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Applications of Burr III-Weibull quantile function in reliability analysis

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ABSTRACT
This paper introduces a new family of distributions defined in terms of quantile function. The quantile function introduced here is the sum of quantile functions of life time distributions called Burr III and Weibull. Different distributional characteristics and reliability properties are included in the study. Method of Least Square and Method of *L*-moments are applied to estimate the parameters of the model. Two real life data sets are used to illustrate the performance of the model.

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1. Introduction

The distribution function or the quantile function can be applied to describe a probability distribution. The concepts and approaches based on distribution functions are traditionally used in the majority areas of statistical theory, despite the fact that both the distribution function and the quantile function provide the same information about the distribution with different interpretations. Quantile-based research has mostly been used when the traditional approach is difficult or unable to generate the required findings. Moreover, quantile functions have a lot of distinct characteristics that distribution functions do not possess, making the former highly interesting in some practical situations. This special feature includes: (a) the sum (product) of two quantile function is again a quantile function; (b) the random numbers can easily be generated from a quantile function; (c) the quantile function of order statistics contains explicit general distribution forms; (d) dealing with inference purposes, statistics based on quantiles are frequently more robust than those based on moments in the distribution function method. In many situations, quantile functions give a clear analysis and in some cases like characterizations, solutions are available only in terms of quantile functions that are not invertible to distribution functions. Hastings Jr et al. (1947) developed a family of distributions via a quantile function, which was an important development in depicting quantile functions to describe statistical data. Further information and application of quantile function can be obtained from Gilchrist (2000), Nair and Sankaran (2009), Nair et al. (2013), Sankaran et al. (2016), Sankaran and Unnikrishnan Nair (2009), and Parzen (1979), etc. Various forms of quantile function were explained by Ramberg and Schmeiser (1972). Govindarajula (1977) made a new quantile function by combining the weighted sum of the quantile functions of two power distributions. Sankaran and Dileep Kumar (2018) developed a new quantile function based on the sum of half logistic and exponential geometric distribution. A quantile-based test for exponentiality against decreasing mean residual quantile function and new better than used in expectation classes of alternatives was conducted by Sreelakshmi et al. (2018).

The importance of Weibull distribution in reliability analysis is due to its ability to describe the various forms of failure time data. Many studies are already conducted to describe the properties, applications and generalizations of Weibull distribution including a review of Weibull distribution given by Hallinan Jr (1993). Various generalizations of Weibull distribution are given by Lai et al. (2003) and Mudholkar and Kollia (1994), etc. Weibull distribution has certain limitations in reliability analysis due to its monotonic behaviour in the failure rate function. This limitation can be overcome by making use of another lifetime distribution called Burr III distribution. The two-parameter Burr type III distribution was initially described by Burr (1942). The Burr type III distribution has a wide range of applications in statistical modelling including survival and reliability study, finance, forestry, environmental studies, etc. In the study of income and wage distribution, it is known as the Dagum distribution (Dagum, 2008), and in the

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meteorological literature, it is known as the kappa-distribution (Mielke, 1973). The hazard rate function of the BIII distribution has a decreasing or unimodal shape.

The primary goal of this study is to offer a new quantile function that can be used in reliability analysis. The proposed quantile function is the sum of quantile functions of Weibull and Burr III distributions. The shapes of the proposed model include decreasing, left skewed, right skewed and symmetrical. The hazard function also contains various shapes such as increasing, decreasing, linear, and upsidedown bathtub shapes for different choices of parameters. The suggested model has a number of advantages over the existing quantile functions, including the ability to give a wide range of distributional forms for various parameter choices. Additionally, the flexible behaviour of the hazard quantile function makes the quantile function suitable for modelling different types of lifetime data. The recommended quantile function is found to be more flexible and tractable than its parent models since it covers almost all shapes for hazard quantile function, that are not encountered in the case of Burr III and Weibull models.

For a non-negative random variable X with the distribution function $F(x)$, the quantile function $Q(u)$ is defined by

$$Q(u) = F^{-1}(u) = \inf\{x : F(x) \geq u\}, \quad 0 \leq u \leq 1.$$

The quantile density function indicated by $q(u)$ is the derivative of quantile function $Q(u)$. If $F(x)$ is right continuous and strictly increasing, then

$$F(Q(u)) = u,$$

so that $F(x) = u$ implies $x = Q(u)$. For the probability density function $f(x)$ of X, we have

$$f(Q(u))q(u) = 1.$$

The suggested quantile function is obtained by taking the sum of quantile functions of Burr III and Weibull distributions. The survival and quantile functions of Burr III (BIII) distribution are respectively given by

$$\bar{F}(x) = 1 - (1 + x^{-c})^{-k}, \quad c > 0,\ k > 0 \tag{1}$$

and

$$Q_1(u) = (u^{-\frac{1}{k}} - 1)^{-\frac{1}{c}}, \quad c > 0,\ k > 0, \tag{2}$$

where c and k are the shape parameters.

The survival and quantile functions of Weibull (W) distribution are respectively given by

$$\bar{G}(x) = \mathrm{e}^{-(\frac{x}{\sigma})^{\lambda}}, \quad \sigma > 0,\ \lambda > 0 \tag{3}$$

and

$$Q_2(u) = \sigma\left[-\log(1-u)\right]^{\frac{1}{\lambda}}, \quad \sigma,\ \lambda > 0. \tag{4}$$

The new class of distributions called Burr III-Weibull (BIIIW) quantile function is obtained by making use of the property that the sum of two positive quantile functions is again a quantile function

$$Q(u) = \sigma\left[-\log(1-u)\right]^{\frac{1}{\lambda}} + \left(u^{-\frac{1}{k}} - 1\right)^{-\frac{1}{c}}, \quad 0 < u < 1, \quad c, k, \sigma, \lambda > 0. \tag{5}$$

The quantile density function is obtained as

$$q(u) = \frac{\sigma\left[-\log(1-u)\right]^{\frac{1}{\lambda}-1}}{\lambda(1-u)} + \frac{\left(u^{-\frac{1}{k}} - 1\right)^{-\frac{1}{c}-1} u^{-\frac{1}{k}-1}}{ck}. \tag{6}$$

The distribution function for the class of distributions (5) cannot be expressed in closed form, hence it has to be numerically evaluated. However we may express (6) using the density function $f(x)$ and distribution function $F(x)$ as

$$f(x) = \frac{ck\lambda(1-F(x))}{ck\sigma[-\log(1-F(x))]^{\frac{1}{\lambda}-1} + \lambda(1-F(x))((F(x))^{-(\frac{1}{k})} - 1)^{-(\frac{1}{c}+1)}(F(x))^{-(\frac{1}{k}+1)}}. \tag{7}$$

The quantile function (5) describes a family of distributions that have varying shapes depending on the parameter values. For various parameter values, the shapes of density functions are shown in Figure 1. It can be observed

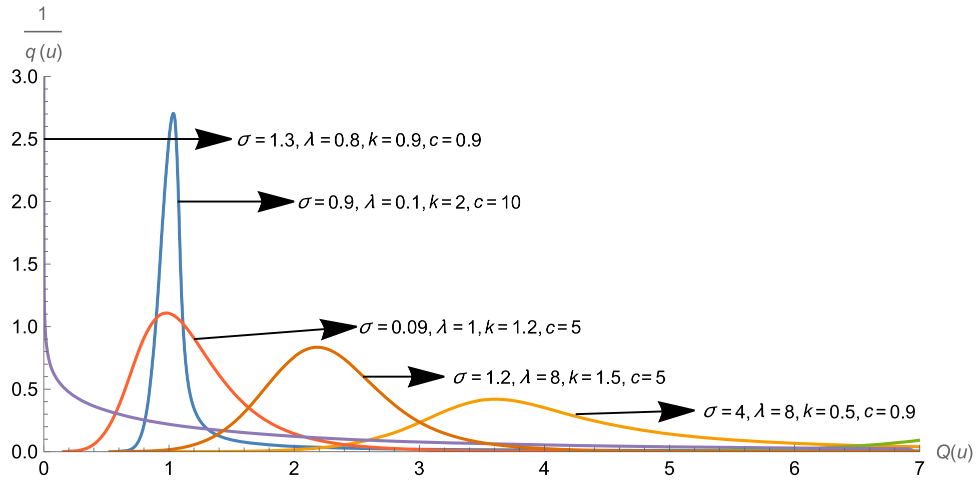


Figure 1. Plots of the density function for various parameter values.

that the family comprises decreasing, unimodal, positive and negatively skewed models for appropriate parameter values.

2. Members of the family

The suggested family of distributions (5) including several well-known distributions for various parameter values is given below.

Case 1: $\sigma = 0, c > 0, k > 0$

$$Q(u) = (u^{-\frac{1}{k}} - 1)^{-\frac{1}{c}}$$

is the quantile function of the Burr III distribution.

Case 2: $c = 1, \sigma = 0$

$$Q(u) = (u^{-\frac{1}{k}} - 1)^{-1}$$

is the quantile function of the Inverse Lomax distribution.

Case 3: $k = 1, \sigma = 0$

$$Q(u) = (u^{-1} - 1)^{-\frac{1}{c}}$$

is the quantile function of the Log-Logistic distribution with scale parameter $\alpha = 1$.

Case 4: $c \to 0$

$$Q(u) = \sigma\left[-\log(1-u)\right]^{\frac{1}{\lambda}}$$

is the quantile function of Weibull distribution.

Also by making use of some transformations mentioned in Gilchrist (2000), we can obtain some other well distributions from the proposed model.

Case 5- Logarithmic Transformation: Using logarithmic transformation in (5), with $\sigma = 0, k = 1$ we have

$$Q(u) = -s\log(u^{-1} - 1),$$

where $s = \frac{1}{c}$, belongs to Logistic distribution with location parameter $\mu = 0$.

Case 6- Reciprocal Transformation : Using reciprocal transformation $Q(u) = \frac{1}{Q(1-u)}$ in (5), with $\sigma = 0, c > 0$ and $k > 0$, the quantile function becomes

$$Q(u) = \left((1-u)^{-\alpha} - 1\right)^{\eta},$$

with $\alpha = \frac{1}{k}$ and $\eta = \frac{1}{c}$. This is the quantile function of Burr XII distribution.

As we already know, the sum of two quantile functions is again a quantile function. But the random variable associated with the quantile function of the sum is not clearly established in the literature. The theorem below can be used to determine the random variable associated with the new model (5).

Theorem 2.1: *If $X \sim W(\lambda,\sigma)$, then the random variable*

$Y = X + ([1 - \mathrm{e}^{-(\frac{X}{\sigma})^{\lambda}}]^{-\frac{1}{k}} - 1)^{-(\frac{1}{c})}$ *has* BIIIW(c, k, σ, λ) *distribution.*

Proof: Let $Q_S(u)$ and $Q_V(u)$ be the appropriate quantile functions for the two random variables S and V with distribution function $F_S(x)$ and $F_V(x)$.

Then, we know that sum of two quantile function is again a quantile function. Hence $Q^*(u) = Q_S(u) + Q_V(u)$.

The random variable corresponding to the quantile function $Q^*(u)$ is $S + Q_V(F_S(S))$ and $V + Q_S(F_V(V))$(given in Sankaran et al., 2016).

Considering $Z\sim$BIII(c, k) and $X \sim$ W(λ, σ), then $X + Q_Z(F_X(X)) \sim$ BIIIW(c, k, σ, λ) distribution.

Given $Q_Z(u) = (u^{-(\frac{1}{k})} - 1)^{-\frac{1}{c}}$ and $F_X(X) = 1 - \mathrm{e}^{-(\frac{X}{\sigma})^{\lambda}}$, then $X + Q_Z(F_X(X)) = X + ([1 - \mathrm{e}^{-(\frac{X}{\sigma})^{\lambda}}]^{-\frac{1}{k}} - 1)^{-(\frac{1}{c})}$. ∎

Theorem 2.2: *If* $Y \sim$ BIII(c, k)*, then the random variable*
$X = Y + \sigma[-\log(1 - (1 + Y^{-c})^{-k})]^{\frac{1}{\lambda}}$ *has* BIIIW(c, k, σ, λ) *distribution.*

Proof: The proof works the same way as Theorem 2.1. ∎

3. Distributional characteristics

The quantile based on measures of the distributional characteristics such as location, dispersion, skewness, and kurtosis has wide applications in statistical analysis. It eliminates the requirement to define a distribution using its moments. These measures are used to estimate model parameters by equating population and sample characteristics. For the model (5), the basic descriptive measures such as the median (Median), interquartile range (IQR), Galton's coefficient of skewness (S), and Moor's coefficient of kurtosis (T) are given below.

The median is a measure of location given by

$$\text{Median} = Q\left(\frac{1}{2}\right) = \sigma\left[\log(2)\right]^{\frac{1}{\lambda}} + \left((0.5)^{-\frac{1}{k}} - 1\right)^{-\frac{1}{c}}. \tag{8}$$

The interquartile range is a measure of dispersion given by

$$\begin{aligned} \text{IQR} &= Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right) \\ &= \sigma\left(\left[\log(4)\right]^{\frac{1}{\lambda}} - \left[-\log(0.75)\right]^{\frac{1}{\lambda}}\right) \\ &\quad + \left((0.75)^{-\frac{1}{k}} - 1\right)^{-\frac{1}{c}} - \left((0.25)^{-\frac{1}{k}} - 1\right)^{-\frac{1}{c}}. \end{aligned} \tag{9}$$

Galton's coefficient of skewness (S) is used to measure skewness

$$\begin{aligned} S &= \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2\text{Median}}{\text{IQR}} \\ &= \frac{\sigma([\log(4)]^{\frac{1}{\lambda}} + [-\log(0.75)]^{\frac{1}{\lambda}} - 2[\log(2)]^{\frac{1}{\lambda}}) + A}{\sigma([\log(4)]^{\frac{1}{\lambda}} - [-\log(0.75)]^{\frac{1}{\lambda}}) + ((0.75)^{-\frac{1}{k}} - 1)^{-\frac{1}{c}} - ((0.25)^{-\frac{1}{k}} - 1)^{-\frac{1}{c}}}, \end{aligned} \tag{10}$$

where $A = ((0.75)^{-\frac{1}{k}} - 1)^{-\frac{1}{c}} + ((0.25)^{-\frac{1}{k}} - 1)^{-\frac{1}{c}} - 2((0.5)^{-\frac{1}{k}} - 1)^{-\frac{1}{c}}$.

The Moor's coefficient of kurtosis (T) is used to measure kurtosis

$$\begin{aligned} T &= \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{\text{IQR}} \\ &= \frac{\sigma([-\log(0.125)]^{\frac{1}{\lambda}} - [-\log(0.375)]^{\frac{1}{\lambda}} + [-\log(0.625)]^{\frac{1}{\lambda}} - [-\log(0.875)]^{\frac{1}{\lambda}}) + B}{\sigma([\log(4)]^{\frac{1}{\lambda}} - [-\log(0.75)]^{\frac{1}{\lambda}}) + ((0.75)^{-\frac{1}{k}} - 1)^{-\frac{1}{c}} - ((0.25)^{-\frac{1}{k}} - 1)^{-\frac{1}{c}}}, \end{aligned} \tag{11}$$

where $B = ((0.875)^{-\frac{1}{k}} - 1)^{-\frac{1}{c}} - ((0.625)^{-\frac{1}{k}} - 1)^{-\frac{1}{c}} + ((0.375)^{-\frac{1}{k}} - 1)^{-\frac{1}{c}} - ((0.125)^{-\frac{1}{k}} - 1)^{-\frac{1}{c}}$.

4. *L*-moments

L-moments are frequently found to be preferable to conventional moments in specifying the characteristics of distributions. Hosking (1990) developed a coherent theory and a comprehensive investigation on *L*-moments, investigating the properties of *L*-moments, their use in summarizing and identifying probability distributions, estimation techniques based on *L*-moments, etc. *L*-moments are the expected value of the linear combinations of order statistics. *L*-moments are more stable against outliers and have lower sample variances. It also provides better asymptotic approximations to sampling distributions.

The r^{th} L moment is given by

$$L_r = \int_0^1 \sum_{k=0}^{r-1} (-1)^{r-1-k} \binom{r-1}{k} \binom{r-1+k}{k} u^k Q(u)\, \mathrm{d}u.$$

The first L-moment (L_1) is the mean of the distribution given by

$$\begin{aligned} L_1 &= \int_0^1 Q(u)\, \mathrm{d}u \\ &= \sigma\Gamma\left(1+\frac{1}{\lambda}\right) + \Gamma\left(1-\frac{1}{c}\right)\left[\frac{\Gamma\left(k+\frac{1}{c}\right)}{\Gamma k}\right]. \end{aligned} \tag{12}$$

The second L-moment is

$$\begin{aligned} L_2 &= \int_0^1 (2u-1)\, Q(u)\, \mathrm{d}u \\ &= \sigma\Gamma\left(1+\frac{1}{\lambda}\right)\left[1-2^{-\frac{1}{\lambda}}\right] + \Gamma\left(1-\frac{1}{c}\right)\left[\frac{\Gamma\left(2k+\frac{1}{c}\right)}{\Gamma(2k)} - \frac{\Gamma\left(k+\frac{1}{c}\right)}{\Gamma k}\right]. \end{aligned} \tag{13}$$

$$\begin{aligned} L_3 &= \int_0^1 \left(6u^2-6u+1\right) Q(u)\, \mathrm{d}u \\ &= \sigma\Gamma\left(1+\frac{1}{\lambda}\right)\left[1-3\times 2^{-\frac{1}{\lambda}}+2\times 3^{-\frac{1}{\lambda}}\right] \\ &\quad + \Gamma\left(1-\frac{1}{c}\right)\left[\frac{\Gamma(k+\frac{1}{c})}{\Gamma k} - \frac{3\Gamma\left(2k+\frac{1}{c}\right)}{\Gamma(2k)} + \frac{2\Gamma\left(3k+\frac{1}{c}\right)}{\Gamma(3k)}\right]. \end{aligned} \tag{14}$$

$$\begin{aligned} L_4 &= \int_0^1 \left(20u^3-30u^2+12u-1\right) Q(u)\, \mathrm{d}u \\ &= \sigma\Gamma\left(1+\frac{1}{\lambda}\right)\left[1-3\times 2^{1-\frac{1}{\lambda}}+10\times 3^{-\frac{1}{\lambda}}-5\times 4^{-\frac{1}{\lambda}}\right] \\ &\quad + \Gamma\left(1-\frac{1}{c}\right) C, \end{aligned} \tag{15}$$

where

$$C = \left[\frac{5\Gamma\left(4k+\frac{1}{c}\right)}{\Gamma(4k)} - \frac{10\Gamma\left(3k+\frac{1}{c}\right)}{\Gamma(3k)} + \frac{6\Gamma\left(2k+\frac{1}{c}\right)}{\Gamma(2k)} - \frac{\Gamma\left(k+\frac{1}{c}\right)}{\Gamma(k)}\right]. \tag{16}$$

The L-coefficient of variation (τ_2), L-coefficient of skewness (τ_3) and L-coefficient of kurtosis (τ_4) for the model (5), are given below.

$$\begin{aligned} \tau_2 &= \frac{L_2}{L_1} \\ &= \frac{\sigma\Gamma\left(1+\frac{1}{\lambda}\right)\left[1-2^{-\frac{1}{\lambda}}\right] + \Gamma\left(1-\frac{1}{c}\right)\left[\frac{\Gamma(2k+\frac{1}{c})}{\Gamma(2k)} - \frac{\Gamma(k+\frac{1}{c})}{\Gamma k}\right]}{\sigma\Gamma\left(1+\frac{1}{\lambda}\right) + \Gamma\left(1-\frac{1}{c}\right)\left[\frac{\Gamma(k+\frac{1}{c})}{\Gamma k}\right]}. \end{aligned} \tag{17}$$

$$\tau_3 = \frac{L_3}{L_2}$$

$$= \frac{\sigma\Gamma\left(1+\frac{1}{\lambda}\right)\left[1-3\times 2^{-\frac{1}{\lambda}}+2\times 3^{-\frac{1}{\lambda}}\right]+\Gamma\left(1-\frac{1}{c}\right)D}{\sigma\Gamma\left(1+\frac{1}{\lambda}\right)\left[1-2^{-\frac{1}{\lambda}}\right]+\Gamma\left(1-\frac{1}{c}\right)\left[\frac{\Gamma(2k+\frac{1}{c})}{\Gamma(2k)}-\frac{\Gamma(k+\frac{1}{c})}{\Gamma k}\right]}, \tag{18}$$

where $D = \left[\frac{\Gamma(k+\frac{1}{c})}{\Gamma k} - \frac{3\Gamma(2k+\frac{1}{c})}{\Gamma(2k)} + \frac{2\Gamma(3k+\frac{1}{c})}{\Gamma(3k)}\right]$.

$$\tau_4 = \frac{L_4}{L_2}$$
$$= \frac{\sigma\Gamma\left(1+\frac{1}{\lambda}\right)\left[1-3\times 2^{1-\frac{1}{\lambda}}+10\times 3^{-\frac{1}{\lambda}}-5\times 4^{-\frac{1}{\lambda}}\right]+\Gamma\left(1-\frac{1}{c}\right)C}{\sigma\Gamma\left(1+\frac{1}{\lambda}\right)\left[1-2^{-\frac{1}{\lambda}}\right]+\Gamma\left(1-\frac{1}{c}\right)\left[\frac{\Gamma(2k+\frac{1}{c})}{\Gamma(2k)}-\frac{\Gamma(k+\frac{1}{c})}{\Gamma k}\right]}, \tag{19}$$

where C is given in (16).

5. Order statistics

Order statistics have a wider application in many areas. One of them is system reliability. Let $X_{i:n}$ be the i^{th} order statistic of a random sample of size n. Then the density function of $X_{i:n}$ is given as follows:

$$f_i(x) = \frac{1}{B(i, n-i+1)} f(x)F^{i-1}(x)(1-F(x))^{n-i},$$

where $B(a, b)$ is the beta function. Substituting (7), we have

$$f_i(x) = \frac{ck\lambda(F(x))^{i-1}(1-F(x))^{n-i+1}}{B(i, n-i+1)\left(ck\sigma[-\log(1-F(x))]^{\frac{1}{\lambda}-1}+\lambda(1-F(x))((F(x))^{-(\frac{1}{k})}-1)^{-(\frac{1}{c}+1)}(F(x))^{-(\frac{1}{k}+1)}\right)}.$$

Hence

$$E(X_{i:n}) = \frac{1}{B(i, n-i+1)}\int_0^\infty x f_i(x)\,\mathrm{d}x.$$

In quantile terms, we have

$$E(X_{i:n}) = \frac{1}{B(i, n-i+1)}\int_0^1 \frac{Q(u)u^{i-1}(1-u)^{n-i}}{q(u)}\,\mathrm{d}u.$$

The quantile function of first order statistic $X_{1:n}$ for the model (5) is given by

$$Q_{1^*}(u) = Q(1-(1-u)^{\frac{1}{n}})$$
$$= \sigma\left[-\log(1-u)^{\frac{1}{n}}\right]^{\frac{1}{\lambda}} + \left[(1-(1-u)^{\frac{1}{n}})^{-\frac{1}{k}}-1\right]^{-\frac{1}{c}}. \tag{20}$$

And the quantile function of n^{th} order statistic $X_{n:n}$ is

$$Q_{n^*}(u) = Q(u^{\frac{1}{n}})$$
$$= \sigma\left[-\log\left(1-u^{\frac{1}{n}}\right)\right]^{\frac{1}{\lambda}} + \left(u^{-\frac{1}{kn}}-1\right)^{-\frac{1}{c}}. \tag{21}$$

6. Reliability properties

There are several functions available for modelling and analysis of lifetime data such as the hazard rate function, the mean residual life function, etc. Nair and Sankaran (2009) defined the hazard quantile function in a quantile structure, which is identical to the hazard rate. Hazard rate can be defined as the conditional probability of a unit failing in the next small interval of time given that the unit has survived age x. The hazard quantile function $H(u)$ is defined as

$$H(u) = h(Q(u)) = \left[(1-u)q(u)\right]^{-1}. \tag{22}$$

Note that $H(u)$ uniquely determines the distribution using the identity

$$Q(u) = \int_0^u \frac{\mathrm{d}p}{(1-p)H(p)}. \tag{23}$$

The hazard quantile function of Weibull and Burr III distribution respectively are

$$H_1(u) = \lambda\sigma^{-1}\left[-\log(1-u)\right]^{1-\frac{1}{\lambda}} \tag{24}$$

$$\text{and} \quad H_2(u) = ck\,(1-u)^{-1}\left(u^{-\frac{1}{k}}-1\right)^{1+\frac{1}{c}} u^{1+\frac{1}{k}}. \tag{25}$$

Since the suggested class of distributions is the sum of quantile functions of Weibull and Burr III quantile functions, Equations (22) and (23) give

$$\frac{1}{H(u)} = \frac{1}{H_1(u)} + \frac{1}{H_2(u)}.$$

For the class of distributions (5) the hazard quantile function has the form

$$H(u) = \frac{ck\lambda}{\lambda(1-u)(u^{-\frac{1}{k}}-1)^{-\frac{1}{c}-1}u^{-\frac{1}{k}-1} + ck\sigma\left[-\log(1-u)\right]^{\frac{1}{\lambda}-1}}, \tag{26}$$

which allows increasing, decreasing, linear, bathtub and upside-down bathtub shapes for different choices of parameters. Figure 2 shows plots of the hazard quantile function for various parameter values.

Using the derivative of $H(u)$ we can obtain the shape of the hazard function as

$$H'(u) = \frac{h'(u)}{\left[\lambda(1-u)(u^{-\frac{1}{k}}-1)^{-\frac{1}{c}-1}u^{-\frac{1}{k}-1} + ck\sigma\left[-\log(1-u)\right]^{\frac{1}{\lambda}-1}\right]^2}.$$

Since $[\lambda(1-u)(u^{-\frac{1}{k}}-1)^{-\frac{1}{c}-1}u^{-\frac{1}{k}-1} + ck\sigma[-\log(1-u)]^{\frac{1}{\lambda}-1}]^2 > 0$ for all values of the parameters, the sign of $H'(u)$ depends on $h'(u)$ given by

$$h'(u) = \frac{(c+1)(u-1)\left(u^{-\frac{1}{k}}-1\right)^{-\frac{1}{c}}\lambda^2}{u^2\left(u^{\frac{1}{k}}-1\right)^2} + \frac{c(k+1)(u-1)\left(u^{-\frac{1}{k}}-1\right)^{-\frac{1}{c}}\lambda^2}{u^2\left(u^{\frac{1}{k}}-1\right)}$$
$$- \frac{ck\left(u^{-\frac{1}{k}}-1\right)^{-\frac{1}{c}}\lambda^2}{u\left(u^{\frac{1}{k}}-1\right)} - \frac{c^2k^2\left(\frac{1}{\lambda}-1\right)\lambda\sigma\left[-\log(1-u)\right]^{-2+\frac{1}{\lambda}}}{1-u}.$$

Let u_0 be the critical point of $H(u)$ satisfying the non-linear equation $h'(u_0) = 0$. Since u_0 has no closed-form, we need to use any mathematical software for its numerical evaluation. Also, we know that the sign of $h''(u_0)$ represents the nature of u_0: u_0 is local minimum if $h''(u_0) > 0$ and local maximum if $h''(u_0) < 0$.

In reliability analysis, the mean residual function is a well-known statistic that has been widely used for modelling lifetime data. The average remaining life of a system given that the system has lasted upto a specific age is called the mean residual life. The quantile version of the mean residual function proposed by Nair and Sankaran (2009) is given by

$$M(u) = \frac{1}{1-u}\int_u^1 \left(Q(p) - Q(u)\right)\mathrm{d}p.$$

For the model (5), $M(u)$ has the form

$$M(u) = \frac{\sigma\Gamma\left(\frac{1}{\lambda}+1, -\log(1-u)\right)}{1-u} - \sigma\left(-\log(1-u)\right)^{\frac{1}{\lambda}}$$

$$+\frac{k}{1-u}\left[B\left(k+\frac{1}{c},1-\frac{1}{c}\right)-B_{u^{\frac{1}{k}}}\left(k+\frac{1}{c},1-\frac{1}{c}\right)\right]$$
$$-\left(u^{-\frac{1}{k}}-1\right)^{-\frac{1}{c}},\quad \mathrm{Re}\left(\frac{1}{c}\right)<1, \tag{27}$$

where $B_u(a,b)=\int_0^u x^{a-1}(1-x)^{b-1}\,\mathrm{d}x$ is the incomplete beta and $\Gamma(s,t)=\int_t^1 x^{s-1}\mathrm{e}^{-x}\,\mathrm{d}x$ is the upper incomplete gamma function.

The hazard quantile function and mean residual quantile function defined in reverse time have the following expression.

$$\Lambda=(uq(u))^{-1}$$
$$=\frac{ck\lambda(1-u)}{u\left[ck\sigma\left(-\log(1-u)\right)^{\frac{1}{\lambda}-1}+\lambda(1-u)\left(u^{-\frac{1}{k}}-1\right)^{-\frac{1}{c}-1}u^{-\frac{1}{k}-1}\right]}. \tag{28}$$

$$R(u)=\frac{1}{u}\int_0^u[Q(u)-Q(p)]\,\mathrm{d}p$$
$$=\sigma\left(-\log(1-u)\right)^{\frac{1}{\lambda}}-\frac{\sigma\gamma(\frac{1}{\lambda}+1,-\log(1-u))}{u}$$
$$+\left(u^{-\frac{1}{k}}-1\right)^{-\frac{1}{c}}-\frac{k}{u}\left[B_{u^{\frac{1}{k}}}\left(k+\frac{1}{c},1-\frac{1}{c}\right)\right],\quad \mathrm{Re}\left(\frac{1}{c}\right)<1, \tag{29}$$

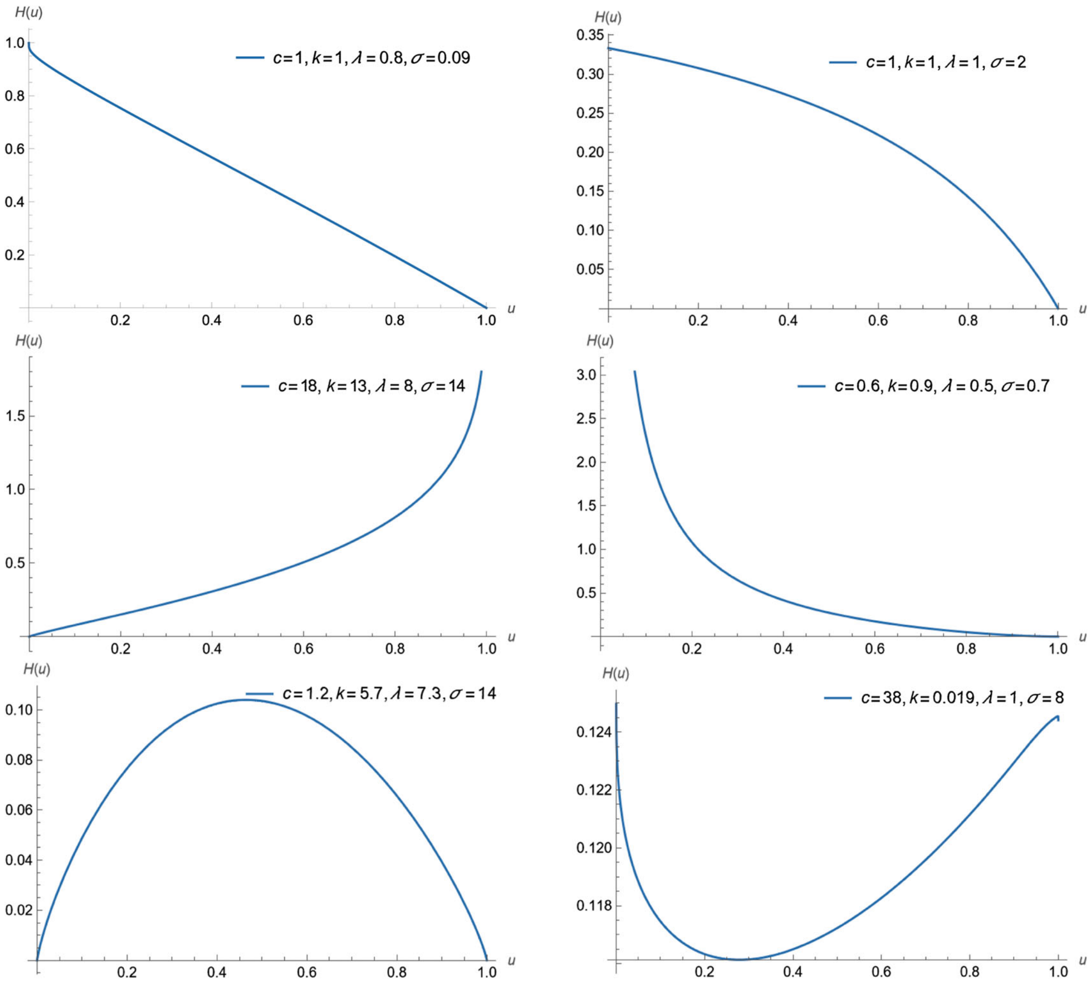


Figure 2. Plots of the hazard quantile function for various parameter values.

where $B_u(a,b) = \int_0^u x^{a-1}(1-x)^{b-1}\,\mathrm{d}x$ is the incomplete beta and $\gamma(s,t) = \int_0^t x^{s-1}\mathrm{e}^{-x}\,\mathrm{d}x$ is the lower incomplete gamma function. $R(u)$ represents the time elapsed since the failure of a unit given that its lifetime is at most x.

The total time on the test transform (TTT) is a well-known statistical method with numerous applications in reliability analysis (Lai & Xie, 2006). The quantile-based TTT proposed by Nair et al. (2008) takes the following form

$$T(u) = \int_0^u (1-p)q(p)\,\mathrm{d}p.$$

Another relationship between total time on test transform and reversed mean residual quantile function (see Nair et al., 2008) has the expression

$$\begin{aligned} T(u) &= Q(u) - uR(u) \\ &= \sigma\left(-\log(1-u)\right)^{\frac{1}{\lambda}}[1-u] + (u^{-\frac{1}{k}}-1)^{-\frac{1}{c}}[1-u] \\ &\quad + \sigma\gamma\left(\frac{1}{\lambda}+1, -\log(1-u)\right) + kB_{u^{\frac{1}{k}}}\left(k+\frac{1}{c}, 1-\frac{1}{c}\right). \end{aligned} \tag{30}$$

The total time in the test statistic is the sum of all observed and incomplete life durations. As the number of units on test approaches infinity, the limit of this statistic is known as the total time on test transform (TTT).

7. Inference and applications

The most commonly used methods for estimating the parameters of quantile function are method of percentiles, method of L-moments, method of least squares, method of maximum likelihood, etc. Here method of least square and method of L-moments are employed to estimate the parameters of the model (5).

7.1. Method of least square estimation (LSE)

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a population of lifetime with quadratic hazard quantile function. Consider $X_{(i)}$ be the i^{th} order statistics of random sample of size n. The random variable $X_{(i)}$ has the same distribution as the random variable $Q(u_i, \hat{\theta})$, where $u_{(i)}$ is the order statistics of the sample following uniform distribution and $\hat{\theta}$ is the estimate of the model's parameter vector. The method of least squares, estimates the unknown parameters $(\theta_1, \theta_2, \ldots, \theta_n)$ by minimising the sum of squares of theoretical and empirical quantile differences (Hankin & Lee, 2006). The function for which we compute the minimum then takes the following form

$$S(\theta_1, \theta_2, \ldots, \theta_n) = \sum_{i=1}^{n}\left(X_{(i)} - Q(u_{(i)}, \theta)\right)^2.$$

7.2. Method of L moments (MLM)

In this method, the sample L-moments are equated to the population L-moments. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the population having quantile function (5). Then the corresponding sample L-moments are

$$\begin{aligned} l_1 &= \frac{1}{n}\sum_{i=1}^{n} X_{(i)}, \\ l_2 &= \left(\frac{1}{2}\right)\binom{n}{2}^{-1}\sum_{i=1}^{n}\left(\binom{i-1}{1} - \binom{n-i}{1}\right)X_{(i)}, \\ l_3 &= \left(\frac{1}{3}\right)\binom{n}{3}^{-1}\sum_{i=1}^{n}\left(\binom{i-1}{2} - 2\binom{i-1}{1}\binom{n-i}{1} + \binom{n-i}{2}\right)X_{(i)}, \\ l_4 &= \left(\frac{1}{4}\right)\binom{n}{4}^{-1}\sum_{i=1}^{n}\left(\binom{i-1}{3} - 3\binom{i-1}{2}\binom{n-i}{1} + 3\binom{i-1}{1}\binom{n-i}{2} - \binom{n-i}{3}\right)X_{(i)}, \end{aligned}$$

Table 1. The sample L-moment values for Dataset-1 and Dataset-2.

	l_1	l_2	l_3	l_4
Dataset-1	4.3253	0.5688	−0.0500	0.0919
Dataset-2	92.0745	51.3599	22.0285	11.062

Table 2. The basic descriptive statistics of the two datasets.

	Median	IQR	Skewness (S)	Kurtosis (T)
Dataset-1	4.3566	1.3232	−0.0358	1.246
Dataset-2	55.1305	107.198	0.3477	1.4241

Table 3. Goodness-of-fit statistics corresponding to the two datasets.

	Dataset-1			Dataset-2		
	Parameters	χ^2(dof)	p-value	Parameters	χ^2(dof)	p-value
LSE	$c = 7.084$	19.2516(13)	0.115	$c = 2.2448$	16.6332(16)	0.409
	$k = 0.5882$			$k = 16.8983$		
	$\lambda = 6.2665$			$\lambda = 0.8285$		
	$\sigma = 3.8404$			$\sigma = 82.7944$		
MLM	$c = 7.4363$	9.3154(13)	0.748	$c = 1.9101$	11.8295(16)	0.7556
	$k = 0.4457$			$k = 16.9201$		
	$\lambda = 5.6163$			$\lambda = 0.7971$		
	$\sigma = 3.7564$			$\sigma = 78.9684$		

Table 4. The chi-square with p-values for the datasets.

	Dataset-1		Dataset-2	
Models	χ^2	p-value	χ^2	p-value
BIIIW	**9.31543**	**0.7487**	**11.8295**	**0.7556**
W	12.4163	0.4938	14.4481	0.5653
BIII	18.8007	0.1294	21.5262	0.1591

where $X_{(i)}$ denotes the i^{th} order statistic. We equate the first four sample L moments with the corresponding population L moments for obtaining the estimates of the parameters c, k, λ, σ. Hence, we have

$$l_r = L_r, \quad r = 1, 2, 3, 4.$$

7.3. Applications

Here we consider two real datasets to compare the proposed estimation techniques and to demonstrate the applicability of the BIIIW quantile function. Chi Square Goodness of fit and Q-Q plot techniques were used to determine the effectiveness of the proposed model. The Q-Q plot indicates the physical closeness of the model. The **first data** taken from Nadarajah and Kotz (2008) represents fracture toughness of Alumina, (Al_2O_3) (in the units of MPa $m^{\frac{1}{2}}$) presented in Table A1 (see Appendix). The **second data** deals with the number of consecutive failures of jet airplanes air conditioning system, (for details see Huang and Oluyede (2014)) given in Table A2.

For the computation of the MLM method, the sample L-moments are calculated for both of the datasets which are described in Table 1. The Newton-Raphson technique is used to solve the four nonlinear equations, which are formed by equating the values of sample L-moments with the corresponding population L-moments given in (12)–(15). The values of the parameters which are obtained using the LSE technique are used as the initial values for the Newton Raphson procedure. The descriptive statistics for the datasets are shown in Table 2. The estimated values, Chi square with degree of freedom (dof) and p-values obtained using the two estimation techniques are shown in Table 3. The results were obtained using the MATHEMATICA software.

From the Chi square values shown in the Table 3, it is evident that the MLM estimation technique is found to be the best fit in both cases. The Figures 3 and 4 represent the Q-Q plots of the datasets corresponding to LSE and MLM techniques. That is, Q-Q plots also guarantees the result obtained using the Chi square values. As a result, of the two described estimation methods, the MLM is found to be the most suitable estimation method for each of the datasets.

The proposed model is then compared against the Weibull (W) and BurrIII (BIII) models. MLM technique is used to find the estimates of comparable models. Figure 7 displays the estimated density and histogram of the

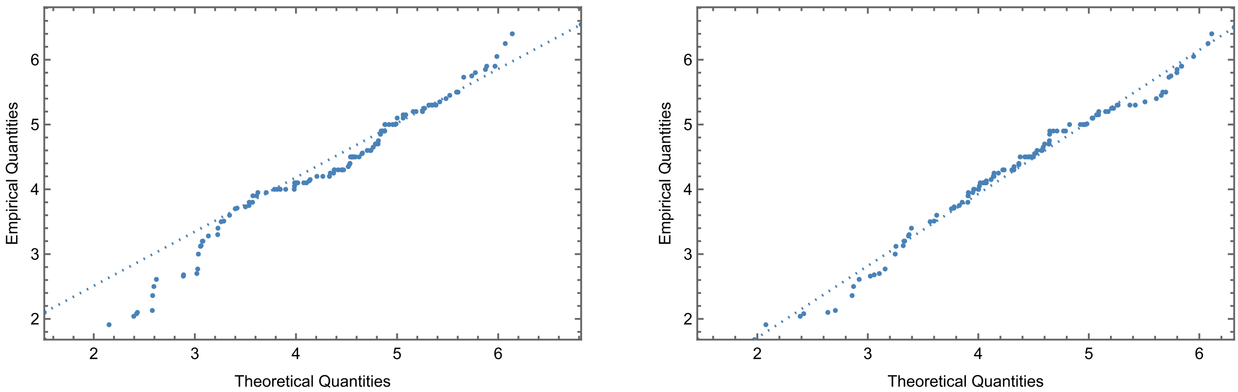


Figure 3. *Q-Q* plot corresponing to LSE and MLM estimates for the first dataset.

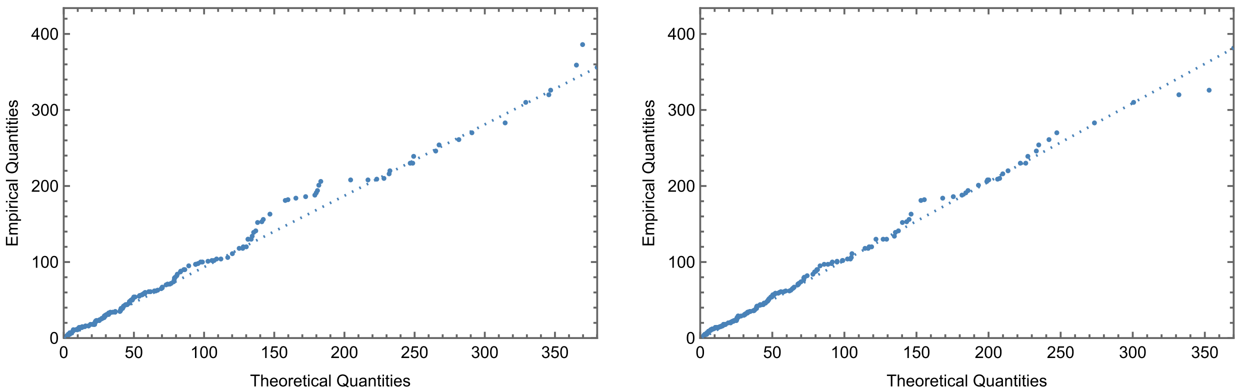


Figure 4. *Q-Q* plot corresponing to LSE and MLM estimates for the second dataset.

datasets for the BIIIW, W and BIII models. This demonstrates that in both cases, our proposed distribution provides a better fit than the other two models. The Chi-square value and p-value obtained for the models BIIIW, W and BIII are given in Table 4. Based on the Chi-square values also, we can conclude that our model gives a better fit. The

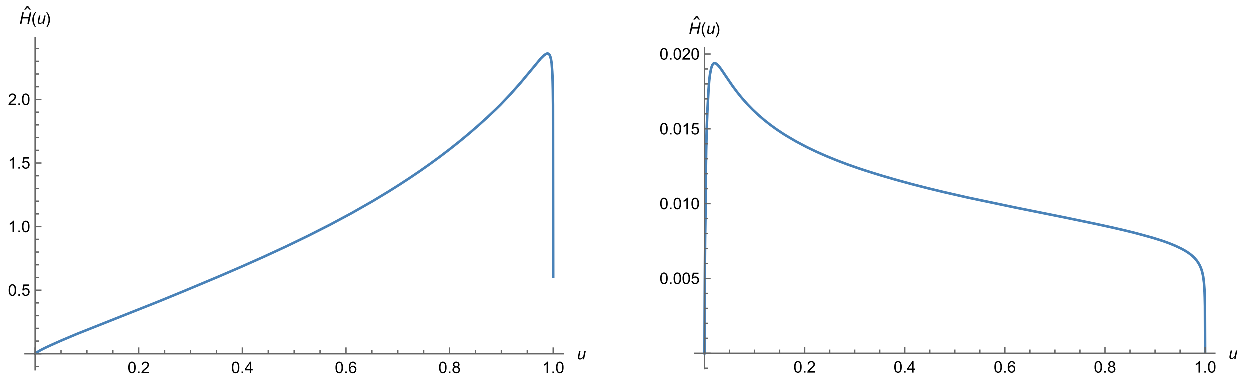


Figure 5. (a) Hazard quantile plot of the first dataset. (b) Hazard quantile plot of the second dataset.

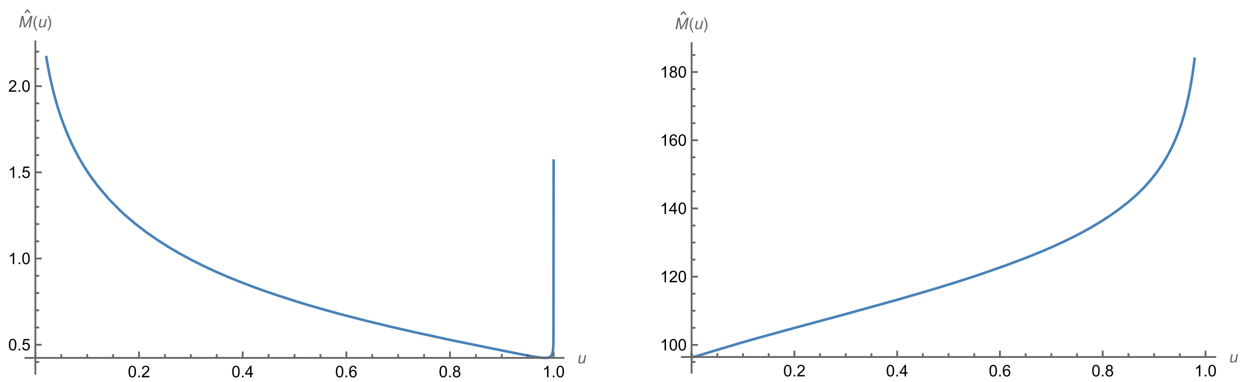
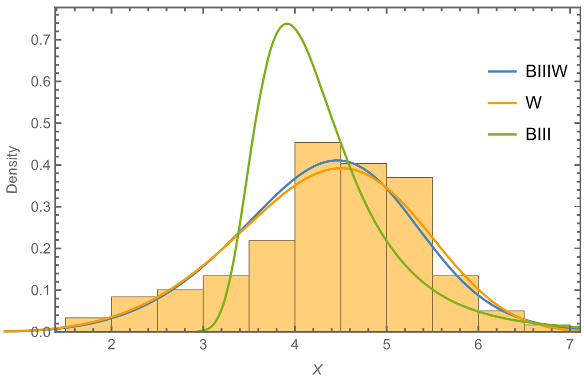


Figure 6. (a) Mean residual plot of the first dataset. (b) Mean residual plot of the second dataset.



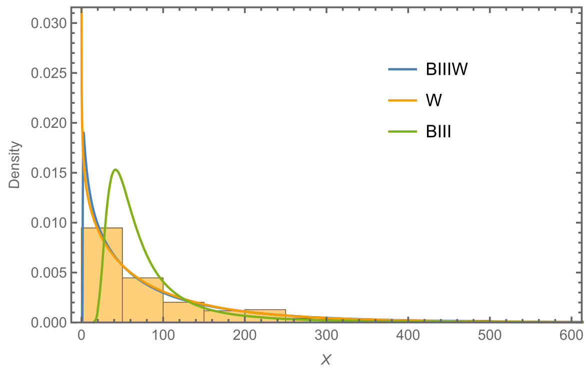


Figure 7. (a) The fitted density of the first dataset. (b) The fitted density of the second dataset.

Figures 5 and 6 describe the flexible nature of the hazard quantile function and mean residual function for different estimates. It is obvious from the shapes of $\hat{H}(u)$, that the new model may be applied to examine a variety of lifetime data.

8. Conclusion

In this paper, we introduced a new quantile function called the BIIIW quantile function, which is the sum of the quantile functions of Burr III and Weibull distributions. The new model has several sub-models such as Inverse-Lomax, Logistic, Weibull, Log-Logistic, etc. Different reliability properties of the suggested class are discussed. The new model achieves several interesting behaviours for the hazard quantile function. The estimation of parameters of the model is conducted using the method of Least square and the method of L-moments. The applications of the proposed model were studied with the use of two real life datasets. From the two applications, it is clear that the proposed model provides a better fit than the other competing models. The flexible nature of the hazard quantile function makes the model effective for fitting many types of lifetime data.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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Appendix

Table A1. Fracture toughness data of Alumina (Al_2O_3).

5.5	5	4.9	6.4	5.1	5.2	5.2	5	4.7	4
4.5	4.2	4.1	4.56	5.01	4.7	3.13	3.12	2.68	2.77
2.7	2.36	4.38	5.73	4.35	6.81	1.91	2.66	2.61	1.68
2.04	2.08	2.13	3.8	3.73	3.71	3.28	3.9	4	3.8
4.1	3.9	4.05	4	3.95	4	4.5	4.5	4.2	4.55
4.65	4.1	4.25	4.3	4.5	4.7	5.15	4.3	4.5	4.9
5	5.35	5.15	5.25	5.8	5.85	5.9	5.75	6.25	6.05
5.9	3.6	4.1	4.5	5.3	4.85	5.3	5.45	5.1	5.3
5.2	5.3	5.25	4.75	4.5	4.2	4	4.15	4.25	4.3
3.75	3.95	3.51	4.13	5.4	5	2.1	4.6	3.2	2.5
4.1	3.5	3.2	3.3	4.6	4.3	4.3	4.5	5.5	4.6
4.9	4.3	3	3.4	3.7	4.4	4.9	4.9	5	

Table A2. Number of failures for the air conditioning system of jet airplanes.

194	413	90	74	55	23	97	50	359	50
130	487	57	102	15	14	10	57	320	261
51	44	9	254	493	33	18	209	41	58
60	48	56	87	11	102	12	5	14	14
29	37	186	29	104	7	4	72	270	283
7	61	100	61	502	220	120	141	22	603
35	98	54	100	11	181	65	49	12	239
14	18	39	3	12	5	32	9	438	43
134	184	20	386	182	71	80	188	230	152
5	36	79	59	33	246	1	79	3	27
201	84	27	156	21	16	88	130	14	118
44	15	42	106	46	230	26	59	153	104
20	206	5	66	34	29	26	35	5	82
31	118	326	12	54	36	34	18	25	120
31	22	18	216	139	67	310	3	46	210
57	76	14	111	97	62	39	30	7	44
11	63	23	22	23	14	18	13	34	16
18	130	90	163	208	1	24	70	16	101
52	208								