Research Article

Nicy Sebastian* and Rudolf Gorenflo

Fractional differentiation of the product of Bessel functions of the first kind

Abstract: Two differential transforms involving the Gauss hypergeometric function in the kernels are considered. They generalize the classical Riemann–Liouville and Erdélyi–Kober fractional differential operators. Formulas of compositions for such generalized fractional differentials with the product of Bessel functions of the first kind are proved. Special cases of products of cosine and sine functions are given. The results are established in terms of a generalized Lauricella function due to Srivastava and Daoust. Corresponding assertions for the Riemann–Liouville and the Erdélyi–Kober fractional integral transforms are presented. Statistical interpretations of fractional-order integrals and derivatives are also established.

Keywords: Fractional differential transforms, Bessel functions of the first kind, generalized hypergeometric series, generalized Lauricella series in several variables, gamma Bessel density

MSC 2010: 26A33, 33C10, 33C20, 33C50, 33C60, 26A09

DOI: 10.1515/anly-2015-5004
Received July 6, 2015; accepted August 6, 2015

Dedicated to the memory of Professor Anatoly Kilbas

1 Introduction

In recent years, considerable interest has been attracted by the so-called fractional calculus, which allows one to consider integration and differentiation of any order, not necessarily integer. There is a revived interest in fractional integrals and fractional derivatives due to their recently found applications in reaction, diffusion and reaction-diffusion problems, in solving certain partial differential equations, in input-output models and in related areas. The fractional integration operators involving various special functions, in particular the Gauss hypergeometric functions, have found significant importance and applications in various subfields of mathematical analysis. This idea provoked many authors to choose a more special case of such kernels and to develop a theory of the corresponding generalized fractional calculus that featured many applications.

In 1978, Saigo defined a pair of fractional integral and differential operators involving the Gauss hypergeometric function as kernel. Let $(\mathcal{I}_{0+}^{\alpha,\beta,\eta}f)(x)$, $(\mathcal{I}_{-}^{\alpha,\beta,\eta}f)(x)$ and $(\mathcal{D}_{0+}^{\alpha,\beta,\eta}f)(x)$, $(\mathcal{D}_{-}^{\alpha,\beta,\eta}f)(x)$ be defined for $x > 0$ and complex $\alpha, \beta, \eta \in \mathbb{C}$ by

$$(\mathcal{I}_{0+}^{\alpha,\beta,\eta}f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\Big(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\Big) f(t)\,\mathrm{d}t, \qquad \Re(\alpha) > 0, \tag{1.1}$$

$$(\mathcal{I}_{-}^{\alpha,\beta,\eta}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\Big(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\Big) f(t)\,\mathrm{d}t, \quad \Re(\alpha) > 0, \tag{1.2}$$

***Corresponding author: Nicy Sebastian:** Department of Statistics, St. Thomas College, Thrissur, Kerala 680 001, India, e-mail: nicyseb@yahoo.com
Rudolf Gorenflo: Department of Mathematics and Informatics, Free University of Berlin, Arnimallee 3, 14195 Berlin, Germany, e-mail: gorenflo@math.fu-berlin.de

and

$$(\mathcal{D}_{0+}^{\alpha,\beta,\eta}f)(x) = (\mathcal{J}_{0+}^{-\alpha,-\beta,\alpha+\eta}f)(x) \equiv \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (\mathcal{J}_{0+}^{-\alpha+n,-\beta-n,\alpha+\eta-n}f)(x), \qquad n = [\mathbb{R}(\alpha)] + 1,\ \mathbb{R}(\alpha) > 0, \tag{1.3}$$

$$(\mathcal{D}_{-}^{\alpha,\beta,\eta}f)(x) = (\mathcal{J}_{-}^{-\alpha,-\beta,\alpha+\eta}f)(x) \equiv (-1)^n\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (\mathcal{J}_{-}^{-\alpha+n,-\beta-n,\alpha+\eta}f)(x), \quad n = [\mathbb{R}(\alpha)] + 1,\ \mathbb{R}(\alpha) > 0, \tag{1.4}$$

respectively. Here, $\mathbb{R}(\alpha)$ denotes the real part of α, $\Gamma(\alpha)$ is the Euler gamma function (see [1]) and ${}_2F_1(a, b; c; z)$ is the Gauss hypergeometric function defined for complex $a, b, c, \in \mathbb{C}$, $c \neq 0, -1, -2, \dots$, by the hypergeometric series (see [1, 2.1(2)])

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{1.5}$$

where $(z)_k$ is the Pochhammer symbol defined for $z \in \mathbb{C}$ and $k \in \mathbb{N}$ by

$$(z)_0 = 1, \quad (z)_k = z(z+1)\cdots(z+k-1), \quad k \in \mathbb{N}_0,\ \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \mathbb{N} = \{1, 2, \dots\}.$$

The series in (1.5) is absolutely convergent for

$$|z| < 1 \quad \text{and} \quad |z| = 1, \quad z \neq 1, \quad \mathbb{R}(c-a-b) > 0.$$

The operators in (1.1), (1.2) and (1.3), (1.4), known as the generalized fractional calculus operators, were introduced by Saigo [10] and their properties were studied by many authors (see [3, Section 7.12]. In particular, the operators in (1.3) and (1.4) are inverse to the ones in (1.1) and (1.2), that is,

$$\mathcal{D}_{0+}^{\alpha,\beta,\eta} = (\mathcal{J}_{0+}^{\alpha,\beta,\eta})^{-1}, \quad \mathcal{D}_{-}^{\alpha,\beta,\eta} = (\mathcal{J}_{-}^{\alpha,\beta,\eta})^{-1}.$$

When $\beta = -\alpha$, the operators in (1.3) and (1.4) coincide with the classical left-sided and right-sided Riemann–Liouville fractional differentiation operators of order $\alpha \in \mathbb{C}$, $\mathbb{R}(\alpha) > 0$, (see [11, Section 5.1])

$$(\mathcal{D}_{0+}^{\alpha,-\alpha,\eta}f)(x) = (\mathcal{D}_{0+}^{\alpha}f)(x) \equiv \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{\Gamma(n-\alpha)}\int_0^x (x-t)^{n-\alpha-1}f(t)\,\mathrm{d}t, \quad x > 0,\ n = [\mathbb{R}(\alpha)+1], \tag{1.6}$$

$$(\mathcal{D}_{-}^{\alpha,-\alpha,\eta}f)(x) = (\mathcal{D}_{-}^{\alpha}f)(x) \equiv \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{\Gamma(n-\alpha)}\int_x^\infty (t-x)^{n-\alpha-1}f(t)\,\mathrm{d}t, \quad x > 0,\ n = [\mathbb{R}(\alpha)+1]. \tag{1.7}$$

If $\beta = 0$ and for complex $\alpha, \eta \in \mathbb{C}$, $\mathbb{R}(\alpha) > 0$ and $n = [\mathbb{R}(\alpha)] + 1$, the operators in (1.3) and (1.4) take the form

$$(\mathcal{D}_{0+}^{\alpha,0,\eta}f)(x) = (D_{\eta,\alpha}^{+}f)(x) \equiv \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (\mathcal{J}_{0+}^{-\alpha+n,-\alpha,\alpha+\eta-n}f)(x), \quad x > 0,\ n = [\mathbb{R}(\alpha)+1], \tag{1.8}$$

$$(\mathcal{D}_{-}^{\alpha,0,\eta}f)(x) = (D_{\eta,\alpha}^{-}f)(x) \equiv \left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n (\mathcal{J}_{-}^{-\alpha+n,-\alpha,\alpha+\eta}f)(x), \quad x > 0,\ n = [\mathbb{R}(\alpha)+1]. \tag{1.9}$$

These operators can be called Erdélyi–Kober fractional differentiation operators as inverse to the corresponding Erdélyi–Kober fractional integration operators (see [11, (18.5) and (18.6)]). For suitable functions f, they can be represented by

$$(\mathcal{D}_{\eta,\alpha}^{+}f)(x) = x^{-\eta}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{\Gamma(n-\alpha)}\int_0^x t^{\alpha+\eta}(x-t)^{n-\alpha-1}f(t)\,\mathrm{d}t, \tag{1.10}$$

$$(\mathcal{D}_{\eta,\alpha}^{-}f)(x) = x^{\eta+\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \frac{1}{\Gamma(n-\alpha)}\int_x^\infty t^{-\eta}(t-x)^{n-\alpha-1}f(t)\,\mathrm{d}t, \tag{1.11}$$

for $x > 0$, $\alpha, \eta \in \mathbb{C}$, $\mathbb{R}(\alpha) > 0$ and $n = [\mathbb{R}(\alpha)] + 1$.

Our paper is devoted to the study of compositions of the generalized fractional differentiation operators (1.3) and (1.4) with the product of Bessel functions $J_\nu(z)$ of the first kind, which is defined for complex $z \in \mathbb{C}$, $z \neq 0$, and $\nu \in \mathbb{C}$, $\Re(\nu) > -1$, by (see [2, 7.2(2)])

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{z}{2})^{\nu+2k}}{\Gamma(\nu+k+1)k!}. \tag{1.12}$$

We prove that such compositions are expressed in terms of the generalized Lauricella function due to Srivastava and Daoust [13], which is defined by

$$\begin{aligned} F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}} \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} &= F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}} \left[\begin{matrix} [(a):\theta',\dots,\theta^{(n)}],\, [(b'):\phi'];\dots;[(b)^{(n)}:\phi^{(n)}]; \\ [(c):\psi',\dots,\psi^{(n)}],\, [(d'):\delta'];\dots;[(d)^{(n)}:\delta^{(n)}]; \end{matrix}\; z_1,\dots,z_n \right] \\ &= \sum_{k_1,\dots,k_n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{k_1\theta'_j+\dots+k_n\theta^{(n)}_j}}{\prod_{j=1}^{C}(c_j)_{k_1\psi'_j+\dots+k_n\psi^{(n)}_j}} \frac{\prod_{j=1}^{B'}(b'_j)_{k_1\phi'_j}\cdots\prod_{j=1}^{B^{(n)}}(b^{(n)}_j)_{k_n\phi^{(n)}_j}}{\prod_{j=1}^{D'}(d'_j)_{k_1\delta'_j}\cdots\prod_{j=1}^{D^{(n)}}(d^{(n)}_j)_{k_n\delta^{(n)}_j}} \frac{z_1^{k_1}}{k_1!}\cdots\frac{z_n^{k_n}}{k_n!}, \end{aligned} \tag{1.13}$$

where the coefficients

$$\theta^{(m)}_j,\ j=1,\dots,A, \quad \phi^{(m)}_j,\ j=1,\dots,B^{(m)}, \quad \psi^{(m)}_j,\ j=1,\dots,C, \quad \delta^{(m)}_j,\ j=1,\dots,D^{(m)},$$

are real and positive for all $m \in \{1,\dots,n\}$ and (a) abbreviates the array of A parameters $a_1,\dots,a_A$, $(b^{(m)})$ abbreviates the array of $B^{(m)}$ parameters $b^{(m)}_j$, $j=1,\dots,B^{(m)}$, for all $m \in \{1,\dots,n\}$, with similar interpretations for (c) and $(d^{(m)})$, $m=1,\dots,n$. The multiple series (1.13) converges absolutely when
(i) $\Delta_i > 0$, $i=1,\dots,n$, for all complex values of $z_1,\dots,z_n$, or
(ii) $\Delta_i = 0$, $i=1,\dots,n$, provided in addition $|z_i| < \varrho_i$, $i=1,\dots,n$,
and diverges when $\Delta_i < 0$, $i=1,\dots,n$, except for the trivial case $z_1=\dots=z_n=0$, where

$$\Delta_i \equiv 1 + \sum_{j=1}^{C}\psi^{(i)}_j + \sum_{j=1}^{D^{(i)}}\delta^{(i)}_j - \sum_{j=1}^{A}\theta^{(i)}_j - \sum_{j=1}^{B^{(i)}}\phi^{(i)}_j, \quad i=1,\dots,n,$$

and

$$\varrho_i = \min_{\mu_1,\dots,\mu_n>0}\{E_i\}, \quad i=1,\dots,n,$$

where

$$E_i = (\mu_i)^{1+\sum_{j=1}^{D^{(i)}}\delta^{(i)}_j - \sum_{j=1}^{B^{(i)}}\phi^{(i)}_j} \frac{\prod_{j=1}^{C}(\sum_{i=1}^{n}\mu_i\psi^{(i)}_j)^{\psi^{(i)}_j}\prod_{j=1}^{D^{(i)}}(\delta^{(i)}_j)^{\delta^{(i)}_j}}{\prod_{j=1}^{A}(\sum_{i=1}^{n}\mu_i\theta^{(i)}_j)^{\theta^{(i)}_j}\prod_{j=1}^{B^{(i)}}(\phi^{(i)}_j)^{\phi^{(i)}_j}}.$$

(For more details, see [13].)

Special cases of (1.13) are established in terms of the generalized hypergeometric function of one and two variables, respectively. For the sake of completeness, we define these functions here. A generalized hypergeometric function ${}_pF_q(z)$ is defined for complex $a_i, b_j \in \mathbb{C}$, $b_j \neq 0,-1,\dots$, $i=1,2,\dots,p$, $j=1,2,\dots,q$, by the generalized hypergeometric series (see [1, 4.1(1)])

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = \sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}.$$

This series is absolutely convergent for all values of $z \in \mathbb{C}$ if $p \le q$ and it is an entire function of z. We define a generalization to the Kampé de Fériet function by means of the double hypergeometric series (see [13])

$$F^{p:q;k}_{l:m;n}\left[\begin{matrix}(a_p):(b_q);(c_k); \\ (\alpha_l):(\beta_m);(\gamma_n);\end{matrix}\; x,y\right] = \sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!}. \tag{1.14}$$

The above double series is absolutely convergent for all values of x and y if $p+q < l+m+1$ and $p+k < l+n+1$. Also, if $p+q = l+m+1$ and $p+k = l+n+1$, we must have any one of the sets of conditions

(i) $p \le l$ for $\max\{|x|, |y|\} < 1$, or
(ii) $p > l$ for $|x|^{1/(p-l)} + |y|^{1/(p-l)} < 1$.

The paper is organized as follows. In Section 2, two preliminary lemmas are presented. In Section 3, formulas of compositions of the differential transforms (1.3) and (1.4) with the product of Bessel functions (1.12) are proved in terms of the generalized Lauricella function (1.13). The corresponding results for compositions of Riemann–Liouville and Erdélyi–Kober fractional integrals (1.6), (1.7) and (1.10), (1.11) with the product of Bessel functions are also presented in Section 3. Special cases of $J_{\nu_j}(a_j t^{\rho_j})$ for $\nu_j = -\frac{1}{2}$ and $\nu_j = \frac{1}{2}$ and $\rho_j = 1$, $j = 1, \dots, n$, giving compositions of fractional integrals with cosine and sine functions are considered in Sections 4 and 5, respectively. Finally, statistical interpretations of fractional-order integrals and derivatives are established in Section 6.

2 Preliminary lemmas

Our main results in Section 3 are based on two preliminary assertions giving composition formulas of the generalized fractional differential operators (1.3) and (1.4) with a power function. These assertions are based on the corresponding statements for the generalized fractional integrals (1.1) and (1.2) obtained in [5]. The left-sided generalized differentiation (1.3) of a power function is given by the following results.

Lemma 2.1 (5, Lemma 3). *Let $\alpha, \beta, \eta \in \mathbb{C}$ be such that*

$$\mathbb{R}(\alpha) > 0, \quad \mathbb{R}(\sigma) > -\min[0,\ \mathbb{R}(\alpha+\beta+\eta)].$$

Then, we have

$$(\mathcal{D}_{0+}^{\alpha,\beta,\eta} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)\Gamma(\sigma+\alpha+\beta+\eta)}{\Gamma(\sigma+\beta)\Gamma(\sigma+\eta)} x^{\sigma+\beta-1}, \quad x > 0. \tag{2.1}$$

In particular, for $x > 0$ we have

$$(\mathcal{D}_{0+}^{\alpha} t^{\sigma-1})(x) = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)} x^{\sigma-\alpha-1}, \qquad \mathbb{R}(\sigma) > 0,\ \mathbb{R}(\alpha) > 0,$$

$$(\mathcal{D}_{\eta,\alpha}^{+} t^{\sigma-1})(x) = \frac{\Gamma(\sigma+\alpha+\eta)}{\Gamma(\sigma+\eta)} x^{\sigma-1}, \quad \mathbb{R}(\alpha) > 0,\ \mathbb{R}(\sigma) > -\mathbb{R}(\alpha+\eta).$$

The composition of the right-sided generalized differentiation (1.4) with a power function is given by the following assertion.

Lemma 2.2 (5, Lemma 4). *Let $\alpha, \beta, \eta \in \mathbb{C}$ be such that*

$$\mathbb{R}(\alpha) > 0, \quad \mathbb{R}(\sigma) < 1, \quad \mathbb{R}(\sigma) < 1 + \min[\mathbb{R}(-\beta-n),\ \mathbb{R}(\alpha+\eta)], \quad n = [\mathbb{R}(\alpha)] + 1.$$

Then, we have

$$(\mathcal{D}_{-}^{\alpha,\beta,\eta} t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma-\beta)\Gamma(\alpha+\eta+1-\sigma)}{\Gamma(1-\sigma)\Gamma(\eta-\beta-\sigma+1)} x^{\sigma+\beta-1}, \quad x > 0.$$

In particular, for $x > 0$ we have

$$(\mathcal{D}_{-}^{\alpha} t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma+\alpha)}{\Gamma(1-\sigma)} x^{\sigma-\alpha-1}, \qquad \mathbb{R}(\alpha) > 0,\ \mathbb{R}(\sigma) < 1 + \mathbb{R}(\alpha) - n,$$

$$(\mathcal{D}_{\eta,\alpha}^{-} t^{\sigma-1})(x) = \frac{\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma+\eta)} x^{\sigma-1}, \quad \mathbb{R}(\alpha) > 0,\ \mathbb{R}(\sigma) < 1 + \mathbb{R}(\alpha+\eta) - n.$$

3 Fractional differentiation of the product of Bessel functions of the first kind

First, we consider the generalized left-sided fractional differentiation (1.3) of the product of Bessel functions. It is given by the following result.

Theorem 3.1. *Let $\alpha, \beta, \eta, \sigma, \nu_j \in \mathbb{C}$, $a_j, \rho_j \in \mathbb{R}_+$, $j = 1, \dots, n$, be such that*

$$\Re(\nu_j) > -1, \quad j = 1, \dots, n, \ \Re(\alpha) > 0, \tag{3.1}$$

and

$$\Re\Big(\sigma + \sum_{j=1}^{n} \rho_j \nu_j\Big) > \max[0, -\Re(\beta), -\Re(\eta) - \Re(\beta + \eta)]. \tag{3.2}$$

Then, we have

$$\Big(\mathcal{D}_{0+}^{\alpha,\beta,\eta} t^{\sigma-1}\Big(\prod_{j=1}^{n} J_{\nu_j}(a_j t^{\rho_j})\Big)\Big)(x) = x^{\sigma+\beta-1}\Big(\prod_{j=1}^{n} \frac{(\frac{a_j x^{\rho_j}}{2})^{\nu_j}}{\Gamma(\nu_j + 1)}\Big)\frac{\Gamma(u)\Gamma(v)}{\Gamma(w)\Gamma(z)}$$
$$\times F_{2:1,\dots,1}^{2:0,\dots,0}\left[\begin{matrix}[u:2\rho_1,\dots,2\rho_n],[v:2\rho_1,\dots,2\rho_n]: \\ [w:2\rho_1,\dots,2\rho_n],[z:2\rho_1,\dots,2\rho_n]:[\nu_1+1:1],\dots,[\nu_n+1:1]:\end{matrix}; -\frac{a_1^2 x^{2\rho_1}}{4}, \dots, -\frac{a_n^2 x^{2\rho_n}}{4}\right], \tag{3.3}$$

where

$$u = \sigma + \sum_{j=1}^{n} \rho_j \nu_j, \quad v = \sigma + \alpha + \beta + \eta + \sum_{j=1}^{n} \rho_j \nu_j, \quad w = \sigma + \beta + \sum_{j=1}^{n} \rho_j \nu_j, \quad z = \sigma + \eta + \sum_{j=1}^{n} \rho_j \nu_j,$$

and $F_{2:1,\dots,1}^{2:0,\dots,0}[\,\cdot\,]$ is given by (1.13).

Proof. First of all, we note that Δ_i in (1.13) is given by $\Delta_i = 1 + n > 0$, $i = 1, \dots, n$, $n = 1, \dots,$ and therefore $F_{2:1,\dots,1}^{2:0,\dots,0}[\,\cdot\,]$ is defined in the right-hand side of (3.3).

To prove (3.3), we have

$$\Big(\mathcal{D}_{0+}^{\alpha,\beta,\eta} t^{\sigma-1}\Big(\prod_{j=1}^{n} J_{\nu_j}(a_j t^{\rho_j})\Big)\Big)(x) = (\mathcal{D}_{0+}^{\alpha,\beta,\eta}\{t^{\sigma-1} J_{\nu_1}(a_1 t^{\rho_1}) \cdots J_{\nu_n}(a_n t^{\rho_n})\})(x)$$
$$= \Big(\mathcal{D}_{0+}^{\alpha,\beta,\eta}\Big\{t^{\sigma-1} \sum_{k_1=0}^{\infty}\Big(\frac{(-1)^{k_1}(\frac{a_1 t^{\rho_1}}{2})^{\nu_1+2k_1}}{\Gamma(\nu_1 + k_1 + 1)k_1!}\Big) \cdots \sum_{k_n=0}^{\infty}\Big(\frac{(-1)^{k_n}(\frac{a_n t^{\rho_n}}{2})^{\nu_n+2k_n}}{\Gamma(\nu_n + k_n + 1)k_n!}\Big)\Big\}\Big)(x)$$

and using (1.3) and (1.13) and changing the orders of integration and summation, we have

$$\Big(\mathcal{D}_{0+}^{\alpha,\beta,\eta} t^{\sigma-1}\Big(\prod_{j=1}^{n} J_{\nu_j}(a_j t^{\rho_j})\Big)\Big)(x) = \sum_{k_1,\dots,k_n=0}^{\infty} \frac{(-1)^{k_1}(\frac{a_1}{2})^{\nu_1+2k_1}}{\Gamma(\nu_1 + 1)(\nu_1 + 1)_{k_1} k_1!} \cdots \frac{(-1)^{k_n}(\frac{a_n}{2})^{\nu_n+2k_n}}{\Gamma(\nu_n + 1)(\nu_n + 1)_{k_n} k_n!}$$
$$\times (\mathcal{D}_{0+}^{\alpha,\beta,\eta}\{t^{\sigma+\nu_1\rho_1+\cdots+\nu_n\rho_n+2\rho_1 k_1+\cdots+2\rho_n k_n-1}\})(x).$$

By (3.1) and (3.2), for any $k_j = 0, \dots, $, $j = 1, \dots, n$, we have

$$\Re\Big(\sigma + \sum_{j=1}^{n} \rho_j \nu_j\Big) > \max[0, -\Re(\beta), -\Re(\eta), -\Re(\beta + \eta) - \Re(\alpha + \beta + \eta)].$$

Finally, by applying Lemma 2.1 and using (2.1) with σ replaced by

$$\sigma + \sum_{j=1}^{n} \rho_j \nu_j + 2\sum_{j=1}^{n} \rho_j k_j, \quad j = 1, \dots, n,$$

we obtain

$$\Big(\mathcal{D}_{0+}^{\alpha,\beta,\eta} t^{\sigma-1}\Big(\prod_{j=1}^{n} J_{\nu_j}(a_j t^{\rho_j})\Big)\Big)(x) = \sum_{k_1,\dots,k_n=0}^{\infty} \frac{(-1)^{k_1}(\frac{a_1}{2})^{\nu_1+2k_1}}{\Gamma(\nu_1 + 1)(\nu_1 + 1)_{k_1} k_1!} \cdots \frac{(-1)^{k_n}(\frac{a_n}{2})^{\nu_n+2k_n}}{\Gamma(\nu_n + 1)(\nu_n + 1)_{k_n} k_n!}$$
$$\times \frac{\Gamma(\sigma + \sum_{j=1}^{n}(\nu_j\rho_j + 2\rho_j k_j))\Gamma(\sigma + \alpha + \beta + \eta + \sum_{j=1}^{n}(\nu_j\rho_j + 2\rho_j k_j))}{\Gamma(\sigma + \beta + \sum_{j=1}^{n}(\nu_j\rho_j + 2\rho_j k_j))\Gamma(\sigma + \eta + \sum_{j=1}^{n}(\nu_j\rho_j + 2\rho_j k_j))}$$
$$\times x^{\sigma+\beta-1+\sum_{j=1}^{n}(\nu_j\rho_j+2\rho_j k_j)}.$$

This in accordance with (1.13) and gives the result in (3.3). □

Corollary 3.2. *Let $\alpha, \sigma, \nu_j \in \mathbb{C}$, $a_j, \rho_j \in \mathbb{R}_+$, $j = 1, \dots, n$, be such that $\Re(\nu_j) > -1$, $\Re(\alpha) > 0$ and*

$$\Re\left(\sigma + \sum_{j=1}^{n} \rho_j \nu_j\right) > 0.$$

Then, we have

$$\left(\mathcal{D}_{0+}^{\alpha} t^{\sigma-1}\left(\prod_{j=1}^{n} J_{\nu_j}(a_j t^{\rho_j})\right)\right)(x) = x^{\sigma-\alpha-1}\left(\prod_{j=1}^{n} \frac{(\frac{a_j x^{\rho_j}}{2})^{\nu_j}}{\Gamma(\nu_j+1)}\right)\frac{\Gamma(\sigma + \sum_{j=1}^{n}\rho_j\nu_j)}{\Gamma(\sigma - \alpha + \sum_{j=1}^{n}\rho_j\nu_j)}$$
$$\times F_{1:1,\dots,1}^{1:0,\dots,0}\left[\begin{matrix}[\sigma+\sum_{j=1}^{n}\rho_j\nu_j:2\rho_1,\dots,2\rho_n]: \\ [\sigma-\alpha+\sum_{j=1}^{n}\rho_j\nu_j:2\rho_1,\dots,2\rho_n]:[\nu_1+1:1],\dots,[\nu_n+1:1]:\end{matrix}; -\frac{a_1^2x^{2\rho_1}}{4}, \dots, -\frac{a_n^2x^{2\rho_n}}{4}\right].$$

Corollary 3.3. *Let $\alpha, \eta, \sigma, \nu_j \in \mathbb{C}$, $a_j, \rho_j \in \mathbb{R}_+$, $j = 1, \dots, n$ be such that $\Re(\nu_j) > -1$, $\Re(\alpha) > 0$ and*

$$\Re\left(\sigma + \sum_{j=1}^{n} \rho_j \nu_j\right) > \max[0, -\Re(\eta)].$$

Then, we have

$$\left(\mathcal{D}_{\eta,\alpha}^{+} t^{\sigma-1}\left(\prod_{j=1}^{n} J_{\nu_j}(a_j t^{\rho_j})\right)\right)(x) = x^{\sigma-1}\left(\prod_{j=1}^{n} \frac{(\frac{a_j x^{\rho_j}}{2})^{\nu_j}}{\Gamma(\nu_j+1)}\right)\frac{\Gamma(\sigma + \alpha + \eta + \sum_{j=1}^{n}\rho_j\nu_j)}{\Gamma(\sigma + \eta + \sum_{j=1}^{n}\rho_j\nu_j)}$$
$$\times F_{1:1,\dots,1}^{1:0,\dots,0}\left[\begin{matrix}[\sigma+\alpha+\eta+\sum_{j=1}^{n}\rho_j\nu_j:2\rho_1,\dots,2\rho_n]: \\ [\sigma+\eta+\sum_{j=1}^{n}\rho_j\nu_j:2\rho_1,\dots,2\rho_n]:[\nu_1+1:1],\dots,[\nu_n+1:1]:\end{matrix}; -\frac{a_1^2x^{2\rho_1}}{4}, \dots, -\frac{a_n^2x^{2\rho_n}}{4}\right].$$

Corollaries 3.2 and 3.3 follow from Theorem 3.1 in the respective cases $\beta = -\alpha$ and $\beta = 0$ if we take (1.6) and (1.8) into account.

Corollary 3.4. *Let $\alpha, \beta, \sigma, \nu_1, \nu_2 \in \mathbb{C}$, $a_1, a_2, \rho_1, \rho_2 \in \mathbb{R}_+$, be such that $\Re(\nu_1) > -1$, $\Re(\nu_2) > -1$, $\Re(\alpha) > 0$ and $\Re(\sigma + \rho_1\nu_1 + \rho_2\nu_2) > \max[0, -\Re(\beta), -\Re(\eta), -\Re(\beta+\eta)]$. Let also $n = 2$, $a_1 = \lambda_1$, $a_2 = \lambda_2$, $\rho_1 = 1$, $\rho_2 = 1$. Then,* (3.3) *reduces to*

$$(\mathcal{D}_{0+}^{\alpha,\beta,\eta} t^{\sigma-1}(J_{\nu_1}(\lambda_1 t)J_{\nu_2}(\lambda_2 t)))(x) = \frac{x^{c-1}\lambda_1^{\nu_1}\lambda_2^{\nu_2}}{2^{\nu_1+\nu_2}\Gamma(\nu_1+1)\Gamma(\nu_2+1)}\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)}$$
$$\times F_{4:1,1}^{4:0,0}\left[\begin{matrix}[\frac{a}{2}:1,1],[\frac{a+1}{2}:1,1],[\frac{b}{2}:1,1],[\frac{b+1}{2}:1,1]: \\ [\frac{c}{2}:1,1],[\frac{c+1}{2}:1,1],[\frac{d}{2}:1,1],[\frac{d+1}{2}:1,1]:[\nu_1+1:1],\dots,[\nu_n+1:1]:\end{matrix}; -\frac{\lambda_1^2x^2}{4}, -\frac{\lambda_2^2x^2}{4}\right], \tag{3.4}$$

where

$$a = \sigma + \nu_1 + \nu_2, \quad b = \sigma + \alpha + \beta + \eta + \nu_1 + \nu_2, \quad c = \sigma + \beta + \nu_1 + \nu_2, \quad d = \sigma + \eta + \nu_1 + \nu_2,$$

and $F[\cdot]$ is defined in (1.14).

Proof. First we note that the convergence conditions $p + q < l + m + 1$ and $p + k < l + n + 1$ from (1.14) take the form $4 < 6$ and therefore $F[\cdot]$ is absolutely convergent.

Now, we prove (3.4). Taking $n = 2$, $a_1 = \lambda_1$, $a_2 = \lambda_2$, $\rho_1 = 1$, $\rho_2 = 1$ in (3.3), we have

$$(\mathcal{D}_{0+}^{\alpha,\beta,\eta} t^{\sigma-1}(J_{\nu_1}(\lambda_1 t)J_{\nu_2}(\lambda_2 t)))(x) = \frac{x^{\sigma+\nu_1+\nu_2+\beta-1}\lambda_1^{\nu_1}\lambda_2^{\nu_2}}{2^{\nu_1+\nu_2}\,\Gamma(\nu_1+1)\Gamma(\nu_2+1)}\frac{\Gamma(\sigma+\nu_1+\nu_2)\Gamma(\sigma+\alpha+\beta+\eta+\nu_1+\nu_2)}{\Gamma(\sigma+\nu_1+\nu_2+\beta)\Gamma(\sigma+\eta+\nu_1+\nu_2)}$$
$$\times F_{2:1,1}^{2:0,0}\left[\begin{matrix}[\sigma+\nu_1+\nu_2:2,2],[\sigma+\alpha+\beta+\eta+\nu_1+\nu_2:2,2]: \\ [\sigma+\beta+\nu_1+\nu_2:2,2],[\sigma+\eta+\nu_1+\nu_2:2,2]:[\nu_1+1:1],[\nu_2+1:1]:\end{matrix}; -\frac{\lambda_1^2x^2}{4}, -\frac{\lambda_2^2x^2}{4}\right].$$

Applying the result

$$(z)_{2k} = 2^{2k}\left(\frac{z}{2}\right)_k\left(\frac{z+1}{2}\right)_k, \quad z \in \mathbb{C},\ k \in \mathbb{N}_0, \tag{3.5}$$

and using equations (1.13) and (3.5), we obtain

$$\begin{aligned}
(\mathcal{D}_{0+}^{\alpha,\beta,\eta} t^{\sigma-1}(J_{\nu_1}(\lambda_1 t)J_{\nu_2}(\lambda_2 t)))(x) &= \frac{x^{\sigma+\beta+\nu_1+\nu_2-1}\lambda_1^{\nu_1}\lambda_2^{\nu_2}}{2^{\nu_1+\nu_2}\Gamma(\nu_1+1)\Gamma(\nu_2+1)} \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)} \\
&\quad \times \sum_{k_1,k_2=0}^{\infty} \frac{(a)_{2(k_1+k_2)}(b)_{2(k_1+k_2)}}{(c)_{2(k_1+k_2)}(d)_{2(k_1+k_2)}(\nu_1+1)_{k_1}(\nu_2+1)_{k_2}} \frac{(-\frac{\lambda_1^2x^2}{4})^{k_1}}{k_1!} \frac{(-\frac{\lambda_2^2x^2}{4})^{k_2}}{k_2!} \\
&= \frac{x^{\sigma+\nu_1+\nu_2-\beta-1}}{2^{\nu_1+\nu_2}\Gamma(\nu_1+1)\Gamma(\nu_2+1)} \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)\Gamma(d)} \\
&\quad \times F_{4:1,1}^{4:0,0}\left[\begin{matrix}[\frac{a}{2}:1,1],[\frac{a+1}{2}:1,1],[(\frac{b}{2}):1,1],[(\frac{b+1}{2}):1,1]: \\ [\frac{c}{2}:1,1],[\frac{c+1}{2}:1,1],[\frac{d}{2}:1,1],[\frac{d+1}{2}:1,1]:[\nu_1+1:1],\ldots,[\nu_n+1:1]:\end{matrix}; -\frac{\lambda_1^2x^2}{4}, -\frac{\lambda_2^2x^2}{4}\right],
\end{aligned}$$

which proves the claim. □

Example 3.5. If $n = 1$, $a_1 = \lambda$, $\rho_1 = 1$, $\nu_1 = \nu$, then (3.3) reduces to

$$(D_{0+}^{\alpha,\beta,\eta} t^{\sigma-1}J_\nu(t))(x) = \frac{x^{\sigma+\nu+\beta-1}\lambda^\nu}{2^\nu} \frac{\Gamma(\sigma+\nu)\Gamma(\sigma+\nu+\alpha+\beta+\eta)}{\Gamma(\sigma+\nu+\beta)\Gamma(\sigma+\nu+\eta)\Gamma(\nu+1)} {}_4F_5\left[\begin{matrix}\frac{\sigma+\nu}{2}, \frac{\sigma+\nu+1}{2}, \frac{\sigma+\nu+\alpha+\beta+\eta}{2}, \frac{\sigma+\nu+\alpha+\beta+\eta+1}{2} \\ \nu+1, \frac{\sigma+\nu+\beta}{2}, \frac{\sigma+\nu+\beta+1}{2}, \frac{\sigma+\nu+\eta}{2}, \frac{\sigma+\nu+\eta+1}{2}\end{matrix}; -\frac{\lambda^2x^2}{4}\right].$$

Indeed, by putting the above values in (3.3), we have

$$(\mathcal{D}_{0+}^{\alpha,\beta,\eta} t^{\sigma-1}J_\nu(t))(x) = x^{\sigma+\nu+\beta-1}\frac{\lambda^\nu}{2^\nu} \frac{\Gamma(\sigma+\nu)\Gamma(\sigma+\nu+\eta+\beta)}{\Gamma(\nu+1)\Gamma(\sigma+\nu+\beta)\Gamma(\sigma+\nu+\eta)} F_{2:1}^{2:0}\left[\begin{matrix}[\sigma+\nu:2],[\sigma+\eta+\beta+\nu:2]: \\ [\sigma+\beta+\nu:2],[\eta+\sigma+\nu:2]:[\nu+1:1]:\end{matrix}; -\frac{\lambda^2x^2}{4}\right].$$

Using (1.13) and (3.5), we obtain

$$\begin{aligned}
(\mathcal{D}_{0+}^{\alpha,\beta,\eta} t^{\sigma-1}J_\nu(t))(x) &= x^{\sigma+\nu+\beta-1}\frac{\lambda^\nu}{2^\nu} \frac{\Gamma(\sigma+\nu)\Gamma(\sigma+\nu+\eta+\beta)}{\Gamma(\nu+1)\Gamma(\sigma+\nu+\beta)\Gamma(\sigma+\nu+\eta)} \frac{(\sigma+\nu)_{2k}(\sigma+\nu+\eta+\beta)_{2k}}{(\sigma+\nu+\beta)_{2k}(\sigma+\nu+\beta)_{2k}} \frac{(-\frac{\lambda^2x^2}{4})^k}{k!} \\
&= \frac{x^{\sigma+\nu+\beta-1}}{2^\nu} \frac{\Gamma(\sigma+\nu)\Gamma(\sigma+\nu+\alpha+\beta+\eta)}{\Gamma(\sigma+\nu+\beta)\Gamma(\sigma+\nu+\eta)\Gamma(\nu+1)} {}_4F_5\left[\begin{matrix}\frac{\sigma+\nu}{2}, \frac{\sigma+\nu+1}{2}, \frac{\sigma+\nu+\alpha+\beta+\eta}{2}, \frac{\sigma+\nu+\alpha+\beta+\eta+1}{2} \\ \nu+1, \frac{\sigma+\nu+\beta}{2}, \frac{\sigma+\nu+\beta+1}{2}, \frac{\sigma+\nu+\eta}{2}, \frac{\sigma+\nu+\eta+1}{2}\end{matrix}; -\frac{\lambda^2x^2}{4}\right]
\end{aligned}$$

and (3.4) is proved. For more special cases, see [4, Section 4].

Remark 3.6. Equation (3.4) was proved in [4, Theorem 3].

4 Fractional differentiation of the cosine function

For $\nu = -\frac{1}{2}$, the Bessel function $J_\nu(z)$ in (1.12) coincides with the cosine function apart from the multiplier

$$\left(\frac{2}{\pi z}\right)^{\frac{1}{2}},$$

that is,

$$J_{-\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos(z).$$

From Theorem 3.1, we obtain the following result for

$$\nu_1 = \cdots = \nu_n = -\frac{1}{2} \quad \text{and} \quad \rho_1 = \cdots = \rho_n = 1.$$

Theorem 4.1. *Let $\alpha, \beta, \eta, \sigma \in \mathbb{C}$, $a_j \in \mathbb{R}_+$, $j = 1, \dots, n$, be such that*

$$\Re(\alpha) > 0 \quad \text{and} \quad \Re(\sigma) > \max[0, -\Re(\beta), -\Re(\eta), -\Re(\beta+\eta)]. \tag{4.1}$$

Let also $\Re(\sigma) > 0$ and $\Re(\sigma + \alpha + \beta + \eta) > 0$. Then, we have

$$\Big(\mathcal{D}_{0+}^{\alpha,\beta,\eta} t^{\sigma-1}\Big(\prod_{j=1}^{n}\cos(a_j t)\Big)\Big)(x) = x^{\sigma+\beta-1}\frac{\Gamma(\sigma)\Gamma(\sigma+\alpha+\beta+\eta)}{\Gamma(\sigma+\beta)\Gamma(\sigma+\eta)} \times F^{2:0,\dots,0}_{2:1,\dots,1}\left[\begin{matrix}[\sigma:2,\dots,2],[\sigma+\alpha+\beta+\eta:2,\dots,2]: \\ [\sigma+\beta:2,\dots,2],[\sigma+\eta:2,\dots,2]:[\frac{1}{2}:1],\dots,[\frac{1}{2}:1]:\end{matrix}; -\frac{a_1^2x^2}{4}, \dots, -\frac{a_n^2x^2}{4}\right]. \tag{4.2}$$

Taking $\beta = -\alpha$ and $\beta = 0$ in Theorem 4.1 and using (1.13) and (4.1) from (4.2), we deduce the following assertions.

Corollary 4.2. *Let $\alpha, \sigma \in \mathbb{C}$, $a_j \in \mathbb{R}_+$, $j = 1, \dots, n$, be such that $\Re(\alpha) > 0$ and $\Re(\sigma) > 0$. Then, we have*

$$\Big(\mathcal{D}_{0+}^{\alpha} t^{\sigma-1}\Big(\prod_{j=1}^{n}\cos(a_j t)\Big)\Big)(x) = x^{\sigma-\alpha-1}\frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)} F^{1:0,\dots,0}_{1:1,\dots,1}\left[\begin{matrix}[\sigma:2,\dots,2]: \\ [\sigma-\alpha:2,\dots,2]:[\frac{1}{2}:1],\dots,[\frac{1}{2}:1]:\end{matrix}; -\frac{a_1^2x^2}{4}, \dots, -\frac{a_n^2x^2}{4}\right].$$

Corollary 4.3. *Let $\alpha, \eta, \sigma \in \mathbb{C}$, $a_j \in \mathbb{R}_+$, $j = 1, \dots, n$, be such that $\Re(\alpha) > 0$ and $\Re(\sigma) > \max[0, -\Re(\eta)]$. Then, we have*

$$\Big(\mathcal{D}_{\eta,\alpha}^{+} t^{\sigma-1}\Big(\prod_{j=1}^{n}\cos(a_j t)\Big)\Big)(x) = x^{\sigma-1}\frac{\Gamma(\sigma+\alpha+\eta)}{\Gamma(\sigma+\eta)} F^{1:0,\dots,0}_{1:1,\dots,1}\left[\begin{matrix}[\sigma+\alpha+\eta:2,\dots,2]: \\ [\sigma+\eta:2,\dots,2]:[\frac{1}{2}:1],\dots,[\frac{1}{2}:1]:\end{matrix}; -\frac{a_1^2x^2}{4}, \dots, -\frac{a_n^2x^2}{4}\right].$$

5 Fractional differentiation of the sine function

For $\nu = \frac{1}{2}$, the Bessel function $J_\nu(z)$ in (1.12) coincides with the sine function apart from the multiplier

$$\Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}},$$

that is,

$$J_{\frac{1}{2}}(z) = \Big(\frac{2}{\pi z}\Big)^{\frac{1}{2}} \sin(z). \tag{5.1}$$

The next statement follows from Theorem 3.1 by setting $\nu_j = \frac{1}{2}$, $\rho_j = 1$, $j = 1, \dots, n$, in (3.3) and using (5.1).

Theorem 5.1. *Let $\alpha, \beta, \eta, \sigma \in \mathbb{C}$, $a_j \in \mathbb{R}_+$, $j = 1, \dots, n$, be such that condition (5.1) be satisfied. Let also $\Re(\sigma) > \max[0, -\Re(\beta), -\Re(\eta), -\Re(\beta+\eta)]$. Then, we have*

$$\Big(\mathcal{D}_{0+}^{\alpha,\beta,\eta} t^{\sigma-n-1}\Big(\prod_{j=1}^{n}\sin(a_j t)\Big)\Big)(x) = \Big(\prod_{j=1}^{n} a_j\Big) x^{\sigma+\beta-1}\frac{\Gamma(\sigma)\Gamma(\sigma+\alpha+\beta+\eta)}{\Gamma(\sigma+\beta)\Gamma(\sigma+\eta)} \times F^{2:0,\dots,0}_{2:1,\dots,1}\left[\begin{matrix}[\sigma:2,\dots,2],[\sigma+\alpha+\beta+\eta:2,\dots,2]: \\ [\sigma+\beta:2,\dots,2],[\alpha+\eta:2,\dots,2]:[\frac{3}{2}:1],\dots,[\frac{3}{2}:1]:\end{matrix}; -\frac{a_1^2x^2}{4}, \dots, -\frac{a_n^2x^2}{4}\right]. \tag{5.2}$$

Taking $\beta = -\alpha$ and $\beta = 0$ in Theorem 5.1 and using (1.13) and (5.1), from (5.2) we deduce the following assertions.

Corollary 5.2. *Let $\alpha, \sigma \in \mathbb{C}$, $a_j \in \mathbb{R}_+$, $j = 1, \dots, n$, be such that $\Re(\alpha) > 0$ and $\Re(\sigma) > 0$. Then, we have*

$$\left(\mathcal{D}^{\alpha}_{0+} t^{\sigma-n-1}\left(\prod_{j=1}^{n} \sin(a_j t)\right)\right)(x) = \left(\prod_{j=1}^{n} a_j\right) x^{\sigma-\beta-1} \frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)} \times F^{1:0,\dots,0}_{1:1,\dots,1}\left[\begin{matrix}[\sigma:2,\dots,2]: \\ [\sigma-\alpha:2,\dots,2],:[\frac{3}{2}:1],\dots,[\frac{3}{2}:1]:\end{matrix}\, ; -\frac{a_1^2 x^2}{4}, \dots, -\frac{a_n^2 x^2}{4}\right].$$

Corollary 5.3. *Let $\alpha, \eta, \sigma \in \mathbb{C}$, $a_j \in \mathbb{R}_+$, $j = 1, \dots, n$, be such that $\Re(\alpha) > 0$ and $\Re(\sigma) > \max[0, -\Re(\eta)]$. Then, we have*

$$\left(\mathcal{D}^{+}_{\eta,\alpha} t^{\sigma-n-1}\left(\prod_{j=1}^{n} \sin(a_j t)\right)\right)(x) = \left(\prod_{j=1}^{n} a_j\right) x^{\sigma-1} \frac{\Gamma(\sigma+\alpha+\eta)}{\Gamma(\sigma+\eta)} \times F^{1:0,\dots,0}_{1:1,\dots,1}\left[\begin{matrix}[\sigma+\alpha+\eta:2,\dots,2]: \\ [\eta+\sigma:2,\dots,2]:[\frac{3}{2}:1],\dots,[\frac{3}{2}:1]:\end{matrix}\, ; -\frac{a_1^2 x^2}{4}, \dots, -\frac{a_n^2 x^2}{4}\right].$$

Remark 5.4. When $n = 1$, $a_1 = \lambda$, all results in Section 4 and Section 5 reduce to [4, Section 5 and Section 6].

The types of single-variable hypergeometric series considered here appear in many physical contexts, including quantum chemistry, the development of few-body and many-body wave functions, and statistical physics. In [6], the authors proved the formulas of compositions for such generalized fractional integrals with the product of Bessel functions of the first kind and observed that the solution is obtained in terms of the multivariable hypergeometric function. Special cases of products of cosine and sine functions are also given in the same paper. It is fairly well known that hypergeometric series in one, two, and more variables occur rather frequently in a wide variety of problems in theoretical physics and applied mathematics (including, for instance, nuclear and neutrino astrophysics), and also in engineering sciences, statistics, and operations research.

6 Statistical interpretations

The traditional special functions (the special functions of classical calculus) are known to be related to the classical fractional calculus and also to the generalized fractional calculus. They have been shown to be representable by fractional-order integration or differentiation of some basic elementary functions. The essentials of fractional calculus according to different approaches which can be useful for our applications in probability theory and in stochastic processes are established with the help of the pathway idea of [8]. The pathway idea was originally developed by Mathai in the 1970s in connection with population models. It was later rephrased and extended to cover scalar as well as matrix cases and was made suitable for modeling data from statistical and physical situations. The main idea behind the pathway model is the switching property of the binomial function to the corresponding exponential function.

By means of the pathway model [8], the pathway fractional integral operator (pathway operator) is defined as

$$(P^{(\eta,q)}_{0+} f)(x) = x^{\eta-1} \int_{0}^{\frac{x}{a(1-q)}} \left[1 - \frac{a(1-q)t}{x}\right]^{\frac{\eta}{(1-q)}-1} f(t)\,dt \tag{6.1}$$

for $f(x) \in L(a,b)$, $\eta \in C$, $\Re(\eta) > 0$, $a > 0$ and $q < 1$, where q is the pathway parameter and $f(t)$ is an arbitrary function. For more details, see [9]. It is also observed that when the pathway parameter, $q = 0$, $a = 1$ and $f(t)$ is replaced by

$${}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{t}{x}\right) f(t),$$

then the pathway operator yields

$$\int_{0}^{x} (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1 - \frac{t}{x}\right) f(t)\,dt = \frac{\Gamma(\alpha)}{x^{-\alpha-\beta}} I^{\alpha,\beta,\eta}_{0+}.$$

Thus, we can obtain all the generalizations, like in [7], [10], etc., of left-sided fractional integrals by suitable substitutions, so that we call it the pathway fractional operator, a path through the pathway parameter q, leading to the above known fractional operators. When $\eta = 1$ and by replacing $f(t)$ by

$$t^{\beta-1}{}_0F_1(\,\cdot\,;\beta;\delta t)$$

in the pathway fractional integral operator, one can deduce the q-analogue of the gamma Bessel density [12] and hence we are essentially dealing with a distribution function under a gamma Bessel-type model in a practical statistical problem. Hence, a connection between statistical distribution theory and fractional calculus is established so that one can make use of the rich results in statistical distribution theory for the further development of fractional calculus and vice versa.

References

[1] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions, Vol. I*, McGraw-Hill, New York, 1953.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions, Vol. II*, McGraw-Hill, New York, 1953.

[3] A. A. Kilbas and M. Saigo, *H-Transforms. Theory and Applications*, Anal. Methods Spec. Funct. 9, Chapman & Hall / CRC, Boca Raton, 2004.

[4] A. A. Kilbas and N. Sebastian, Generalized fractional differentiation of Bessel function of the first kind, *Math. Balkanica (N.S.)* **22** (2008), no. 3–4, 323–346.

[5] A. A. Kilbas and N. Sebastian, Generalized fractional integration of Bessel function of the first kind, *Integral Transforms Spec. Funct.* **19** (2008), no. 12, 869–883.

[6] A. A. Kilbas and N. Sebastian, Fractional integration of the products of Bessel functions of the first kind, *Fract. Calc. Appl. Anal.* **13** (2010), no. 2, 159–175.

[7] H. Kober, On fractional integrals and derivatives, *Quart. J. Math. Oxford Ser.* **11** (1940), 193–211.

[8] A. M. Mathai, A pathway to matrix-variate gamma and normal densities, *Linear Algebra Appl.* **396** (2005), 317–328.

[9] S. S. Nair, Pathway fractional integration operator, *Fract. Calc. Appl. Anal.* **12** (2009), no. 3, 237–252.

[10] M. Saigo, A remark on integral operators involving the Gauss hypergeometric functions, *Math. Rep. Kyushu Univ.* **11** (1977/78), no. 2, 135–143.

[11] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[12] N. Sebastian, A generalized gamma model associated with a Bessel function, *Integral Transforms Spec. Funct.* **22** (2011), no. 9, 631–645.

[13] H. M. Srivastava, M. C. Daoust, Certain generalized Neumann expansions associated with the Kampé de Fériet function, *Nederl. Akad. Wetensch. Proc. Ser. A* **72** (1969), no. 5, 449–457.