-(A_1A_3H^2+\pi^2)]}, i=1,2,3,4$$

$$K_{i5} = M_{i5} = \frac{m_2B_3A_1A_2ReD_i}{D[(r_i+m_2)^2-A_1A_2Re(r_i+m_2)-(A_1A_3H^2+\pi^2)]}, i=1,2,3,4$$

$$\alpha_3 = -\sum_{i=1}^{4}(K_{i1}+K_{i2}+K_{i3}+K_{i4}+K_{i5})$$

$$\beta_3 = -\sum_{i=1}^{4}(K_{i1}e^{r_i+m_3}+K_{i2}e^{r_i+m_4}+K_{i3}e^{r_i+a}+K_{i4}e^{r_i+m_1}+K_{i5}e^{r_i+m_2})$$

$$\alpha_4 = -\sum_{i=1}^{4}(M_{i1}+M_{i2}+M_{i3}+M_{i4}+M_{i5})$$

$$\beta_4 = -\sum_{i=1}^{4}(M_{i1}e^{r_i+m_3}+M_{i2}e^{r_i+m_4}+M_{i3}e^{r_i+a}+M_{i4}e^{r_i+m_1}+M_{i5}e^{r_i+m_2})$$