POWER GENERALIZED DUS TRANSFORMATION IN WEIBULL AND LOMAX DISTRIBUTIONS

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Abstract

A strong need for an appropriate lifetime model arises in reliability analysis. A large number of lifetime distributions are available in the literature. To analyze reliability data, a more suitable lifetime distribution is plausible. Power Generalized DUS (PGDUS) transformation of the lifetime model gives a solution to fit the data with more precision. PGDUS transformation of the exponential distribution is the first attempt in this regard. This new class of distributions can be used for model series systems in which the components are distributed as DUS transformations of some lifetime model. This paper introduces two novel classes of distributions using PGDUS transformation, which is a generalization of DUS transformation, with Weibull and Lomax distributions as the baseline distributions. Some analytical properties like moments, moment generating function, characteristic function, cumulant generating function, quantile function, distribution of order statistics, and Rényi entropy are derived. The maximum likelihood estimation procedure is employed to estimate the unknown parameters. Moreover, a simulation study has been conducted, and data has been analyzed for each of the proposed distributions to demonstrate how well the distributions would perform in a real-life situation. In comparison with some other recent new models, the proposed distribution is found to be a better model.

Keywords: PGDUS transformation, Weibull Distribution, Moments, Lomax Distribution, Maximum likelihood estimator, Simulation

1. Introduction

A large number of distributions are available in the literature to model monotone failure rate data. The Weibull distribution is confined to model data that exhibit monotone failure rate behavior. Due to the inability to handle non-monotone failure rate behavior, various modifications and generalizations are made to the existing Weibull distribution. The generalized Weibull distribution is widely applied in survival analysis and reliability engineering due to its simplicity and relative flexibility. Xie and Lai [24] introduced an additive Weibull model by adding two Weibull survival functions having a bathtub-shaped failure rate function. Theoretical investigations of the exponentiated Weibull family were carried out by Mudholkar and Srivastava ([18], [19]). Bagdonavicius and Nikulin [3] proposed a power-generalized Weibull distribution as an extension of the Weibull distribution. Xie et al. [25] proposed a modified Weibull bathtub-shaped failure rate distribution.

The Lomax distribution (also called Pareto-II distribution) is a heavily skewed probability distribution that plays an imperative role in the analysis of lifetime data sets in business, actuarial science, computer science, queueing theory, Internet traffic modeling, economics, income and wealth inequality, and reliability modeling. A few generalizations and extensions of the Lomax distribution can be seen in the literature, such as the Marshall-Olkin extended Lomax distribution (Ghitany et al. [9], Gupta et al. [10]), exponentiated Lomax distribution (Abdul-Moniem and Abdel-Hameed [1]), Beta-Lomax distribution (BL), Kumaraswamy Lomax

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distribution and McDonald-Lomax distribution (Lemonte and Cordeiro [15]), Gamma-Lomax distribution (Cordeiro et al. [6]), transmuted Weibull Lomax distribution (Afify et al. [2]) and the generalized transmuted Lomax distribution (Nofal et al. [20]).

With application to survival data analysis, Kumar et al. (2015) proposed a method, called DUS transformation, for getting a new distribution based on exponential baseline distributions. In terms of computation and interpretation, this transformation produces a parsimonious result since it does not include any new parameters other than those involved in the baseline distribution. In the case where F(x) is the CDF of the baseline distribution, then the CDF of the new DUS transformed distribution is as follows:

$$G(x) = \frac{1}{e-1}[e^{F(x)} - 1]$$

Maurya et al. [17] introduced the DUS transformation of the Lindley distribution. Tripathi et al. [23] studied the DUS transformation of an exponential distribution and its inference based on the upper record values. Recent studies using the DUS transformation can be seen in the works of Deepthi and Chacko [7], Kavya and Manoharan [11], and Gauthami and Chacko [8]. Recently, Thomas and Chacko [22] introduced an exponentiated generalization of the DUS transformation called the power generalized DUS transformation. When researchers deal with series systems with components distributed as DUS-transformed lifetime distributions, the PGDUS transformation is highly useful. So the investigation of the PGDUS transformation of various lifetime distributions is relevant in the sense of the selection of appropriate lifetime models for series systems.

The main goal of this study is to introduce two novel distributions using the power generalized DUS (PGDUS) transformation. Let X be a random variable with baseline cumulative distribution function (CDF) G(x) and corresponding probability density function (PDF) g(x). Then the CDF of the proposed PGDUS distribution is defined as:

$$F(x) = \left(\frac{e^{G(x)} - 1}{e-1}\right)^{\theta}, \theta > 0, x > 0. \tag{1}$$

and the PDF is,

$$f(x) = \frac{\theta}{(e-1)^{\theta}}(e^{G(x)} - 1)^{\theta-1}e^{G(x)}g(x), \theta > 0, x > 0. \tag{2}$$

The survival function is,

$$R(x) = 1 - (\frac{e^{G(x)} - 1}{e-1})^{\theta}, \theta > 0, x > 0.$$

The failure rate function is,

$$h(x) = \frac{\theta g(x)e^{G(x)}(e^{G(x)} - 1)^{\theta-1}}{(e-1)^{\theta} - (e^{G(x)} - 1)^{\theta}}, \theta > 0, x > 0.$$

The paper is organized as follows. In Section 2, the distribution based on PGDUS transformation with Weibull distribution as baseline distribution is proposed. Moments, moment generating function, characteristic function, cumulant generating function, quantile function, distribution of order statistics, and Rényi entropy are derived. Parameter estimation based on the maximum likelihood method, simulation study, and real data application are also discussed. In section 3, a different distribution using the Lomax distribution as the baseline distribution in the PGDUS transformation is proposed. As in section 2, properties of PGDUS transformation of Lomax distribution are derived. Parameter estimation using the maximum likelihood method, simulation study, and real data application are also discussed. Finally, concluding remarks are given in Section 4.

2. PGDUS Weibull Distribution

In this section, the Weibull distribution is used as the baseline distribution for PGDUS transformation and investigated the distributional properties. The CDF of Weibull distribution with parameters α and β is

$$G(x) = 1 - e^{-(x\beta)^{\alpha}}, \alpha, \beta > 0, x > 0. \tag{3}$$

and corresponding PDF is

$$g(x) = \alpha\beta(x\beta)^{\alpha-1}e^{-(x\beta)^{\alpha}}, \alpha, \beta > 0, x > 0 \tag{4}$$

Using Eq.3 in Eq.1, the CDF of PGDUS transformation of Weibull distribution is obtained as

$$F(x) = \left(\frac{e^{1-e^{-(x\beta)^{\alpha}}} - 1}{e-1}\right)^{\theta}, \alpha, \beta > 0, \theta > 0, x > 0. \tag{5}$$

and the corresponding PDF is given as

$$f(x) = \frac{\theta\alpha\beta^{\alpha}}{(e-1)^{\theta}}x^{\alpha-1}(e^{1-e^{-(x\beta)^{\alpha}}} - 1)^{\theta-1}e^{1-(x\beta)^{\alpha}-e^{-(x\beta)^{\alpha}}}, \alpha, \beta, \theta > 0, x > 0. \tag{6}$$

Then, the failure rate function associated to Eq.5 and Eq.6 is,

$$h(x) = \frac{\theta\alpha\beta^{\alpha}x^{\alpha-1}(e^{1-e^{-(x\beta)^{\alpha}}} - 1)^{\theta-1}e^{1-(x\beta)^{\alpha}-e^{-(x\beta)^{\alpha}}}}{(e-1)^{\theta} - (e^{1-e^{-(x\beta)^{\alpha}}} - 1)^{\theta}}, \alpha, \beta, \theta > 0, x > 0. \tag{7}$$

The distribution with CDF Eq.5 and PDF Eq.6 is referred to as Power generalized DUS Weibull distribution with parameters α, β and θ and is denoted as $PGDUSW(\alpha, \beta, \theta)$. Figures 1 and 2 provide the graphical representation of the PDF and failure rate function respectively for various parameter values

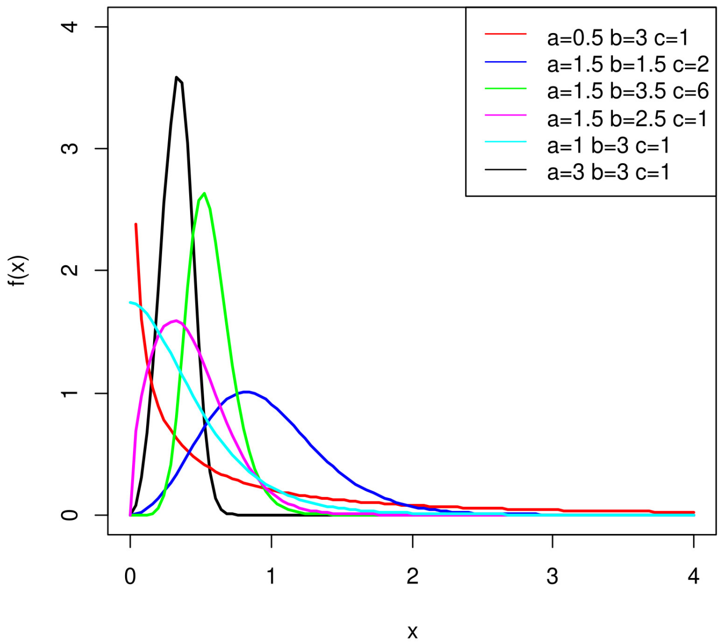

Figure 1: *PGDUSW distribution density plot for various parameter values.*

2.1. Analytical Properties

Moments, moment generating function (MGF), characteristic function (CF), cumulant generating function (CGF), quantile function, distribution of order statistics, and Rényi entropy of the proposed $PGDUSW(\alpha, \beta, \theta)$ distribution are derived.

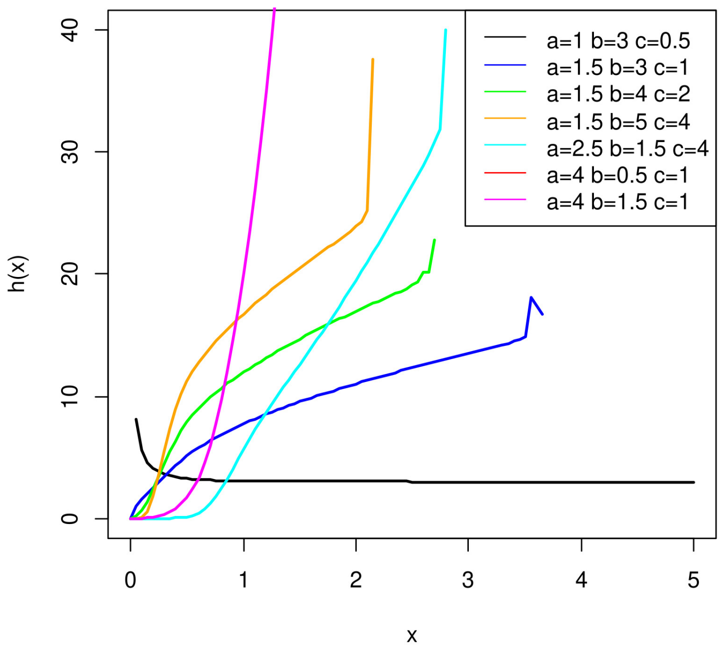

Figure 2: *PGDUSW distribution failure rate plot for various parameter values.*

2.1.1 Moments

The r^{th} raw moment of the $PGDUSW(\alpha, \beta, \theta)$ distribution is given by

$$\mu'_r = \int_0^\infty x^r \frac{\theta\alpha\beta^\alpha x^{\alpha-1}}{(e-1)^\theta}(e^{1-e^{-(x\beta)^\alpha}}-1)^{\theta-1}e^{1-(x\beta)^\alpha - e^{-(x\beta)^\alpha}}dx$$

$$= \frac{\theta\alpha\beta^\alpha e}{(e-1)^\theta}\int_0^\infty x^{r+\alpha-1}e^{-(x\beta)^\alpha}e^{-e^{-(x\beta)^\alpha}}\sum_{k=0}^\infty \binom{\theta-1}{k}(e^{1-e^{-(x\beta)^\alpha}})^{\theta-k-1}(-1)^k dx$$

$$= \frac{\theta\alpha\beta^\alpha e}{(e-1)^\theta}\sum_{k=0}^\infty \binom{\theta-1}{k}(-1)^k e^{\theta-k-1}\int_0^\infty x^{r+\alpha-1}e^{-(x\beta)^\alpha}e^{-(\theta-k)e^{-(x\beta)^\alpha}}dx$$

$$= \frac{\theta\alpha\beta^\alpha e}{(e-1)^\theta}\sum_{k=0}^\infty \binom{\theta-1}{k}(-1)^k e^{\theta-k-1}\sum_{m=0}^\infty \frac{(-1)^m}{m!}(\theta-k)^m\int_0^\infty x^{r+\alpha-1}e^{-(1+m)(x\beta)^\alpha}dx$$

$$= \frac{\theta\beta^{-r}e}{(e-1)^\theta}\sum_{k=0}^\infty\sum_{m=0}^\infty \frac{(-1)^{m+k}}{m!}e^{\theta-k-1}\binom{\theta-1}{k}(\theta-k)^m\frac{\Gamma(\frac{r}{\alpha}+1)}{(1+m)^{\frac{r}{\alpha}+1}}$$

2.1.2 Moment Generating Function

The MGF of $PGDUSW(\alpha, \beta, \theta)$ distribution is

$$M_X(t) = \frac{\theta\alpha\beta^\alpha e}{(e-1)^\theta}\int_0^\infty x^{\alpha-1}e^{tx}e^{-(x\beta)^\alpha}e^{-e^{-(x\beta)^\alpha}}(e^{1-e^{-(x\beta)^\alpha}}-1)^{\theta-1}dx$$

$$= \frac{\theta\alpha\beta^{\alpha}e}{(e-1)^{\theta}}\int_{0}^{\infty} x^{\alpha-1}e^{tx}e^{-(x\beta)^{\alpha}}e^{-e^{-(x\beta)^{\alpha}}}\sum_{k=0}^{\infty}\binom{\theta-1}{k}(-1)^{k}(e^{1-e^{-(x\beta)^{\alpha}}})^{\theta-k-1}dx$$

$$= \frac{\theta\alpha\beta^{\alpha}e}{(e-1)^{\theta}}\sum_{k=0}^{\infty}\binom{\theta-1}{k}(-1)^{k}e^{\theta-k-1}\int_{0}^{\infty} x^{\alpha-1}e^{tx}e^{-(x\beta)^{\alpha}}e^{-(\theta-k)e^{-(x\beta)^{\alpha}}}dx$$

$$= \frac{\theta\alpha\beta^{\alpha}}{(e-1)^{\theta}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!}\binom{\theta-1}{k}e^{\theta-k}(\theta-k)^{m}\int_{0}^{\infty} x^{\alpha-1}e^{tx}e^{-(1+m)(x\beta)^{\alpha}}dx$$

$$= \frac{\theta\alpha\beta^{\alpha}}{(e-1)^{\theta}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+m+n}}{m!n!}\binom{\theta-1}{k}e^{\theta-k}(\theta-k)^{m}(1+m)^{n}\beta^{\alpha n}\int_{0}^{\infty} x^{\alpha+\alpha n-1}e^{tx}dx$$

$$= \frac{\theta\alpha}{(e-1)^{\theta}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+m+n}}{m!n!}\binom{\theta-1}{k}e^{\theta-k}(\theta-k)^{m}(1+m)^{n}\beta^{\alpha+\alpha n}\frac{\Gamma(\alpha+\alpha n)}{t^{\alpha+\alpha n}}$$

2.1.3 Characteristic Function and Cumulant Generating Function

The CF of $PGDUSW(\alpha, \beta, \theta)$ is given by

$$\phi_X(t) = \frac{\theta\alpha}{(e-1)^{\theta}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+m+n}}{m!n!}\binom{\theta-1}{k}e^{\theta-k}(\theta-k)^{m}(1+m)^{n}\beta^{\alpha+\alpha n}\frac{\Gamma(\alpha+\alpha n)}{(it)^{\alpha+\alpha n}},$$

and the CGF of $PGDUSW(\alpha, \beta, \theta)$ is given by

$$\begin{aligned} K_X(t) &= \log\phi_X(t) \\ &= \log\left[\frac{\theta\alpha}{(e-1)^{\theta}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{k+m+n}}{m!n!}\binom{\theta-1}{k}e^{\theta-k}(\theta-k)^{m}(1+m)^{n}\beta^{\alpha+\alpha n}\frac{\Gamma(\alpha+\alpha n)}{(it)^{\alpha+\alpha n}}\right] \end{aligned}$$

where $i = \sqrt{-1}$ is the unit imaginary number.

2.1.4 Quantile Function

The p^{th} quantile $Q(p)$ of the $PGDUSW(\alpha, \beta, \theta)$ is the real solution of the following equation

$$((e^{1-e^{-(\beta Q(p))^{\alpha}}}-1)/(e-1))^{\theta} = p$$

where $p \sim Uniform(0, 1)$. Solving the above equation for $Q(p)$, we have

$$Q(p) = \frac{-1}{\beta^{\alpha}}\log[1-\log{(e-1)p^{\frac{1}{\theta}}}+1]^{\frac{1}{\alpha}}. \tag{8}$$

The median is obtained by setting $p = 0.5$ in the Eq.8. Thus,

$$Median = \frac{-1}{\beta^{\alpha}}\log[1-\log{(e-1)0.5^{\frac{1}{\theta}}}+1]^{\frac{1}{\alpha}}.$$

Similarly, the quartiles Q_1 and Q_3 are obtained respectively by setting $p = 0.25$ and $p = 0.75$ in Eq.8.

2.1.5 Distribution of Order Statistic

Let $X_1, X_2, \ldots, X_m$ be m independent random variables from the $PGDUSW(\alpha,\beta,\theta)$ distribution with CDF Eq.5 and PDF Eq.6. Then the PDF of r^{th} order statistics $X_{(r)}$ of the $PGDUSW(\alpha,\beta,\theta)$ distribution is given by

$$f_{X_{(r)}} = \frac{m!}{(r-1)!(m-r)!}\frac{\theta\alpha\beta^{\alpha}x^{\alpha-1}}{(e-1)^{\theta m}}\left(e^{1-e^{-(x\beta)^{\alpha}}}-1\right)^{\theta r-1}e^{1-(x\beta)^{\alpha}-e^{-(x\beta)^{\alpha}}}$$
$$\left[(e-1)^{\theta}-(e^{1-e^{-(x\beta)^{\alpha}}})^{\theta}\right]^{m-r}, r=1,2,\ldots,m. \tag{9}$$

Then, the PDF of $X_{(1)}$ and $X_{(m)}$ are obtained by setting $r=1$ and $r=m$ respectively in Eq.9. This can be used to analyze the reliability of serial and parallel systems.

2.1.6 Rényi Entropy

Rényi entropy introduced by Rényi [21] is defined as

$$\beth_R(\nu) = \frac{1}{1-\nu}\log\left(\int f^{\nu}(x)dx\right)$$

where $\nu > 0$ and $\nu \neq 1$.

$$\int_0^{\infty} f^{\nu}(x)dx = \frac{(\theta\alpha\beta^{\alpha}e)^{\nu}}{(e-1)^{\theta\nu}}\int_0^{\infty} x^{\nu\alpha-\nu}e^{-\nu(x\beta)^{\alpha}}e^{-\nu e^{-(x\beta)^{\alpha}}}\sum_{k=0}^{\infty}\binom{\nu\theta-\nu}{k}(-1)^k(e^{1-e^{-(x\beta)^{\alpha}}})^{\nu\theta-\nu-k}dx$$

$$= \frac{(\theta\alpha\beta^{\alpha}e)^{\nu}}{(e-1)^{\theta\nu}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!}\binom{\nu\theta-\nu}{k}(\nu\theta-k)^m e^{\nu\theta-\nu-k}\int_0^{\infty} x^{\nu\alpha-\nu}e^{-(\nu+m)(x\beta)^{\alpha}}dx$$

$$= \frac{(\theta\alpha)^{\nu}}{(e-1)^{\theta\nu}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!}\binom{\nu\theta-\nu}{k}(\nu\theta-k)^m e^{\nu\theta-k}\frac{\Gamma(\nu-\frac{\nu}{\alpha}+1)}{(\nu+m)^{\nu-\frac{\nu}{\alpha}+1}\beta^{\alpha-\nu}}$$

Then the Rényi entropy becomes

$$\beth_R(\nu) = \frac{1}{1-\nu}\log\left[\frac{(\theta\alpha)^{\nu}}{(e-1)^{\theta\nu}}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{k+m}}{m!}\binom{\nu\theta-\nu}{k}(\nu\theta-k)^m e^{\nu\theta-k}\frac{\Gamma(\nu-\frac{\nu}{\alpha}+1)}{(\nu+m)^{\nu-\frac{\nu}{\alpha}+1}\beta^{\alpha-\nu}}\right]$$

2.2. Estimation

The method of Maximum likelihood estimation is used to estimate the unknown parameters of the $PGDUSW(\alpha,\beta,\theta)$. For this, consider a random sample of size n from $PGDUSW(\alpha,\beta,\theta)$ distribution. Therefore, the likelihood function is given by,

$$L(\alpha,\beta,\theta|x) = \prod_{i=1}^{n} f(x) = \prod_{i=1}^{n}\frac{\theta\alpha\beta^{\alpha}}{(e-1)^{\theta}}x^{\alpha-1}e^{1-(x_i\beta)^{\alpha}-e^{-(x_i\beta)^{\alpha}}}(e^{1-e^{-(x_i\beta)^{\alpha}}}-1)^{\theta-1} \tag{10}$$

Applying the natural logarithm to Eq.10, the log-likelihood function becomes

$$\log L = n\log(\theta)+n\log(\alpha)+\alpha n\log(\beta)-\theta n\log(e-1)+n+\sum_{i=0}^{n}(\alpha-1)\log(x_i)$$
$$-\sum_{i=0}^{n}(x_i\beta)^{\alpha}-\sum_{i=0}^{n}e^{-(x_i\beta)^{\alpha}}+(\theta-1)\sum_{i=0}^{n}\log(e^{1-e^{-(x_i\beta)^{\alpha}}}-1).$$

Computing the first order partial derivatives, we get

$$\begin{aligned}\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=0}^{n}(x_i\beta)^{\alpha}\log(x_i\beta) + \sum_{i=0}^{n}\log(x_i) + \sum_{i=0}^{n}(x_i\beta)^{\alpha}e^{-(x_i\beta)^{\alpha}}\log(x_i\beta) \\ +n\log(\beta) + \frac{(\theta-1)(x_i\beta)^{\alpha}}{(e^{1-e^{-(x_i\beta)^{\alpha}}}-1)}\log(x_i\beta)e^{1-(x_i\beta)^{\alpha}-e^{-(x_i\beta)^{\alpha}}},\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \beta} = \frac{n\alpha}{\beta} - \sum_{i=0}^{n}\frac{\alpha(x_i\beta)^{\alpha}}{\beta} + \sum_{i=0}^{n}\frac{\alpha(x_i\beta)^{\alpha}}{\beta}e^{-(x_i\beta)^{\alpha}} \\ +(\theta-1)\sum_{i=0}^{n}\frac{\alpha}{\beta}(x_i\beta)^{\alpha}\frac{e^{1-(x_i\beta)^{\alpha}-e^{-(x_i\beta)^{\alpha}}}}{(e^{1-e^{-(x_i\beta)^{\alpha}}}-1)},\end{aligned} \tag{12}$$

and

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - n\log(e-1) + \sum_{i=0}^{n}\log(e^{1-e^{-(x_i\beta)^{\alpha}}}-1). \tag{13}$$

Equations 11, 12, and 13 are not in closed form. The solution of these explicit equations can be obtained analytically and can be solved numerically using R software by taking arbitrary initial values.

2.3. Simulation Study

In order to illustrate the performance of the maximum likelihood method for $PGDUSW(\alpha,\beta,\theta)$ distribution, the inversion transformation method is used. For different values of α, β, and θ, samples of sizes n = 100, 250, 500, 750, and 1000 are generated from the $PGDUSW(\alpha,\beta,\theta)$ model. For 1000 repetitions, the bias and mean square error (MSE) of the estimated parameters are computed. The selected parameter values are $\alpha = 0.5, \beta = 0.5$ and $\theta = 0.5$, $\alpha = 0.5, \beta = 1$ and $\theta = 0.5$ and $\alpha = 1, \beta = 1$ and $\theta = 0.5$. From Tables 1, 2, and 3, it is noted that bias and MSE decrease for the selected parameter values as sample size increases.

Table 1: *Estimate, Biases and MSEs for PGDUSW model at $\alpha = 0.5, \beta = 0.5$ and $\theta = 0.5$*

n	Estimated value of Parameters	Bias	MSE
	$\hat{\alpha}$=0.5668	0.0668	0.0473
100	$\hat{\beta}$=0.7541	0.2541	1.0617
	$\hat{\theta}$=0.5021	0.0031	0.0413
	$\hat{\alpha}$=0.5251	0.0251	0.0118
250	$\hat{\beta}$=0.5832	0.0832	0.1488
	$\hat{\theta}$=0.5032	0.0022	0.0165
	$\hat{\alpha}$=0.5297	0.0189	0.0057
500	$\hat{\beta}$=0.4929	0.0177	0.0318
	$\hat{\theta}$=0.4922	0.0007	0.0068
	$\hat{\alpha}$=0.5188	0.0188	0.0034
750	$\hat{\beta}$=0.4936	-0.0065	0.0223
	$\hat{\theta}$=0.5026	0.0003	0.0050
	$\hat{\alpha}$=0.5165	0.0165	0.0025
1000	$\hat{\beta}$=0.4795	-0.0205	0.0160
	$\hat{\theta}$=0.4922	-0.0078	0.0035

Table 2: *Estimate, Biases and MSEs for PGDUSW model at $\alpha = 0.5, \beta = 1$ and $\theta = 0.5$*

n	Estimated value of Parameters	Bias	MSE
	$\hat{\alpha}$=0.5730	0.0730	0.0460
100	$\hat{\beta}$=1.4827	0.4827	3.7354
	$\hat{\theta}$=0.5134	0.0434	0.0485
	$\hat{\alpha}$=0.5019	0.0019	0.0083
250	$\hat{\beta}$=1.2852	0.2852	0.6372
	$\hat{\theta}$=0.5333	0.0393	0.0169
	$\hat{\alpha}$=0.4943	-0.0057	0.0041
500	$\hat{\beta}$=1.2236	0.2236	0.2915
	$\hat{\theta}$=0.5399	0.0339	0.0102
	$\hat{\alpha}$=0.4886	-0.0109	0.0023
750	$\hat{\beta}$=1.1045	0.1814	0.1353
	$\hat{\theta}$=0.5244	0.0244	0.0050
	$\hat{\alpha}$=0.4822	-0.0178	0.0022
1000	$\hat{\beta}$=1.1814	0.1045	0.1195
	$\hat{\theta}$=0.5207	0.0207	0.0042

Table 3: *Estimate, Biases and MSEs for PGDUSW model at $\alpha = 1, \beta = 1$ and $\theta = 0.5$*

n	Estimated value of Parameters	Bias	MSE
	$\hat{\alpha}$=1.1273	0.1273	0.1628
100	$\hat{\beta}$=1.1460	0.1460	0.8851
	$\hat{\theta}$=0.5223	0.0223	0.0545
	$\hat{\alpha}$=1.0184	0.0184	0.0450
250	$\hat{\beta}$=1.0889	0.0889	0.1068
	$\hat{\theta}$=0.5205	0.0205	0.0177
	$\hat{\alpha}$=1.0109	0.0109	0.0185
500	$\hat{\beta}$=1.0490	0.0490	0.0447
	$\hat{\theta}$=0.5151	0.0151	0.0085
	$\hat{\alpha}$=1.0056	0.0056	0.0107
750	$\hat{\beta}$=1.0381	0.0381	0.0260
	$\hat{\theta}$=0.5095	0.0095	0.0049
	$\hat{\alpha}$=0.9851	-0.0149	0.0074
1000	$\hat{\beta}$=1.0239	0.0239	0.0167
	$\hat{\theta}$=1.0012	0.0012	0.0035

2.4. Application

A real data analysis is carried out to determine the performance of the proposed model. For this, the data on the number of million revolutions before the failure of 23 ball bearings put on test is considered (Lawless [13]), see Table 4.

Different distributions namely, Inverse Weibull (IW) distribution, DUS Exponential (DUSE) distribution, and Kavya-Manoharan Weibull (KMW) distribution are used to compare the performance with the proposed $PGDUSW(\alpha, \beta, \theta)$ distribution. In order to perform the necessary numerical evaluations, the software R is used.

Table 4: *Lawless Data*

17.88	28.92	33.00	41.52	42.12	45.60
48.80	51.84	51.96	54.12	55.56	67.80
68.64	68.64	68.88	84.12	93.12	98.64
105.12	105.84	127.92	128.04	173.40	

Table 5: *Findings for PGDUSW Distribution*

Model	MLEs	log L	AIC	CAIC	KS	p-value
IW	$\hat{\lambda}$ = 1.8341 $\hat{\theta}$ = 0.0206	-115.7887	235.5774	236.1774	0.1328	0.8118
DUSE	$\hat{a}$ = 0.0182	-127.4622	256.9244	257.1149	0.2774	0.0580
KMW	$\hat{\lambda}$ = 2.3169 $\hat{\kappa}$ = 0.0107	-113.4076	230.8152	231.4152	0.1421	0.7419
PGDUSW	$\hat{\alpha}$ = 0.9362 $\hat{\beta}$ = 0.0383 $\hat{\theta}$ = 4.4478	-113.0114	230.0228	230.6228	0.10921	0.9467

To check the acceptability of the $PGDUSW(\alpha, \beta, \theta)$ distribution for the given data set Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), log-likelihood value, and Kolmogorov-Smirnov goodness of fit test statistic (KS) with the p-value are used and the computed values are provided in Table 5. It is worth noting that in the goodness of fit test, the purpose is to determine whether the sets of data with the distribution function $F(y)$ and the hypothesised distribution $F_{PGDUSW}(y)$ are compatible. This problem can be formulated as $H_0 : F(y) = F_{PGDUSW}(y)$ versus the alternative $H_1 : F(y) = F_{PGDUSW}(y)$.

From Table 5, it is noted that the $PGDUSW(\alpha, \beta, \theta)$ distribution fits well for the given data set. To facilitate a better understanding of the results, the plot of the empirical CDF (ECDF) is shown in the figure along with other CDFs of the distributions for the Lawless dataset. Furthermore, our proposed distribution is found to fit better than those of the other distributions.

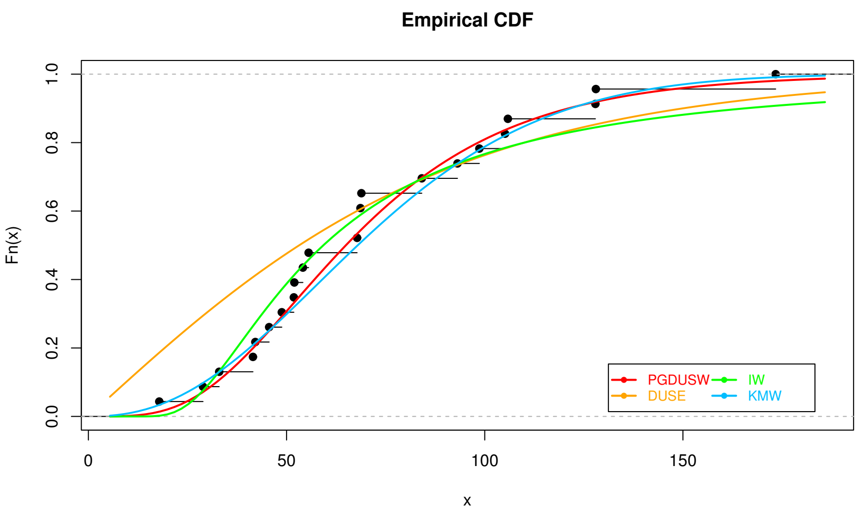

Figure 3: *ECDF plot for various distributions.*

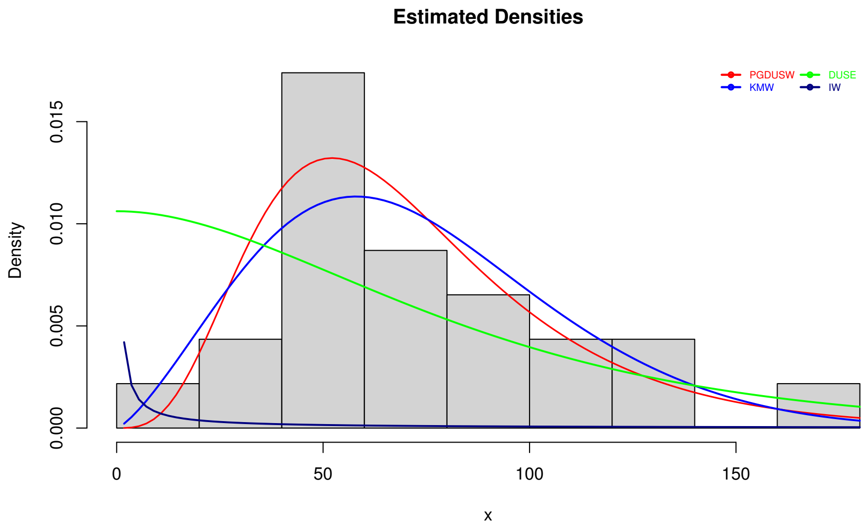

Figure 4: *ECDF plot for various distributions.*

3. PGDUS Lomax Distribution

Power Generalized DUS Lomax distribution denoted as $PGDUSL(\alpha, \beta, \theta)$, is obtained using PGDUS transformation with Lomax distribution as baseline distribution. Then the CDF of the $PGDUSL(\alpha, \beta, \theta)$ distribution using Eq.1 is given by

$$F(x) = \left(\frac{e^{1-(1+x\beta)^{-\alpha}} - 1}{e - 1} \right)^{\theta}, \alpha, \beta > 0, \theta > 0, x > 0. \tag{14}$$

Then the PDF is

$$f(x) = \frac{\theta \alpha \beta}{(e-1)^{\theta}} (e^{1-(1+x\beta)^{-\alpha}} - 1)^{\theta - 1} e^{1-(1+x\beta)^{-\alpha}} (1 + x\beta)^{-(\alpha+1)}. \tag{15}$$

The failure rate function is

$$h(x) = \frac{\theta \alpha \beta (e^{1-(1+x\beta)^{-\alpha}} - 1)^{\theta - 1} e^{1-(1+x\beta)^{-\alpha}} (1 + x\beta)^{-(\alpha+1)}}{(e-1)^{\theta} - (e^{1-(1+x\beta)^{-\alpha}} - 1)^{\theta}}$$

3.1. Properties of PGDUSL Distribution

Here, we explore a few properties of the PGDUSL distribution.

3.1.1 Moments

The r^{th} raw moments of $PGDUSL(\alpha, \beta, \theta)$ is

$$\mu'_r = \frac{\theta \alpha}{(e-1)^{\theta}} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{k+m+n}}{n!} \binom{\alpha + k}{k} \binom{\theta - 1}{m} \beta^{k+1} e^{\theta - m} (\theta - m)^n$$
$$B(r + k + 1, \alpha n - r - k - 1)$$

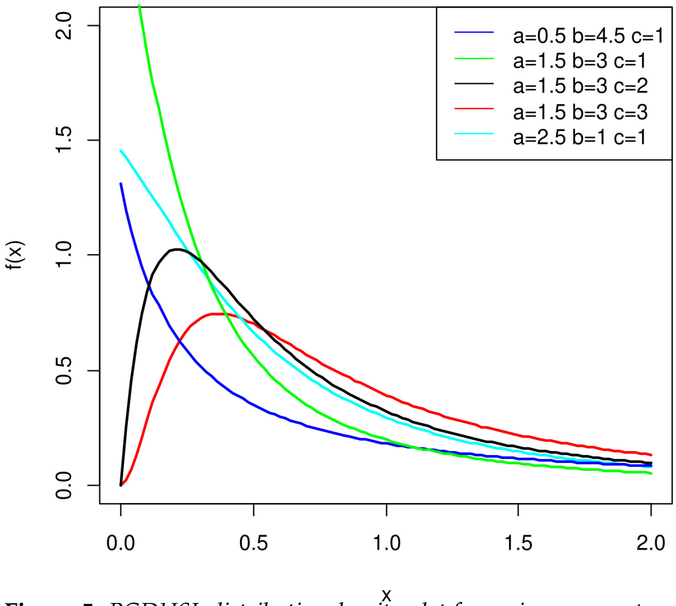

Figure 5: *PGDUSL distribution density plot for various parameter values.*

3.1.2 Quantile Function

The p^{th} quantile $Q(p)$ of the $PGDUSL(\alpha, \beta, \theta)$ is the real solution of the following equation

$$((e^{1-1+(\beta Q(p))^{\alpha}} - 1)/(e-1))^{\theta} = p$$

where $p \sim Uniform(0,1)$. Solving the above equation for $Q(p)$, we have

$$Q(p) = \frac{1}{\beta}\{\left[1 - \log\left[p^{\frac{1}{\theta}}(e-1)+1\right]\right]^{\frac{-1}{\alpha}} - 1\}.$$

The median is obtained by setting $p = 0.5$ in the above equation. Thus,

$$Median = \frac{1}{\beta}\{\left[1 - \log\left[0.5^{\frac{1}{\theta}}(e-1)+1\right]\right]^{\frac{-1}{\alpha}} - 1\}$$

3.2. Estimation of PGDUSL Distribution

The method of maximum likelihood estimation is used to estimate the unknown parameters of $PGDUSL(\alpha, \beta, \theta)$. For this, consider a random sample of size n from $PGDUSL(\alpha, \beta, \theta)$ distribution. Therefore, the likelihood function is given by,

$$L(\alpha, \beta, \theta | x) = \prod_{i=1}^{n} f(x) = \frac{(\theta\alpha\beta)^n}{(e-1)^{\theta n}} \prod_{i=1}^{n} (e^{1-(1+x_i\beta)^{-\alpha}} - 1)^{\theta-1} e^{1-(1+x_i\beta)^{-\alpha}} (1+x_i\beta)^{-\alpha+1} \tag{16}$$

The log-likelihood function becomes

$$\log L = n\log(\theta) + n\log(\alpha) + n\log(\beta) - \theta n \log(e-1) + n - \sum_{i=1}^{n}(1+x_i\beta)^{-\alpha} - (\alpha+1)\sum_{i=1}^{n}\log(1+x_i\beta) + (\theta-1)\sum_{i=1}^{n}\log(e^{1-(1+x_i\beta)^{-\alpha}} - 1) \tag{17}$$

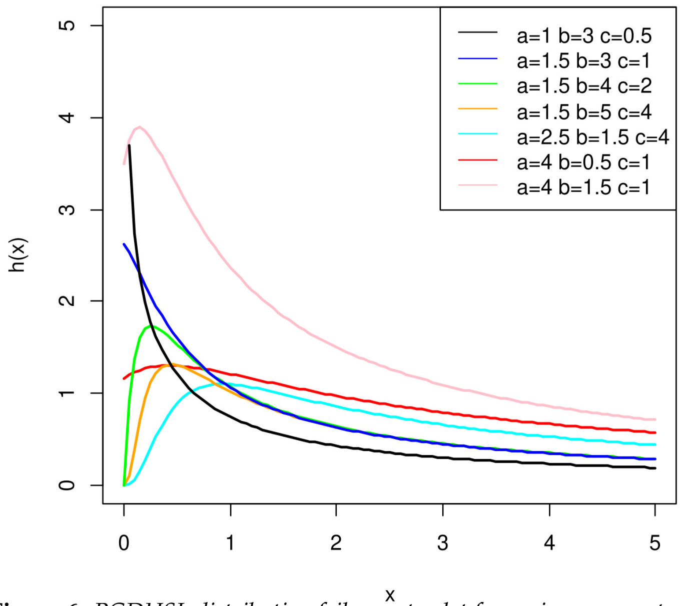

Figure 6: *PGDUSL distribution failure rate plot for various parameter values.*

Computing the first order partial derivatives of Eq.17, we get

$$\begin{aligned}\frac{\partial \log L}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} \log(1+x_i\beta)(1+x_i\beta)^{-\alpha} - \sum_{i=1}^{n} \log(1+x_i\beta) \\ &\quad + \sum_{i=1}^{n} \frac{(\theta-1)\log(1+x_i\beta)e^{1-(1+x_i\beta)^{-\alpha}}(1+x_i\beta)^{-\alpha}}{(e^{1-(1+x_i\beta)^{-\alpha}}-1)},\end{aligned} \tag{18}$$

$$\begin{aligned}\frac{\partial \log L}{\partial \beta} &= \frac{n}{\beta} - \sum_{i=1}^{n} \alpha x_i(1+x_i\beta)^{-(\alpha+1)} - (\alpha+1)\sum_{i=1}^{n} \frac{x_i}{1+x_i\beta} \\ &\quad - \sum_{i=1}^{n} \frac{\alpha x_i(\theta-1)(1+x_i\beta)^{-(\alpha+1)}}{(e^{1-(1+x_i\beta)^{-\alpha}}-1)}\end{aligned} \tag{19}$$

and

$$\frac{\partial \log L}{\partial \theta} = \frac{n}{\theta} - n\log(e-1) + \sum_{i=1}^{n} \log(e^{1-(1+x_i\beta)^{-\alpha}}-1) \tag{20}$$

Equations 18, 19, and 20 are not in closed form. The solution to these explicit equations can be obtained analytically and can be solved numerically using R software by taking arbitrary initial values.

3.3. Simulation Study

In order to illustrate the performance of the maximum likelihood method for $PGDUSL(\alpha,\beta,\theta)$ distribution, the inverse transformation method is used. For different combinations of values of α,β, and θ, samples of sizes n = 250, 500, 750, and 1000 are generated from the $PGDUS-L(\alpha,\beta,\theta)$ model. For 1000 repetitions, the bias and mean square error (MSE) of the estimated parameters are computed. The selected parameter values are $\alpha = 0.5, \beta = 0.5$ and $\theta = 0.5$, $\alpha = 1, \beta = 1.5$ and $\theta = 0.5$ and $\alpha = 1, \beta = 1.5$ and $\theta = 1$. From Tables 6, 7, and 8, it is observed that bias and MSE decrease for the selected parameter values as sample size increases.

Table 6: *Estimate, Biases and MSEs for PGDUSL model at $\alpha = 0.5, \beta = 0.5$ and $\theta = 0.5$*

n	Estimated value of Parameters	Bias	MSE
	$\hat{\alpha}$=0.5100	0.0100	0.0031
250	$\hat{\beta}$=0.5520	0.0720	0.0665
	$\hat{\theta}$=0.5218	0.0218	0.0049
	$\hat{\alpha}$=0.4921	-0.0039	0.0016
500	$\hat{\beta}$=0.5926	0.0526	0.0422
	$\hat{\theta}$=0.5197	0.0197	0.0023
	$\hat{\alpha}$=0.4960	-0.0079	0.0010
750	$\hat{\beta}$=0.5313	0.0343	0.0181
	$\hat{\theta}$=0.5088	0.0088	0.0013
	$\hat{\alpha}$=0.4889	-0.0111	0.0008
1000	$\hat{\beta}$=0.5343	0.0313	0.0134
	$\hat{\theta}$=0.5046	0.0046	0.0009

Table 7: *Estimate, Biases and MSEs for PGDUSL model at $\alpha = 1, \beta = 1.5$ and $\theta = 0.5$*

n	Estimated value of Parameters	Bias	MSE
	$\hat{\alpha}$=1.0268	0.0268	0.0314
250	$\hat{\beta}$=1.6452	0.1800	0.4484
	$\hat{\theta}$=0.5217	0.0217	0.0037
	$\hat{\alpha}$=1.0140	0.0140	0.0131
500	$\hat{\beta}$=1.6800	0.1452	0.2215
	$\hat{\theta}$=0.5187	0.0187	0.0017
	$\hat{\alpha}$=0.9838	-0.0070	0.0080
750	$\hat{\beta}$=1.6374	0.1374	0.1404
	$\hat{\theta}$=0.5040	0.0050	0.0008
	$\hat{\alpha}$=0.9930	-0.0162	0.0059
1000	$\hat{\beta}$=1.6070	0.1070	0.0906
	$\hat{\theta}$=0.5050	0.0040	0.0006

Table 8: *Estimate, Biases and MSEs for PGDUSL model at $\alpha = 1, \beta = 1.5$ and $\theta = 1$*

n	Estimated value of Parameters	Bias	MSE
	$\hat{\alpha}$=1.0284	0.0284	0.0194
250	$\hat{\beta}$=1.69386	0.19386	0.71071
	$\hat{\theta}$=1.05298	0.05297	0.03426
	$\hat{\alpha}$=1.0179	0.0179	0.0082
500	$\hat{\beta}$=1.5999	0.0999	0.1999
	$\hat{\theta}$=1.0472	0.0472	0.0144
	$\hat{\alpha}$=0.9917	-0.0083	0.0049
750	$\hat{\beta}$=1.5596	0.0596	0.1101
	$\hat{\theta}$=1.0145	0.0145	0.0068
	$\hat{\alpha}$=0.9836	-0.0164	0.0033
1000	$\hat{\beta}$=1.5187	0.0187	0.0755
	$\hat{\theta}$=0.9967	-0.0033	0.0051

From Tables 6, 7, and 8, it is observed that bias and MSE are getting closer to zero, as the sample size increases. Therefore, it can be concluded that the proposed model is more consistent and the performance of MLE is highly adequate.

3.4. Real Data Application

Real data analysis is used to determine the applicability of the PGDUSL model. The data set shown in Table 9 is an uncensored data set. As reported by Lee and Wang [14], Table 9 shows the number of months in which 128 bladder cancer patients experienced remission. Different distributions namely, Lomax distribution (LD), DUS Exponential distribution (DUSE), and DUS Lomax distribution (DUSL) are used to compare the performance with the proposed $PGDUSL(\alpha, \beta, \theta)$ distribution. In order to perform the necessary numerical evaluations, the software R is used.

Table 9: ***Remission Times in Months of Blood Cancer Patients***

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.20	2.23
0.52	4.98	6.97	9.02	13.29	0.40	2.26	3.57	5.06	7.09
0.22	13.80	25.74	0.50	2.46	3.64	5.09	7.26	9.47	14.24
0.82	0.51	2.54	3.70	5.17	7.28	9.74	14.76	26.31	0.81
0.62	3.82	5.32	7.32	10.06	14.77	32.15	2.64	3.88	5.32
0.39	10.34	14.83	34.26	0.90	2.69	4.18	5.34	7.59	10.66
0.96	36.66	1.05	2.69	4.23	5.41	7.62	10.75	16.62	43.01
0.19	2.75	4.26	5.41	7.63	17.12	46.12	1.26	2.83	4.33
0.66	11.25	17.14	79.05	1.35	2.87	5.62	7.87	11.64	17.36
0.40	3.02	4.34	5.71	7.93	11.79	18.10	1.46	4.40	5.85
0.26	11.98	19.13	1.76	3.25	4.50	6.25	8.37	12.02	2.02
0.31	4.51	6.54	8.53	12.03	20.28	2.02	3.36	6.76	12.07
0.73	2.07	3.36	6.93	8.65	12.63	22.69	5.49		

To check the acceptability of the $PGDUSL(\alpha, \beta, \theta)$ distribution for the given data set AIC, Consistent AIC (CAIC), log-likelihood value, and KS statistic with the p-value are used and the computed values are provided in Table 10. It is worth noting that in the goodness of fit test, the purpose is to determine whether the sets of data with the distribution function $F(y)$ and the hypothesised distribution $F_{PGDUSL}(y)$ are compatible. This problem can be formulated as $H_{04} : F(y) = F_{PGDUSL}(y)$ versus the alternative $H_{14} : F(y) \neq F_{PGDUSL}(y)$.
Similarly the following hypotheses are tested.
$H_{01} : F(y) = F_{LD}(y)$ Vs $H_{11} : F(y) \neq F_{LD}(y)$
$H_{02} : F(y) = F_{DUSE}(y)$ Vs $H_{12} : F(y) \neq F_{DUSE}(y)$
$H_{03} : F(y) = F_{DUSL}(y)$ Vs $H_{13} : F(y) \neq F_{DUSL}(y)$
From Table 10, it is clear that $PGDUSL(\alpha, \beta, \theta)$ distribution fits well for the given data set. To facilitate a better understanding of the results, the plot of the empirical CDF (ECDF) is shown in the Fig.7 along with other CDFs of the distributions for the blood cancer patients dataset. Also, the plot of fitted densities for the blood cancer patients dataset are given. Furthermore, our proposed distribution is found to fit better than those of the other distributions.

Table 10: *Findings for PGDUSL distribution*

Model	MLEs	log L	AIC	CAIC	KS	p- value
LD	$\hat{\lambda} = 15.2817$ $\hat{\theta} = 0.0074$	-414.98	833.960	834.056	0.094	0.208
DUSE	$\hat{\mu} = 0.1342$	-433.139	868.278	868.309	0.081	0.366
DUSL	$\hat{\lambda} = 6.471$ $\hat{\theta} = 0.0253$	-413.077	830.153	830.249	0.075	0.463
PGDUSL	$\hat{\alpha} = 3.842$ $\hat{\beta} = 0.0605$ $\hat{\theta} = 1.3984$	-411.019	828.039	828.2324	0.035	0.998

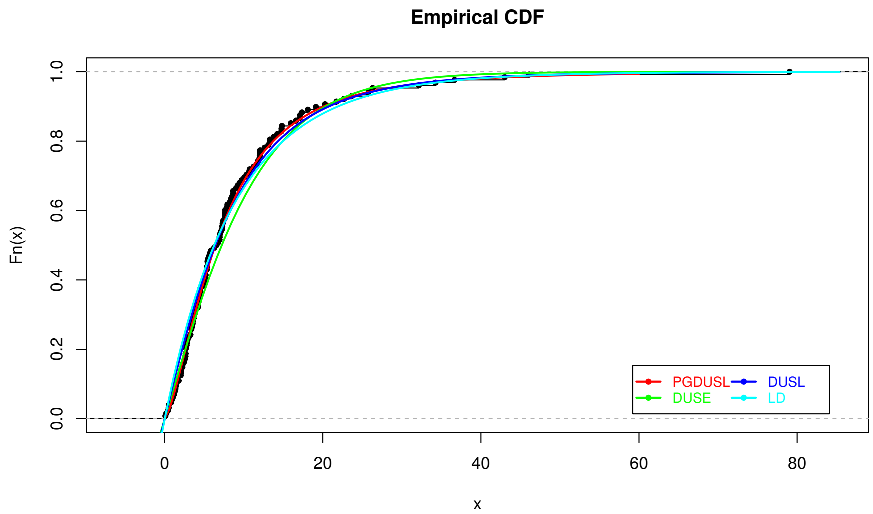

Figure 7: *ECDF plot of the models for blood cancer patients dataset.*

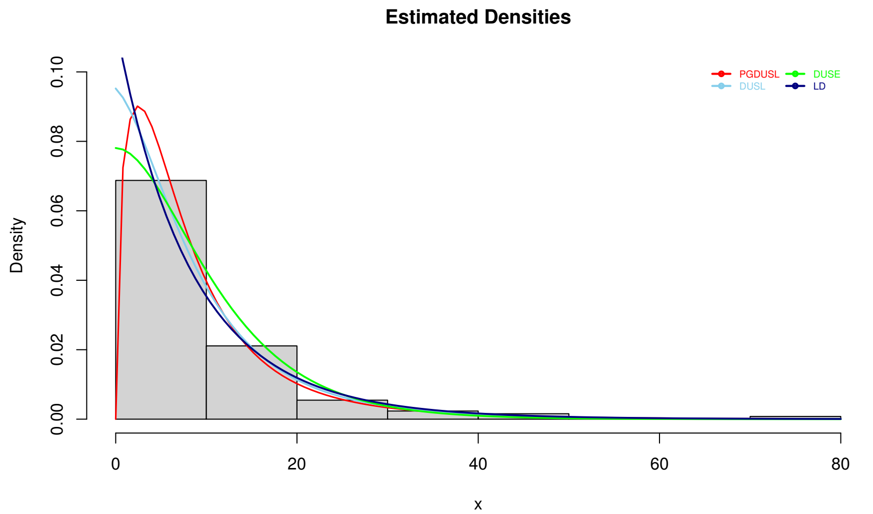

Figure 8: *Estimated densities of the models for the blood cancer patients dataset.*

4. Discussion

This paper proposes the power generalized DUS transformation of Weibull and Lomax distributions. Moments, MGF, CF, CGF, quantile function, distribution of order statistics, and Rényi entropy are derived. The parameter estimation has been done using the maximum likelihood method. By using a simulation study, it is observed that the estimates of the proposed distributions have smaller bias and mean square error when the sample size is larger. Real-world applications have been performed to determine the applicability of the proposed model. Furthermore, the newly developed models are compared with a few existing models, and it is found that the newly developed distributions perform better than the few existing models. When conducting reliability analysis with a series system where each of the components has a specific lifetime distribution, the PGDUS approach is highly useful.

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