Chapter 1

Introduction

1.1 Fluid

A substance capable of flowing is termed as a fluid. Fluids are of two types, namely liquids and gases. The study of fluid's behaviour at rest (termed fluid statics) and in motion (termed fluid dynamics) is combinedly known as fluid mechanics.

1.2 Density

The ratio of fluid's mass to its volume is defined as the density (denoted by ρ) of the fluid.

1.3 Viscosity

The internal friction that exhibits resistance to the alterations in fluid's form is known as viscosity. Viscosity can also be conceptualized as the thickness of a fluid.

1.4 Newton's Law of Viscosity

The Newton's law of viscosity states that the shear stress is directly proportional to the velocity gradient. Mathematically,

$$\tau = \mu \frac{du}{dx} \tag{1.4.1}$$

where τ is the shear stress, u is the velocity and μ is the constant of proportionality known as dynamic viscosity. The ratio of dynamic viscosity to the density is known as kinematic viscosity.

1.5 Thermal Conductivity

The measure that quantifies the material's ability to conduct heat is known as thermal conductivity. Alternatively, thermal conductivity can also be defined as the ratio of heat flux to the temperature gradient.

1.6 Types of Fluid

1.6.1 Incompressible Fluid

A fluid whose density remains constant when a force is applied is known as incompressible fluid. Liquids are, generally, regarded as incompressible fluids.

1.6.2 Compressible Fluid

A fluid whose density alters when a force is applied is known as compressible fluid. Gases are, generally, regarded as compressible fluids.

1.6.3 Ideal Fluid

Ideal fluid is an imaginary fluid that is incompressible and having zero viscosity.

1.6.4 Real Fluid

Any fluid having viscosity is known as real fluid. Realistically, all fluids are real fluids.

1.6.5 Newtonian Fluid

The fluid that follows Newton's law of viscosity is known as Newtonian fluid. Alcohol, gases, gasoline and water are a few examples of Newtonian fluid.

1.6.6 Non-Newtonian Fluid

The fluid that doesn't follow Newton's law of viscosity is known as non-Newtonian fluid. Blood, paint, ketchup and toothpaste are a few examples of non-Newtonian fluid.

1.6.7 Ideal Plastic Fluid

A Newtonian fluid whose shear stress exceeds the resultant value is known as ideal plastic fluid.

1.7 Dilatant Fluid

Dilatant (or shear thickening) fluid is a type of non-Newtonian fluid that exhibits a positive functional relationship between shear stress and viscosity. Examples of dilatant fluids include oobleck and quicksand.

1.8 Pseudoplastic Fluid

Pseudoplastic (or shear thinning) fluid is a type of non-Newtonian fluid that exhibits an inverse functional relationship between shear stress and viscosity. Examples of pseudoplastic fluids include blood and ketchup.

1.9 Casson Fluid

Casson fluid is a shear thinning fluid that assumes infinite viscosity at zero shear-rate, and vice versa. The rheological equation for an isotropic and incompressible flow of a Casson fluid is given by (see Isa et al., 2017; Suresh Reddy & Panda, 2022):

$$\tau_{ij} = \begin{cases} 2\left(\mu_B + \frac{P_y}{\sqrt{2\pi}}\right) e_{ij}\ , \ \pi > \pi_c \\ \\ 2\left(\mu_B + \frac{P_y}{\sqrt{2\pi_c}}\right) e_{ij},\ \pi < \pi_c \end{cases} \tag{1.9.1}$$

where where μ_B is plastic dynamic viscosity of the non-Newtonian fluid, P_y is the yield stress of the fluid, τ_{ij} is $(i,j)^{th}$ component of the stress tensor, $\pi = e_{ij}e_{ij},\ e_{ij}$ is the $(i,j)^{th}$ component of the deformation rate and π_c is a critical value of π based on the non-Newtonian model. *Human blood*, honey, tomato sauce, jelly, and concentrated fruit juices are examples of Casson fluid.

1.10 Carreau Fluid

Carreau fluid is a generalized Newtonian fluid whose viscosity μ depends on the shear rate, which is based on the equation:

$$\mu(\dot{\gamma}) = \mu_\infty + (\mu_0 - \mu_\infty)\left(1 + (\lambda\dot{\gamma})^2\right)^{\frac{n-1}{2}} \tag{1.10.1}$$

where λ is the relaxation time, $\dot{\gamma}$ is the generalised shear rate, n is the power-law index, μ_0 represents the zero shear viscosity, and μ_∞ is the infinite shear viscosity. It is noteworthy to mention that the above model with $n < 1$ shows a shear thinning nature and $n > 1$ shows a shear thickening nature. *Human blood* is an example of Carreau fluid with shear thinning nature.

1.11 Laminar Fluid Flow

Laminar fluid flow corresponds to the smooth movement of fluid particles. It is characterized by the movement of particles in laminas with zero disruption and negligible lateral mixing.

1.12 Steady and Unsteady Fluid Flow

The fluid flow in which the fluid properties (like pressure, velocity, density, etc.) at a point remains unaltered with time is known as steady fluid flow. The case in which the fluid properties at a point alter with time corresponds to the unsteady fluid flow.

1.13 Rotational and Irrotational flow

The curl of velocity (known as vorticity, Ω) measures the fluid's rotation. The type of flow in which the fluid particles rotate about their own axis (when $\Omega \neq 0$) is known as rotational flow. The type of flow in which the fluid particles do not rotate about their own axis (when $\Omega = 0$) is known as irrotational flow.

1.14 Mathematical Model

The description of a system using mathematical concepts and language is known as mathematical model and the process of developing a mathematical model is known as mathematical modelling. Mathematical models find its use in natural sciences, engineering disciplines and social sciences. A mathematical model helps to explain a system and to study the effects of different components and also to make predictions about its behaviour.

1.15 Fundamental Equations of Fluid Flow

1.15.1 Equation of Continuity

The equation of continuity is derived from the law of conservation of mass which states that the rate of increase in the fluid's mass within a considered region is equal to the rate of flow through its boundary. In vector notation it can be written as (see Bansal, 1977):

$$\frac{\partial \rho}{\partial t} + \nabla.\left(\rho \vec{V}\right) = 0 \tag{1.15.1}$$

where $\vec{V}$ is the velocity vector and ρ is the fluid density.

Note:

- In the case of steady compressible flow, the equation of continuity reduces to:

$$\nabla.\left(\rho\vec{V}\right) = 0 \tag{1.15.2}$$

- In the case of incompressible flow, the equation of continuity reduces to:

$$\nabla.\vec{V} = 0 \tag{1.15.3}$$

1.15.2 Equation of Momentum (Navier-Stokes' Equation)

The equation of momentum is derived from the Newton's second law of motion which states that the total force amounts to the rate of change of linear momentum. In vector notation it can be written as (see Bansal, 1977):

$$\rho\frac{D\vec{V}}{Dt} = \rho\vec{F} + \mu\nabla^2\vec{V} - \nabla P \tag{1.15.4}$$

where $\vec{V}$ is the velocity vector, ρ is the fluid density, P is the pressure, $\vec{F}$ is the force, μ is the dynamic viscosity, and $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \left(\vec{V}.\nabla\right)$ is known as the material derivative.

1.15.3 Equation of Energy

The equation of energy is derived from the law of conservation of energy which states that the total energy in an isolated system is conserved. It can be written as (see Bansal, 1977):

$$\rho\frac{DE_t}{Dt} = \rho\left(v_i\frac{Dv_i}{Dt} + \frac{DI}{Dt} + v_j\frac{\partial K}{\partial x_j}\right) \tag{1.15.5}$$

where $(v_i\ , v_j)$ are the velocity components, ρ is the fluid density, I is the internal energy, K is the potential energy, and E_t is the total energy.

1.16 Boundary Conditions

The constraints necessary to solve a boundary value problem (BVP) are termed as boundary conditions (see Wazwaz, 2002; Strauss, 2007). BVP is a differential equation (or system of differential equations) that must be solved in a domain with specified boundaries. BVP is different from the initial value problem (IVP) in which only one extreme of the interval is known.

1.16.1 Dirichlet Boundary Condition

Dirichlet boundary condition (DBC), named after Peter Gustav Lejeune Dirichlet, is a type of boundary condition that specifies the value that the unknown function must take along the domain's boundary.

1.16.2 Neumann Boundary Condition

Neumann boundary condition (NBC), named after Carl Neumann, is a type of boundary condition that specifies the value that the derivative of a solution will take on the domain's boundary.

1.16.3 Robin Boundary Condition

Robin boundary condition (RBC), named after Victor Gustave Robin, is a type of boundary condition that consists of a linear combination of the field's values and its derivative on the boundary.

1.16.4 Mixed Boundary Condition

Mixed boundary condition (MBC) is a type of boundary condition that applies multiple types of boundary conditions in different parts of the domain. It is worth noting that the boundary condition must be applied on the entire boundary. RBC consists of applying different types of boundary conditions to the same boundary region whereas MBC consists of applying different types of boundary conditions to different parts of the boundary.

1.16.5 Cauchy Boundary Condition

Cauchy boundary condition (CBC) is a type of boundary condition that applies on both the unknown field and its derivative. CBC implies the imposition of two constraints (1 DBC + 1 NBC) whereas RBC implies only one constraint.

1.16.6 Slip Boundary Constraint

The relative movement of the fluid with the boundary is characterized with the aid of slip constraint. Multiple slip corresponds to the case when more than one slip (velocity, thermal, or solutal) condition is considered.

The Wu's velocity slip boundary constraint for arbitrary Knudsen number is given by (see Wu, 2008; Fang & Aziz, 2010):

$$U_{slip} = \frac{2}{3}\left[\frac{(3-\alpha f^3)}{\alpha} - \frac{3}{2}\frac{(1-f^2)}{K_n}\right]\lambda\,\frac{\partial u}{\partial y} - \frac{1}{4}\left[f^4 - \frac{2}{K_n^2}\left(1-f^2\right)\right]\lambda^2\,\frac{\partial^2 u}{\partial y^2}$$

$$= A\,\frac{\partial u}{\partial y} + B\,\frac{\partial^2 u}{\partial y^2}$$

where α is the momentum accommodation coefficient with $0 \leq \alpha \leq 1$, λ is the molecular mean free path, K_n is the Knudsen number defined as the mean free path λ divided by a characteristic length for the flow and $f = min\,[1/K_n, 1]$. Based on the definition of f, it is seen that for any given value of K_n, we have $0 \leq f \leq 1$. The molecular mean free path is always positive. Thus, $B < 0$ and A is a positive number.

1.16.7 Newtonian Boundary Constraint

The Newton thermal boundary condition accounts for one of the most common boundary constraints encountered in general practice. The existence of convective heating (or cooling) at the surface due to the surface energy balance is termed Newtonian (or convective) boundary constraint.

1.16.8 Passive Control of Nanoparticles

The boundary constraint when the normal flux of nanoparticles is zero is termed as zero mass flux or passive control of nanoparticles.

1.16.9 Stefan Blowing

The diffusion of nanoparticles (or species) from the lengthening surface to the ambient region creates a blowing effect. This blowing effect, derived from the Stefan problem, is dissimilar from the mass injection or blowing resulting from transpiration on a permeable surface.

1.17 Bioconvection

Bioconvection refers to the convection due to the erratic movement of microorganisms initiated by the unstable density stratification of microorganisms.

1.18 Magnetic Field

The vector field that describes the magnetic influence on a moving electric charge or a magnetic material is termed as magnetic field. This force is measured using the Lorentz force whose formula is given by:

$$\vec{J} = \sigma\left(\vec{E} + \vec{V} \times \vec{B}\right) \tag{1.18.1}$$

The electromagnetic force $\vec{F_m}$ that must be accounted in the momentum equation is given by:

$$\vec{F_m} = \vec{J} \times \vec{B} \tag{1.18.2}$$

where $\vec{J}$ is the current density, σ is the electrical conductivity, $\vec{V}$ is the velocity, $\vec{E}$ is the electrical field intensity and $\vec{B}$ is the magnetic field.

1.19 Induced Magnetic Field

Induced magnetic field is the additional magnetic field that gets induced on electrically conducting fluid in the presence of an external magnetic field. This phenomenon is due to the impact of a larger magnetic Reynolds number.

1.20 Dimensional Analysis

Dimensional analysis is the study of the mathematical correspondence on the quantities of a physical problem based on its units and dimensions.

1.21 Non-Dimensional Parameters

The ratio of one force to the other force yields a dimensionless number known as the non-dimensional parameter.

1.21.1 Reynolds Number

The ratio of inertial force to the viscous force is known as Reynolds number. The flow pattern is determined by the Reynolds number. A high Reynolds number implies an increased inertial force that corresponds to a turbulent flow pattern. A low Reynolds number implies an increased viscous force that corresponds to a laminar flow pattern.

1.21.2 Magnetic Parameter

The ratio of electromagnetic force to viscous force is known as the magnetic parameter. An increase in the magnetic parameter generates Lorentz force that affects the flow profiles.

1.21.3 Eckert Number

The ratio of advective mass transport to the heat dissipation potential is known as Eckert number. The phenomenon of self-heating of a fluid due to viscous dissipation is characterized by Eckert number.

1.21.4 Schmidt Number

The ratio of kinematic viscosity to mass diffusivity is known as Schmidt number.

1.21.5 Prandtl Number

The ratio of kinematic viscosity to thermal diffusivity is known as Prandtl number. It measures the relative importance of heat conduction and fluid's viscosity.

1.21.6 Lewis Number

The ratio of thermal diffusivity to mass diffusivity is known as Lewis number. It can be alternatively defined as the ratio of Schmidt number to Prandtl number.

1.21.7 Péclet Number

The product of Reynolds number and Prandtl number is known as Péclet number.

1.21.8 Biot Number

The ratio of resistance to heat transport inside a body and at the surface of a body is known as Biot number.

1.21.9 Skin Friction Coefficient

The dimensionless shearing stress on the surface of a body due to a fluid motion is known as the skin friction coefficient.

1.21.10 Nusselt Number

The ratio of convective heat transport to conductive heat transport is known as Nusselt number. It measures the effectiveness of heat convection at the surface.

1.21.11 Sherwood Number

The ratio of convective mass transport to diffusive mass transport is known as Sherwood number. It measures the effectiveness of mass convection at the surface.

1.21.12 Motile Density Number

The ratio of convective transport of microorganisms to the diffusive transport of microorganisms is known as motile density number.

1.22 Body Fluid

The fluid produced and circulated within the human body or secreted outside the human body is known as body fluid. Blood, saliva, urine, tears, sweat, and breast milk are a few examples of body fluid. Water is the basis of all body fluids and the human body is composed of about 60% of water.

1.23 Nanofluid

Nanofluid is a colloidal mixture in which a base fluid (water, oil, ethylene glycol, etc.) is mixed with nanometer-sized particles (metals, carbides, oxides or carbon nanotubes). Fluids constituting two nanometer-sized particles are termed hybrid nanofluids. Choi & Eastman, 1995 observed that the nanofluid tends to upgrade and stabilize the thermal properties of the fluid which marked a revolution in the field of fluid dynamics.

1.24 Volume Fraction of Nanoparticle

Nanoparticle volume fraction is the ratio between the volume of nanoparticles and the volume of all constituents of the mixture (or nanofluid). It is dimensionless and expressed as a number between 0 and 1.

1.25 Thermophysical Properties of Nanofluid

1.25.1 Dynamic Viscosity of Nanofluid

The effective dynamic viscosity of nanofluid:

- Based on Einstein, 1906 model, is given by

$$\frac{\mu_{nf}}{\mu_f} = 1 + 2.5\ \phi_{np} \tag{1.25.1}$$

- Based on Brinkman, 1952 model, is given by

$$\frac{\mu_{nf}}{\mu_f} = \frac{1}{(1-\phi_{np})^{2.5}} \tag{1.25.2}$$

- Based on Graham, 1981 model, is given by

$$\frac{\mu_{nf}}{\mu_f} = 1 + 2.5\ \phi_{np} + 4.5 \left\{ \frac{1}{\left(\frac{h}{R_{np}}\right)\left(2+\frac{h}{R_{np}}\right)\left(1+\frac{h}{R_{np}}\right)^2} \right\} \tag{1.25.3}$$

where μ_{nf} is the dynamic viscosity of the nanofluid, μ_f is the dynamic viscosity of the base fluid, ϕ_{np} is the nanoparticle volume fraction, h is the inter-particle spacing and R_{np} is the nanoparticle radius.

1.25.2 Thermal Conductivity of Nanofluid

The effective thermal conductivity of nanofluid:

- Based on Maxwell, 1873 model (applicable for spherical-shaped nanoparticles), is given by

$$\frac{k_{nf}}{k_f} = \frac{k_{np} + 2k_f - 2\phi_{np}\,(k_f - k_{np})}{k_{np} + 2k_f + \phi_{np}\,(k_f - k_{np})} \tag{1.25.4}$$

- Based on Hamilton & Crosser, 1962 model (applicable for non-spherical shaped nanoparticles), is given by

$$\frac{k_{nf}}{k_f} = \frac{k_{np} + (\Lambda-1)\,k_f - (\Lambda-1)\,\phi_{np}\,(k_f - k_{np})}{k_{np} + (\Lambda-1)\,k_f + \phi_{np}\ \ (k_f - k_{np})} \tag{1.25.5}$$

- Based on Xue, 2005 model (applicable for carbon nanotubes), is given by

$$\frac{k_{nf}}{k_f} = \frac{(1-\phi_{np}) + 2\phi_{np}\left(\frac{k_{np}}{k_{np}-k_f}\right)\ ln\left(\frac{k_{np}+k_f}{2k_f}\right)}{(1-\phi_{np}) + 2\phi_{np}\left(\frac{k_f}{k_{np}-k_f}\right)\ ln\left(\frac{k_{np}+k_f}{2k_f}\right)} \tag{1.25.6}$$

where k_{nf} is the thermal conductivity of the nanofluid, k_f is the thermal conductivity of the base fluid, k_{np} is the thermal conductivity of the nanoparticle, ϕ_{np} is the nanoparticle volume fraction and Λ is the nanoparticle shape factor.

1.25.3 Density of Nanofluid

Based on mixture theory, the density of nanofluid is given by

$$\rho_{nf} = (1 - \phi_{np})\,\rho_f + \phi_{np}\ \rho_{np} \tag{1.25.7}$$

where ρ_{nf} is the density of the nanofluid, ρ_f is the density of the base fluid, ρ_{np} is the density of the nanoparticle and ϕ_{np} is the nanoparticle volume fraction.

1.25.4 Specific Heat of Nanofluid

Based on mixture theory, the specific heat of nanofluid is given by

$$\left(\rho C_p\right)_{nf} = (1 - \phi_{np})\left(\rho C_p\right)_f + \phi_{np}\ \left(\rho C_p\right)_{np} \tag{1.25.8}$$

where $\left(\rho C_p\right)_{nf}$ is the specific heat of the nanofluid, $\left(\rho C_p\right)_f$ is the specific heat of the base fluid, $\left(\rho C_p\right)_{np}$ is the specific heat of the nanoparticle and ϕ_{np} is the nanoparticle volume fraction.

1.25.5 Electrical Conductivity of Nanofluid

Based on mixture theory, the effective electrical conductivity of nanofluid is given by

$$\frac{\sigma_{nf}}{\sigma_f} = 1 + \frac{3\left(\frac{\sigma_{np}}{\sigma_f} - 1\right)\phi_{np}}{\left(\frac{\sigma_{np}}{\sigma_f} + 2\right) - \left(\frac{\sigma_{np}}{\sigma_f} - 1\right)\phi_{np}} \tag{1.25.9}$$

where σ_{nf} is the electrical conductivity of the nanofluid, σ_f is the electrical conductivity of the base fluid, σ_{np} is the electrical conductivity of the nanoparticle and ϕ_{np} is the nanoparticle volume fraction.

1.26 Khanafer-Vafai-Lightstone Model

Khanafer, Vafai, & Lightstone, 2003 proposed a nanofluid transport model accounting the nanoparticle dispersion. It was developed by considering the effective thermophysical properties of nanofluid.

1.27 Buongiorno Model

The non-homogeneous two-component nanomaterial model describes that the base fluid and nanoparticle have different temperatures and velocities. Buongiorno, 2006 proposed a two-phase nanofluid model comprising seven slip mechanisms (diffusiophoresis, gravity, Brownian diffusion, fluid drainage, thermophoresis, inertia, and Magnus effect). Among the considered slip mechanisms, only Brownian diffusion and thermophoresis were observed to be significant.

1.28 Modified Buongiorno Model

The Buongiorno model excludes the augmentation in the effective thermophysical properties of the fluid due to the addition of nanoparticles. Different nanoparticles exhibit varied changes in the thermophysical properties of nanofluids and hence these attributes must be also heeded. Therefore, the Buongiorno model was modified by including effective thermophysical properties which is known as the modified Buongiorno model (see Yang, Li, & Nakayama, 2013).

1.29 Solution Methodology

The numerical computation on the modelled system has elucidated with the aid of following techniques.

1.29.1 Finite Difference based *bvp5c* Scheme

The *bvp5c* scheme employs a code based on finite differences that implements the Lobatto IIIA formula of the sixth order in four steps (see Kierzenka & Shampine, 2008) which is a positioning formula whose polynomial positioning provides a C^1-continuous solution with uniform precision of the fifth order in the considered closed interval. Lobatto IIIA's sixth-order four-step formula is implemented as an implicit Runge-Kutta formula. The *bvp5c* scheme solves the algebraic equations directly and checks the true error (positive difference between real and approximate solutions). This process is continued until the appropriate level of error tolerance is reached.

1.29.2 Runge-Kutta based *ode45* Scheme

The *ode45* scheme (see Shampine & Reichelt, 1997) is a single-step method that employs a code based on an explicit Runge-Kutta method of orders 4 and 5.

1.29.3 Runge-Kutta-Fehlberg Method & Adaptive Runge-Kutta Method

The general algorithm (based on the Runge-Kutta formulas of orders 5 and 4) is defined as:

$$K_1 = h\ f\,(x, y) \tag{1.29.1}$$

$$K_i = h\ f\left(x + A_i\ h, y + \sum_{j=1}^{i-1} B_{ij}\ K_j\right);\quad i = 2, 3, \ldots, 6 \tag{1.29.2}$$

$$y_5\,(x + h) = y\,(x) + \sum_{i=1}^{6} C_i\ K_i \tag{1.29.3}$$

$$y_4\,(x + h) = y\,(x) + \sum_{i=1}^{6} D_i\ K_i \tag{1.29.4}$$

The coefficient matrices appearing in the above formulas are not unique.

For Runge-Kutta-Fehlberg method (see Md Basir et al., 2019), the coefficient matrices are given by:

$$A = \begin{bmatrix} 0 & 1/4 & 3/8 & 12/13 & 1 & 1/2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 0 & 0 \\ 3/32 & 9/32 & 0 & 0 & 0 \\ 1932/2197 & -7200/2197 & 7296/2197 & 0 & 0 \\ 439/216 & -8 & 3860/513 & -845/4104 & 0 \\ -8/27 & 2 & -3544/2565 & 1859/4104 & -11/40 \end{bmatrix}$$

$$C = \begin{bmatrix} 16/135 & 0 & 6656/12825 & 28561/56430 & -9/50 & 2/55 \end{bmatrix}$$

$$D = \begin{bmatrix} 25/216 & 0 & 1408/2565 & 2197/4104 & -1/5 & 0 \end{bmatrix}$$

For adaptive Runge-Kutta method (see Kiusalaas, 2005), the coefficient matrices are given by:

$$A = \begin{bmatrix} 0 & 1/5 & 3/10 & 3/5 & 1 & 7/8 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1/5 & 0 & 0 & 0 & 0 \\ 3/40 & 9/40 & 0 & 0 & 0 \\ 3/10 & -9/10 & 6/5 & 0 & 0 \\ -11/54 & 5/2 & -70/27 & 35/27 & 0 \\ 1631/55296 & 175/512 & 575/13824 & 44275/110592 & 253/4096 \end{bmatrix}$$

$$C = \begin{bmatrix} 37/378 & 0 & 250/621 & 125/594 & 0 & 512/1771 \end{bmatrix}$$

$$D = \begin{bmatrix} 2825/27648 & 0 & 18575/48384 & 13525/55296 & 277/14336 & 1/4 \end{bmatrix}$$

1.29.4 Bulirsch-Stoer Algorithm

The Bulirsch-Stoer algorithm is a numerical procedure based on the Richardson extrapolation and the midpoint method (see Kiusalaas, 2005 Bulirsch, Stoer, & Stoer, 2002). In place of the original Bulirsch and Stoer technique, a simplified version that holds on to the fundamental ideas of Bulirsch and Stoer has been utilized. The automated step size adjustment has been omitted in the algorithm for increased efficiency. For computations requiring high accuracy solutions involving smooth functions, the Bulirsch-Stoer algorithm has an upper hand.
To compute the solution of the ODE $y'(x) = F(x, y)$ with initial point (x_0) and terminal point $(x_0 + nh)$, where h is the step size and n is the number of steps, the following computations are carried out using the midpoint method:

$$\begin{aligned} y_1 &= y_0 + hF_0 \\ y_2 &= y_0 + 2hF_1 \\ y_3 &= y_1 + 2hF_2 \\ &\vdots \\ y_n &= y_{n-2} + 2hF_{n-1} \end{aligned} \tag{1.29.5}$$

where $y_i = y(x_i)$ and $F_i = F(x_i, y_i)$, $i = 1$ *to* n. Furthermore, the final value $y(x_0 + nh)$ is calculated using the formula $y(x_0 + nh) = \frac{1}{2}[y_n + (y_{n-1} + hF_n)]$ that has been finalized by averaging the $y_n's$ obtained through the midpoint formula and the Euler formula. The initial application of the midpoint method uses two integration steps. The number of steps is increased by 2 in successive integrations, each integration being followed by Richardson extrapolation. The procedure is stopped when the difference of successive solutions is less than the required tolerance.

1.30 Motivation

Heat and mass transfer plays a pivotal role in many practical applications like cancer therapy, automotive radiators, air conditioning, refrigeration, microwave ovens, blow moulding, and petrochemical refining. Over the years, scientists have been thriving in finding ways to enhance these transfer capabilities. The use of nanofluid in place of conventional fluid by Choi & Eastman, 1995 marked a major breakthrough in the field of fluid dynamics. Nanofluids play a crucial role in medical and engineering applications due to their ability to enhance heat and mass transfer capabilities. Nanofluid involving microorganisms is an advancing field that has intrigued researchers due to its relevance in antibiotics, biofuel, toxin removal, targeted drug delivery and food digestion.

Nanomedicine is a new concept which combines nanotechnology and medicine. Nanomedicine is the application of nanoscale technologies to the practice of medicine. It is used for diagnosis, prevention and treatment of disease and to gain increased understanding of underlying disease mechanisms. The nascent field of nanomedicine has evoked enormous interest among physical and biological scientists. The great appeal of nanomedicine lies in its promise of using the unique properties of nanoscale materials to address some of the most challenging problems of medical diagnosis and therapy. The novel procedure that integrates therapy and diagnosis in a single platform is termed theranostics.

A mathematical model is a description of a system using mathematical concepts and language. The process of developing a mathematical model is termed mathematical modelling. Mathematical models are used in the natural sciences and engineering disciplines as well as social sciences. A model may help to explain a system and to study the effects of different components and to make predictions about behaviour. Based on literature review and practical applications, very less attention is given to the theoretical studies (through mathematical modelling) concerning biological variations in body fluids due to the application of nanofluid/hybrid nanofluid in various geometries.

1.31 Objectives

The main objectives of this research work are:

- To mathematically model nanofluid flow problems considering their biological application.
- To solve and analyze the modelled flow problems using different numerical methods.
- To work on the mathematical and physical interpretation of results generated using the graphical data.
- To study and evaluate the impact of nanofluids in the biological systems.
- To statistically model the interactive effects of pertinent parameters on the physical quantity of choice.
- To explain the significance of the derived results and to elucidate their biological meaning.

1.32 Overview of the Thesis

The thesis is arranged into 12 chapters. Chapter 1 introduces the basic concepts, preliminaries and definitions to the reader. An extensive review of related literature has been presented in Chapter 2. Owing to their practical applications, nine fluid flow problems are modelled and investigated in this thesis whose summaries are stated below.

Bioconvective stagnation-point flow due to induced magnetic field

For its applications in biomedical imaging, hyperthermia, targeted drug delivery, and cancer therapy, the bioconvective stagnation point flow involving carbon nanotubes along a lengthening sheet subject to induced magnetic field and multiple stratification effects is investigated. Additionally, chemical reaction and viscous dissipation effects are also heeded. Relevant similarity formulas are effectuated in converting the modelled equations into a first-order system of ODEs and are further treated in MATLAB using *ode45* and Newton Raphson method. Illustrations on the consequence of effectual parameters on the physical quantities and the flow profiles are achieved with the aid of graphs.

Bioconvectice hybrid stagnation-point flow due to induced magnetic field
For its applications in biomedical imaging, hyperthermia, targeted drug delivery, and cancer therapy, the dynamics of water conveying single-wall carbon nanotubes (SWCNTs) and magnetite nanoparticles on the bioconvective stagnation-point flow along a stretching sheet subject to chemical reaction, viscous dissipation, induced magnetic field, and stratification effects is investigated. Relevant similarity formulas are effectuated in converting the governing equations into a system of ODEs and are further treated numerically using the Runge-Kutta-Fehlberg method with the shooting technique. Illustrations on the consequence of effectual parameters on the physical quantities and the flow profiles are achieved with the aid of graphs.

Significance of nanoparticle shape on stagnation-point flow in the presence of induced magnetic field
Non-spherical nanoparticles have gained popularity for their ability in changing the thermophysical properties of a nanofluid. The significance of multiple slip and nanoparticle shape on stagnation point flow of blood-based silver nanofluid considering chemical reaction, induced magnetic field, thermal radiation, and linear heat source which is beneficial in cancer therapy, biomedical imaging, hyperthermia, and tumor therapy is investigated. Relevant similarity transformations are effectuated in converting the mathematically modelled governing equations into a system of ODEs and are then numerically resolved in MATLAB employing the adaptive Runge-Kutta method and the Newton Raphson method. Observations on the consequence of differing parameters on varying attributes are achieved via tables and graphs. Additionally, the shape effect of nanoparticles on various attributes is also evaluated.

Bioconvective EMHD nanofluid flow past a stretching sheet
Carbon nanotubes (CNTs) are highly recognized for their diverse biomedical applications. The present study aims to numerically and statistically study the stratification

effects of bioconvective electromagnetohydrodynamic (EMHD) flow past a stretching sheet using water-based CNT. The current study, with applications ranging from biomedical imaging, targeted drug delivery, and cancer therapy, provides a theoretical perspective that is beneficial in biomedical engineering. The mathematically modelled system of partial differential equations (PDEs) is then transmuted into a system of ordinary differential equations (ODEs) using apposite transformations which are then resolved numerically using *bvp5c* (MATLAB built-in function) algorithm. The impact of influential parameters on concentration, velocity, microbial concentration, temperature and physical quantities are illustrated with the aid of graphs and tables. Further, statistical techniques like correlation, the slope of linear regression, probable error and multiple linear regression are employed in scrutinizing the consequence of influential parameters on physical quantities.

Hydromagnetic Carreau nanoliquid flow over an elongating cylinder

The focal concern of this chapter is to numerically scrutinize the consequences of multiple slip, linear radiation and chemically reactive species on MHD convective Carreau nanoliquid flow over an elongating cylinder. The present study has applications in metallurgical procedures, blow molding, glass fibers, and extrusion processes. Relevant transformations are implemented in converting the governing equations into a system of ODEs and are further treated numerically using the Bulirsch-Stoer algorithm coupled with the shooting technique in MATLAB. The Bulirsch-Stoer method is based on two ideas (midpoint method & Richardson extrapolation). Illustrations on the effect of flow profiles due to the varying parameter values are achieved using graphs. Characteristics of skin friction are quantified using linear regression slope. Simultaneous impact of radiation and thermal slip on Nusselt number and the response of solutal slip and chemically reactive species on mass transfer rate is analysed using three-dimensional surface plots. Moreover, statistical scrutiny on the impact of Hartmann number ($1 \leq M \leq 2$), thermal radiation ($1 \leq Rd \leq 2$) and thermal slip parameter ($0.1 \leq b_2 \leq 0.3$) over heat transfer rate employing Response Surface Methodology (RSM) and sensitivity analysis is also performed.

Bioconvective stagnation-point flow over a rotating stretchable disk

Owing to its practical application in pharmaceuticals, biosensors, medical instruments, bio-chromatography, microchip pump, biomedical science, micro-actuators, and aerodynamics, the bioconvective stagnation-point flow over a whirling extendible disk is investigated. Ferro-nanofluid has been particularly chosen in this work since water-based magnetite nanofluid and rotating disk share similar applications. The nanomaterial flow has been modelled using the modified Buongiorno nanofluid model (MBNM). The impact of the stratification constraints and magnetic field are also accounted. Von Kármán's similarity transformations are employed and the transmuted nonlinear ODEs are resolved using the finite-difference based *bvp5c* routine. MATLAB generated flow profiles have been analyzed for augmentations in the influential parameter values. The influence of magnetic field parameter ($0.2 \leq M \leq 1.8$), thermal stratification parameter ($0.1 \leq S_1 \leq 0.5$), volume fraction of magnetite nanoparticles ($0.01 \leq \phi \leq 0.09$), and velocity ratio parameter ($0.1 \leq S \leq 0.5$) on the heat transfer rate has been scrutinized statistically using a five-level four-factor response surface optimized model.

Nanoliquid flow with irregular heat source and realistic boundary conditions

Titanium dioxide plays a vital role in cancer therapy methods (including photothermal therapy and photodynamic therapy), skincare products, heat exchangers, and car radiators. Therefore, the dynamics of the $TiO_2 - H_2O$ nanomaterial over a nonlinearly stretched surface is investigated. For realistic nanoliquid modelling, the conventional Buongiorno model has been improvised (called modified Buongiorno model) by incorporating the effective thermophysical properties of the nanoliquid. Experimentally derived correlations of the thermal conductivity and dynamic viscosity of the nanomaterial are utilized. The significance of passive control of nanoparticles is also studied. The heat transfer analysis includes the mechanism of Rosseland heat flux and exponential heat source. Similarity theory is used to obtain nonlinear ordinary differential equations (ODEs) from the governing partial differential equations which are solved numerically using *bvp5c*, a finite difference-based routine in MATLAB. Further, the heat transfer rate is statistically scrutinized for the consequence of magnetic field ($0.5 \leq M \leq 1.5$), thermal radiation ($0.5 \leq R_d \leq 1.5$) and exponential heat source ($0.2 \leq Q_E \leq 0.4$) by employing

Response Surface Methodology (RSM) and sensitivity analysis.

Bioconvective magnetized nanomaterial flow subjected to convective thermal heating and Stefan blowing

Theranostic is a novel procedure that integrates therapy and diagnosis in a single platform. For its application in theranostic and photothermal therapy for melanoma skin cancer, the hydromagnetic bioconvective flow of a nanomaterial over a lengthening surface is investigated. Realistic nanomaterial modelling is achieved by incorporating passive control of the nanoparticles at the boundary. The impact of the Newtonian heating and Stefan blowing constraints are also accounted. Apposite transformations are employed and then transmuted nonlinear ODEs are resolved using the Bulirsch-Stoer and Newton-Raphson methods. The influence of Stefan blowing parameter ($-3 \leq Sb \leq 3$), the magnetic field parameter ($0.8 \leq M \leq 1.2$) and the Biot number ($0.2 \leq Bi \leq 0.4$) on the heat transfer rate has been scrutinized and optimized utilizing the response surface methodology (RSM). The sensitivity of heat transport rate is computed.

Electro-magnetohydrodynamic Casson nanomaterial flow over a nonlinearly stretched surface

For its biomedical applicability, the dynamics of electro-magnetohydrodynamic flow of blood-gold nanomaterial over a nonlinearly stretching surface utilizing the Casson model has been elucidated numerically. The impact of second-order hydrodynamic-slip, nanoparticle radius, first-order thermal-slip, inter-particle spacing and non-uniform heat source are also accounted. The modelled flow equations are transmuted into a nonlinear system of first-order ODEs (with the aid of apposite similarity variables) which are then resolved numerically utilizing the finite-difference based $bvp5c$ scheme. The current chapter finds its application in radiofrequency ablation, magnetic resonance imaging, cancer therapy, and targeted drug delivery.

Lastly, Chapter 12 presents the concluding remarks of the thesis and proposals for future work. An extensive bibliography follows this chapter.