ESTIMATION OF STRESS-STRENGTH RELIABILITY FOR AKASH DISTRIBUTION

Akhila K Varghese, V. M. Chacko
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Department of Statistics
St. Thomas' College (Autonomous)
Thrissur, Kerala, 680 001, India
akhilavarghesek@gmail.com
chackovm@gmail.com

Abstract

In this paper, we consider the estimation of the stress–strength parameter R = P[Y < X], when X and Y are following one-parameter Akash distributions with parameter θ_1 *and* θ_2 *respectively. It is assumed that they are independently distributed. The maximum likelihood estimator (MLE) of R and its asymptotic distribution are obtained. Asymptotic distributions of the maximum likelihood estimator is useful for constructing confidence interval of P[Y < X]. The Bootstrap confidence interval of P[Y < X] is also computed. The illustrative part consists of the analysis of two real data sets, (i) simulated and (ii) real.*

Keywords: stress–strength model; maximum-likelihood estimator; bootstrap confidence intervals; asymptotic distributions

I. Introduction

In reliability analysis, estimation of stress-strength reliability is one of the important and difficult but tractable problem, while using various distributions. In the statistical literature, estimating the stress–strength parameter, R, is quite useful. For example, if X is a measure of a system's strength when it is subjected to a stress Y, then R is a measure of system performance that naturally occurs in a system's mechanical dependability. The system fails if and only if the applied stress exceeds its strength at any point. In reliability analysis, a variety of lifespan distributions are used. In dependability analysis, terms like exponential, Weibull, log-Normal, and their generalizations are frequently used. A number of academics have recently proposed several distributions, with the new ones demonstrating a superior fit than current well-known distributions. While using better fitted models in stress-strength analysis, one may have to inspect its estimation procedure, since, if the estimation procedure fails with available techniques, one may not be able to solve the problem with new models. So estimation of various reliability parameters is vital and researchers have to give more concentration of estimation while using better fitted models.

The estimation of reliability or survival probability of a stress-strength model when X and Y have specified distributions has been discussed in literature. The survival probabilities of a single component stress-strength (SSS) model have been considered by several authors for different distributions, see Raqab and Kundu [13], Kundu and Gupta [9,10], Constantine and Karson [6] and Downtown [7]. Several authors have studied the problem of estimating R. Church and Harris [5] derived the MLE of R when X and Y are independently Normally distributed. The MLE of R, when X and Y have bivariate exponential distributions has been considered by Awad et. al. [2]. Awad and Gharraf [3] provided a simulation study to compare three estimates of R when X and Y are independent but not identically distributed Burr random variables. Ahmad et. al. [1} and Surles and Padgett [15,14] provided estimates for R when X and Y are having Burr Type X distribution.

In this paper, we consider the problem of estimating the stress–strength reliability parameter R = P(Y < X), when X and Y be independent strength and stress random variables having Akash distribution with parameters θ_1 and θ_2 respectively. Rama Shanker [12] introduced Akash distribution by considering a two-component mixture of an Exponential distribution having scale parameter θ and a Gamma distribution having shape parameter 3 and scale parameter θ. The probability density function (pdf) of Akash distribution can be defined as

$$f(x;\theta)=\frac{\theta^3}{\theta^2+2}(1+x^2)e^{-\theta x};x>0,\theta>0.$$

The corresponding cumulative distribution function (cdf) is given by

$$F(x)=1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x};x>0,\theta>0.$$

The estimation of the stress–strength parameter R = P[Y < X], when X and Y are having one-parameter Akash distributions with parameter θ_1 and θ_2 respectively, is an unsolved problem. Statistical inference on stress-strength parameters is important in reliability analysis. It is observed that the maximum likelihood estimators can be obtained implicitly by solving two nonlinear equations, but they cannot be obtained in closed form. So, MLE's of parameters are derived numerically. It is not possible to compute the exact distributions of the maximum likelihood estimators, and we used the asymptotic distribution and we constructed approximate confidence intervals of the unknown parameters.

The rest of the paper is organized as follows. In Section 2, the MLE of R is computed. The asymptotic distribution of the MLEs are provided in Section 3. Bootstrap confidence interval is presented in Section 4. In Section 5, simulation study is given. Theoretical results are verified by analyzing one data set in Section 6 and conclusions are given in Section 7.

II. Maximum Likelihood Estimator of R

In this section, the procedure of estimating the reliability of P[Y<X] models using Akash distributions, is considered. It is clear that

$$R=P(Y<X)=\int_{X<Y}f(x,y)dx\,dy=\int_0^\infty f(x;\theta_1)F(x;\theta_2)dx$$

where f (x,y), is the joint pdf of random variables X and Y, having Akash distributions. If the r.v's X and Y are independent, then f (x, y) = f (x) g(y), where f (x) and g(y) are the marginal pdfs of X and Y, so that

$$R=\int_0^\infty \frac{\theta_1^3}{\theta_1^2+2}(1+x^2)e^{-\theta_1 x}1-\left[\frac{1+\theta_2 x(\theta_2 x+2)}{\theta_2^2+2}\right]e^{-\theta_2 x}dx.$$

On simplification we get.

$$R = 1 - \frac{\theta_1^3[\theta_2^6+4\theta_1\theta_2^5+6\theta_1^2\theta_2^4+4\theta_1^3\theta_2^3++22\theta_1\theta_2^3+\theta_1^4\theta_2^2+22\theta_1^2\theta_2^2+4\theta_1^2+2\theta_1^4+20\theta_1\theta_2+10\theta_2^3\theta_2+40\theta_2^2+8\theta_2^4]}{(\theta_1^2+2)\left(2+\theta_2^2\right)(\theta_1+\theta_2)^5}$$

If we have two ordered random samples representing strength (X1,X2,...,Xn) and stress (Y1, Y2,...,Ym) of sizes n and m respectively, following Akash distribution with parameters θ_1 and θ_2, respectively. Then the likelihood function of the combined random sample can be obtained as follows

$$L = \prod_{i=1}^{n} \frac{\theta_1^3}{\theta_1^2+2}(1+x_i^2)e^{-\theta_1 x_i} \prod_{i=1}^{m} \frac{\theta_2^3}{\theta_2^2+2}\left(1+y_j^2\right)e^{-\theta_2 y_j}.$$

The log-likelihood function is

$$l = logL = 3n\,log\ \theta_1 - (\theta_1^2+2) - \theta_1\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} log\ (1+x_i{}^2) + 3m\,log\ \theta_2 - mlog\,(\theta_2^2+2) - \theta_2\sum_{i=1}^{m} y_i + \sum_{i=1}^{m} log\left(1+y_j^2\right).\text{-----------------}(1)$$

The solution of the following non-linear equations yield the MLE of the parameters θ_1 and θ_2. Differentiating (1) with respect to parameters θ_1 and θ_2, we get

$$\frac{\partial l}{\partial\theta_1} = \frac{3n}{\theta_1} - \frac{n2\theta_1}{(\theta_1^2+2)} - \sum_{i=1}^{n} x_i$$

and

$$\frac{\partial l}{\partial\theta_2} = \frac{3m}{\theta_2} - \frac{m2\theta_2}{(\theta_2^2+2)} - \sum_{i=1}^{m} y_j.$$

The second partial derivatives of (1) with respect to parameters θ_1 and θ_2, are

$$\frac{\partial^2 l}{\partial\theta_1^2} = \frac{-3n}{\theta_1^2} - \frac{2n(2-\theta_1^2)}{(\theta_1^2+2)^2}$$

and

$$\frac{\partial^2 l}{\partial\theta_2^2} = \frac{-3m}{\theta_2^2} - \frac{2m(2-\theta_2^2)}{(\theta_2^2+2)^2}$$

MLE of R is obtained as

$$\hat{R} = 1 - \frac{\hat\theta_1^3[\hat\theta_2^6+4\hat\theta_1\hat\theta_2^5+6\hat\theta_1^2\hat\theta_2^4+4\hat\theta_1^3\hat\theta_2^3++22\hat\theta_1\hat\theta_2^3+\hat\theta_1^4\hat\theta_2^2+22\hat\theta_1^2\hat\theta_2^2+4\hat\theta_1^2+2\hat\theta_1^4+20\hat\theta_1\hat\theta_2+10\hat\theta_2^3\hat\theta_2+40\hat\theta_2^2+8\hat\theta_2^4]}{(\hat\theta_1^2+2)\left(2+\hat\theta_2^2\right)\left(\hat\theta_1+\hat\theta_2\right)^5}$$

This can be used in estimation of stress-strength for the given data.

III. Asymptotic Distribution and Confidence Intervals

In this section, the asymptotic distribution and confidence interval of the MLE of R are obtained. To find an asymptotic variance of the MLE $R^{\wedge ML}$, let us denote the Fisher information matrix of $\theta = (\theta_1, \theta_2)$ as $I(\theta) = [I_{ij}(\theta); i,j = 1,2]$, i.e.,

$$I(\theta) = E\begin{bmatrix} \frac{-\partial^2 l}{\partial\theta_1^2} & \frac{-\partial^2 l}{\partial\theta_1\partial\theta_2} \\ \frac{-\partial^2 l}{\partial\theta_2\partial\theta_1} & \frac{-\partial^2 l}{\partial\theta_2^2} \end{bmatrix}$$

To establish the asymptotic Normality, we define

$$d(\theta) = \left(\frac{\partial R}{\partial\theta_1}, \frac{\partial R}{\partial\theta_2}\right)' = (d_1, d_2)',$$

where

$$\frac{\partial R}{\partial\theta_1} = -\left\{\left(\frac{\theta_1^3}{(\theta_1^2+2)(\theta_2^2+2)(\theta_1+\theta_2)^5}\right)\left(4\theta_2^5 + 12\theta_1\theta_2^4 + 12\theta_1^2\theta_2^3 + 22\theta_2^3 + 4\theta_1^3\theta_2^2 + 44\theta_1\theta_2^2 + 30\theta_1^2 + 20\theta_2 + 8\theta_1^3 + 8\theta_1\right) + \left(\frac{(\theta_1^2+2)(\theta_2^2+2)(\theta_1+\theta_2)^5 3\theta_1^2 - \theta_1^3(\theta_2^2+2)[(\theta_1^2+2)5(\theta_1+\theta_2)^4 + 2\theta_1]}{(\theta_1^2+2)^2(\theta_2^2+2)^2(\theta_1+\theta_2)^{10}}\right)\left(\theta_2^6 + 4\theta_1\theta_2^5 + 6\theta_1^2 + \theta_2^4 + 8\theta_2^4 + 4\theta_1^3\theta_2^3 + 22\theta_1\theta_2^3 + \theta_1^4\theta_2^2 + 22\theta_1^2\theta_2^2 + 40\theta_2^2 + 10\theta_1^3\theta_2 + 20\theta_1\theta_2 + 2\theta_1^4 + 4\theta_1^2\right)\right\}$$

$$\frac{\partial R}{\partial\theta_2} = \left\{\left(\frac{\theta_1^3}{(\theta_1^2+2)(\theta_2^2+2)(\theta_1+\theta_2)^5}\right)\left(6\theta_2^5 + 20\theta_1\theta_2^4 + 24\theta_1^2\theta_2^3 + 32\theta_2^3 + 12\theta_1^3\theta_2^2 + 66\theta_1\theta_2^2 + 2\theta_1^4\theta_2 + 44\theta_1^2\theta_2 + 80\theta_2 + 10\theta_1^3 + 20\theta_1\right) + \left(\frac{-\theta_1^3(\theta_1^2+2)[(\theta_2^2+2)5(\theta_1+\theta_2)^4 + (\theta_1+\theta_2)2\theta_2]}{(\theta_1^2+2)^2(\theta_2^2+2)^2(\theta_1+\theta_2)^{10}}\right)\left(\theta_2^6 + 4\theta_1 + \theta_2^5 + 6\theta_2^2 + 22\theta_2^2 + 8\theta_2^2 + \theta_2^2 + +\theta_1^3\theta_2^3 + 22\theta_1\theta_2^3 + \theta_2^2 + 10\theta_1^3\theta_2 + 20\theta_1\theta_2 + 2\theta_1^4 + 4\theta_1^2\right)\right\}.$$

We obtain the asymptotic distribution of $R^{\wedge ML}$ as

$$\sqrt{n+m}(R^{\wedge ML} - R) \to^d N(0, d'(\theta)I^{-1}(\theta)d(\theta)).$$

The asymptotic variance of $R^{\wedge ML}$ is obtained as

$$AV(R^{\wedge ML}) = \frac{1}{n+m} d'(\theta)^{-1} I(\theta) d(\theta).$$

$$i.e., AV(R^{\wedge ML}) = V(\theta^{\wedge}{}_1)d_1^2 + V(\theta^{\wedge}{}_1)d_2^2 + 2d_1 d_2\left(\theta^{\wedge}{}_1, \theta^{\wedge}{}_1\right).$$

Asymptotic $100(1-\gamma)\%$ confidence interval for R can be obtained as

$$R^{\wedge ML} \pm Z_{\frac{\gamma}{2}}\sqrt{AV(R^{\wedge ML})}.$$

IV. Bootstrap Confidence Intervals

In this section, we use confidence intervals based on the parametric percentile bootstrap methods (we call it from now on as Boot-p), Kundu et. al. [11]. Bootstrapping is a statistical approach that resamples a single dataset in order to build up a huge proportion of simulated samples. To estimate confidence intervals of R in this methods, the following steps are used.

1.Estimate θ and ,say $\hat{\theta}$, from the sample using maximum likelihood estimate method
2. Generate a bootstrap sample $(x_{1:n,}x_{2:n,}, x_{2:n,}, \ldots x_{n:n,})$ using $\hat{\theta}$, where $x_{i:n}$ represents the ith observation when there are n observations in the experiment. Obtain the bootstrap estimate of θ, say $\hat{\theta}*$ using the bootstrap sample.
3. Repeat Step [2] NBOOT times.
4. Let $C\bar{D}F(x) = P(\hat{\theta}^{\wedge *} \leq x)$, be the cumulative distribution function of $\lambda^{\wedge *}$. Define $\hat{\theta}^{\wedge}{}_{Boot-p}\,(x) = \widehat{CD}F^{-1}(x)$ for a given x. The approximate $100(1-\alpha)\%$ confidence interval for θ is given by

$$\left(\hat{\theta}^{\wedge}{}_{Boot-p}\left(\frac{\alpha}{2}\right), \hat{\theta}^{\wedge}{}_{Boot-p}\left(1-\frac{\alpha}{2}\right)\right)$$

V. SIMULATION STUDY

In this section, we present some results based on inversion method to assess the performance of estimators of R. For this purpose, we have generated 1000 samples from independent Akash (θ_1) and Akash (θ_2) distributions. We considered sets of parameter values 1.25 and 1.75 which correspond to the R values 0.6491261. The bias and the mean square error (MSE) of the parameter estimates are calculated. In Tables 1, Maximum likelihood estimate of R (R(ML)), the average biase, MSE, asymptotic confidence Intervals (AS(CI)) and Bootstrap confidence interval (BT(CI)) corresponding to different (n,m) values are calculated by the method explained in section 3.

Table 1

(n, m)	$R(ML)$	$BIAS$	MSE	AS(Cl)	BT(CI)
$(7, 7)$	0.712703	0.06357	0.007773	0.5929103,0.8324957	0.6775518,0.8844710
$(15, 15)$	0.714086	0.06495	0.006078	0.6295406,0.7986314	0.6777669,0.8394092
$(30, 30)$	0.713589	0.06446	0.005118	0.652743,0.7744354	0.64231680.7997702
$(30, 35)$	0.713068	0.06399	0.004900	0.6571613,0.7689664	0.5521229,0.7371387
$(40, 40)$	0.713401	0.06426	0.004835	0.6614061,0.7654101	0.6363909,0.7394242
$(50, 50)$	0.713477	0.06341	0.004699	0.667147,0.7598043	0.6572927,0.7705276

From the simulation results, it is observed that as the sample size (n,m) increases the biases and the MSEs decrease. Thus the consistency properties of all the methods are verified.

VI. DATA ANALYSIS

In this section, we consider two real data sets of the breaking strengths of jute fiber at two different gauge lengths (see Xia et. al. [16]). Two sets of real data are shown as follows.

Data set I: Breaking strength of jute fiber length 10 mm (variable X) 693.73, 704.66, 323.83, 778.17, 123.06, 637.66, 383.43, 151.48, 108.94, 50.16, 671.49, 183.16, 257.44, 727.23, 291.27, 101.15, 376.42, 163.40, 141.38, 700.74, 262.90, 353.24, 422.11, 43.93, 590.48, 212.13, 303.90, 506.60, 530.55, 177.25.

Data set II: Breaking strength of jute fiber length 20 mm (variable Y) 71.46, 419.02, 284.64, 585.57, 456.60, 113.85, 187.85, 688.16, 662.66, 45.58, 578.62, 756.70, 594.29, 166.49, 99.72, 707.36, 765.14, 187.13, 145.96, 350.70, 547.44, 116.99, 375.81, 581.60, 119.86, 48.01, 200.16, 36.75, 244.53, 83.55

These data were first used by Xia et al. [16] and later by Saracoglu et. al. [4]. Shahsanaei and Daneshkhah [8] used the data to study the estimation of stress-strength parameter for generalized linear failure rate distribution (GLFRD) under progressive type-II censoring and studied the validity of GLFRD for both data sets.

The table 2 gives the result of goodness of fit test. The test used to check whether the considered distribution is a good fit to the data.

Table 2

PLANE	MLEs	K-S Statistic	P-value
length 10 $mm(X)$	0.00831404	0.13641	0.5847
length 20 $mm(Y)$	0.00880360	0.20925	0.1248

MLEs of parameters of Akash (θ_1) and Akash (θ_2) distributions are 0.00831404 and 0.00880360. Reliability P(Y<X) value for the data is 0.526798. The 95% asymptotic interval of R is (0.36446, 0.7335) and 95% bootstrap confidence interval is (0.376799,0.676798).

VII. CONCLUSION

In this paper, we considered the problem of estimation of P(Y<X) using Akash distribution. The MLE of SSS reliability, R is obtained. Also, asymptotic $100(1-v)\%$ CI for the reliability parameter is computed. Bootstrap confidence interval is also obtained. When the sample size is increased, MSE caused by the estimates comes nearer to zero by extensive simulation. Finally, real data sets are analyzed.

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