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POWER WEIBULL QUANTILE FUNCTION AND IT'S RELIABILITY ANALYSIS

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Abstract

In this article, we propose a new class of distributions defined by a quantile function, which is the sum of the quantile functions of the Power and Weibull distributions. Various distributional properties and reliability characteristics of the class are discussed. To examine the usefulness of the model, the model is applied to a real life datasets. Parameters are estimated using maximum likelihood estimation technique.

Keywords: Power distribution; Weibull distribution; L-moments; Hazard quantile function; Mean residual quantile function; Residual variance quantile function; Reversed hazard quantile function.

1. Introduction

In modelling and analysis of statistical data, probability distribution can be specified either in terms of distribution function or by the quantile function. Quantile functions have several interesting properties that are not shared by distributions, which makes it more convenient for analysis. For example, the sum of two quantile functions is again a quantile function. For a nonnegative random variable X with distribution function F(x), the quantile function $Q(u)$ is defined by Nair and Sankaran [7]

$$Q(u) = F^{-1}(u) = inf\{x : F(x) \geq u\}, \quad 0 \leq u \leq 1 \tag{1}$$

For every -$\infty < x < \infty$ and $0 < u < 1$, we have

$$F(x) \geq u \text{ if and only if } Q(u) \leq x.$$

Thus, if there exists an x such that $F(x) = u$, then $F(Q(u)) = u$ and $Q(u)$ is the smallest value of x satisfying $F(x) = u$. Further, if $F(x)$ is continuous and strictly increasing, $Q(u)$ is the unique value x such that $F(x) = u$, and so by solving the equation $F(x) = u$, we can find x in terms of u which is the quantile function of X.

If $f(x)$ is the probability function of X, then $f(Q(u))$ is called the density quantile function. The derivative of $Q(u)$,

$$q(u) = Q^{'}(u),$$

is known as the quantile density function of X. If $F(x)$ is right continuous and strictly increasing, we have

$$F(Q(u)) = u \tag{2}$$

so that F(x) = u implies x = Q(u). When f(x) is the probability density function (PDF) of X; we have from (2)

$$q(u)f(Q(u)) = 1 \tag{3}$$

Quantile function has several properties that are not shared by distribution function. See Nair and Sankaran [7] for details. For example, the sum of two quantile functions is again a quantile function. Further, the product of two positive quantile functions is again a quantile function in the nonnegative setup. There are explicit general distribution forms for the quantile function of order statistics. It is easier to generate random numbers from the quantile function. A major development in portraying quantile functions to model statistical data is given by Hastings *et al.* [5], who introduced a family of distributions by a quantile function. This was refined later by Tukey [13] to form a symmetric distribution, called the Tukey lambda distribution.

This model was generalized in different ways, referred as lambda distributions. These include various forms of quantile functions discussed in Ramberg and Schmeiser [10], Ramberg [8], Ramberg *et al.* [9] and Freimer *et al.* [1]. Govindarajulu [3] introduced a new quantile function by taking the weighted sum of quantile functions of two power distributions. Hankin and Lee [4] presented a new power - Pareto distribution by taking the product of power and Pareto quantile functions. Van Staden and Loots [14] developed a four-parameter distribution, using a weighted sum of the generalized Pareto and its reflection quantile functions. Sankaran *et al.* [12] developed a new quantile function based on the sum of quantile functions of generalized Pareto and Weibull quantile functions. Sankaran and Dileep [11] developed a new quantile function based on the sum of quantile functions of half logistic and exponential geometric distributions.

The aim of the present work is to introduce a new quantile function that is useful in reliability analysis. The proposed quantile function is derived by taking the sum of quantile functions of power and Weibull distributions. The survival function and quantile function of power distribution are respectively given by

$$\bar{F}(x) = 1 - \left(\frac{x}{\alpha}\right)^{\beta} \qquad 0 \leq x \leq \alpha; \alpha, \beta > 0$$

and

$$Q_1(u) = \alpha u^{\frac{1}{\beta}} \qquad 0 \leq u \leq 1, \qquad \alpha, \beta > 0. \tag{4}$$

The survival function and quantile function of Weibull distribution are respectively given by

$$\bar{F}(x) = exp\left[-\left(\frac{x}{\sigma}\right)^{\lambda}\right] \qquad x > 0; \lambda, \sigma > 0$$

and

$$Q_2(u) = \sigma(-log(1-u))^{\frac{1}{\lambda}} \qquad 0 \leq u \leq 1, \quad \alpha, \lambda > 0 \tag{5}$$

We now propose a new class of distributions defined by a quantile function, which is the sum of quantile functions of power and Weibull distributions.

2. Power–Weibull (PW) quantile function

Let X and Y be two nonnegative random variables with distribution functions F(x) and G(x) with quantile functions $Q_1(u)$ and $Q_2(u)$, respectively. Then

$$Q(u) = Q_1(u) + Q_2(u), \tag{6}$$

is also a quantile function. We now introduce a class of distributions given by the quantile function,

$$Q(u) = \alpha u^{\frac{1}{\beta}} + \sigma(-log(1-u))^{\frac{1}{\lambda}} \qquad 0 \leq u \leq 1, \quad \alpha, \beta, \sigma, \lambda > 0 \tag{7}$$

Thus Q(u) is the sum of (4) and (5). It is named as Power–Weibull (PW) quantile function. The quantile density function is obtained as

$$q(u) = \frac{\alpha u^{\frac{1}{\beta}-1}}{\beta} + \frac{\sigma(-log(1-u))^{\frac{1}{\lambda}-1}}{\lambda(1-u)}. \tag{8}$$

For the proposed class of distribution, the density function $f(x)$ can be written in terms of the distribution function as

$$f(x) = \frac{\beta\lambda(1-F(x))}{\alpha\lambda(1-F(x))F(x)^{\frac{1}{\beta}-1} + \sigma\beta(-log^{\frac{1}{\lambda}-1}(1-F(x)))}. \tag{9}$$

For all values of the parameters, the density is strictly decreasing in x and it tends to zero as x $\to \infty$.

The quantile function (7) represents a family of distributions with a variety of shapes for its probability density function. Plots of the density function for different combinations of parameters are shown in figure 1, 2 and 3.

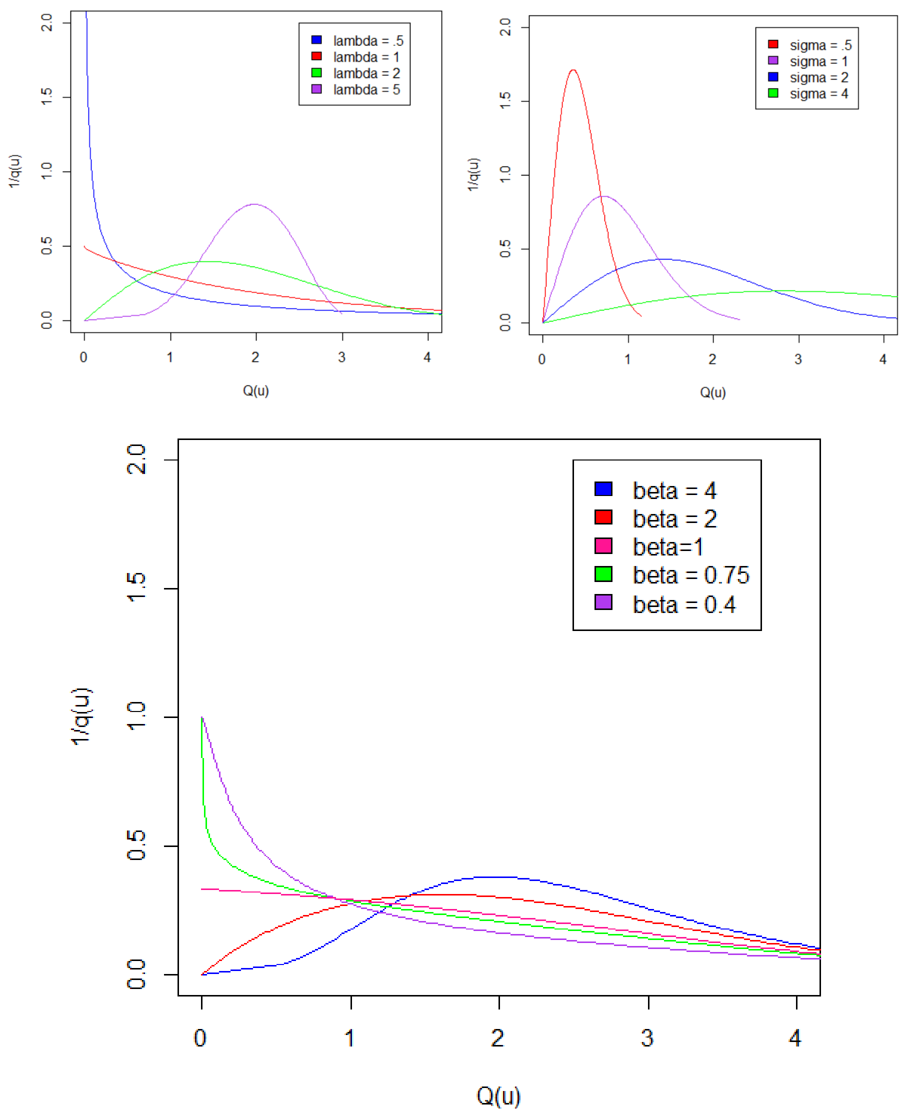

Figure 1: *Plot of the density function for various values of λ , σ and β*

3. Members of the family

The proposed family of distributions (7) includes several well-known distributions for various values of the parameters.

Case 1. $\alpha = 0, \sigma > 0$.

$$Q(u) = \sigma(-log(1-u))^{\frac{1}{\lambda}} \tag{10}$$

is the quantile function of the Weibull distribution, which contains the exponential distribution with mean σ for $\lambda = 1$ and the Rayleigh distribution when $\lambda = 2$.

Case 2. $\sigma = 0, \alpha > 0, \beta > 0$.

$$Q(u) = \alpha u^{\frac{1}{\beta}} \tag{11}$$

is the quantile function of the power distribution.

Case 3. $\alpha = 0, \lambda = 1, \sigma > 0$.

$$Q(u) = \alpha log\left(\frac{1-pu}{1-u}\right) \tag{12}$$

which is quantile function of exponential geometric distribution with $p = 0$ and $\alpha = -\sigma$.

We can derive some well-known distributions from the proposed model by making use of various transformations described in Gilchrist [2].

Case 4. Consider the power u-transformation $T(u) = u^{\frac{1}{\theta}}$ with $\alpha = 0$. Then,

$$Q(u) = \sigma(-log(1-u^{\frac{1}{\theta}}))^{\frac{1}{\lambda}} \tag{13}$$

is the quantile function of exponentiated Weibull distribution. If $\lambda = 1$,

$$Q(u) = \sigma(-log(1-u^{\frac{1}{\theta}})) \tag{14}$$

is the quantile function of generalized exponential distribution.

Case 5. By reciprocal transformation with σ=0 then,

$$Q(u) = \frac{1}{Q(1-u)} = \sigma(1-u)^{-\frac{1}{\alpha}} \tag{15}$$

which is the quantile function of Pareto distribution with $\sigma = \frac{1}{\alpha}$ and $\alpha = \beta$.

4. Distributional characteristics

The quantile based measures of the distributional characteristics like location, dispersion, skewness, and kurtosis are popular in statistical analysis. These measures are also useful for estimating parameters of the model by matching population characteristics with corresponding sample characteristics. For the model (7), we have,

$$Median = Q(\frac{1}{2}) = \alpha(0.5)^{\frac{1}{\beta}} + \sigma(log(2))^{\frac{1}{\lambda}}. \tag{16}$$

The inter-quartile-range, IQR is obtained as,

$$\begin{aligned} IQR &= Q(\frac{3}{4}) - Q(\frac{1}{4}) \\ &= \alpha[0.75^{\frac{1}{\beta}} - 0.25^{\frac{1}{\beta}}] + \sigma[(log(4))^{\frac{1}{\lambda}} - (-log(0.75))^{\frac{1}{\lambda}}]. \end{aligned} \tag{17}$$

The Galton's coefficient of skewness, S is given by,

$$S = \frac{Q(\frac{3}{4}) + Q(\frac{1}{4}) - 2Median}{IQR}$$
$$= \frac{\alpha[0.75^{\frac{1}{\beta}} + 0.25^{\frac{1}{\beta}} - 2(0.5)^{\frac{1}{\beta}}] + \sigma[(log4)^{\frac{1}{\lambda}} + (-log0.75)^{\frac{1}{\lambda}} - 2(log2)^{\frac{1}{\lambda}}]}{\alpha[0.75^{\frac{1}{\beta}} - 0.25^{\frac{1}{\beta}}] + \sigma[(log4)^{\frac{1}{\lambda}} - (-log0.75)^{\frac{1}{\lambda}}]} \quad (18)$$

and the Moor's coefficient of kurtosis,

$$T = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{IQR}$$
$$= \frac{\alpha(7^{\frac{1}{\beta}} - 5^{\frac{1}{\beta}} + 3^{\frac{1}{\beta}} - 1)8^{\frac{-1}{\beta}} + \sigma[log^{\frac{1}{\lambda}}8 - log^{\frac{1}{\lambda}}(8/3) + log^{\frac{1}{\lambda}}(8/5) - log^{\frac{1}{\lambda}}(8/7)]}{\alpha[0.75^{\frac{1}{\beta}} - 0.25^{\frac{1}{\beta}}] + \sigma[(log4)^{\frac{1}{\lambda}} - (-log0.75)^{\frac{1}{\lambda}}]}. \quad (19)$$

5. L-moments

The L-moments are often found to be more desirable than the conventional moments in describing the characteristics of the distributions as well as for inference. A unified theory and a systematic study on L-moments have been presented by Hosking [6]. The L-moments have generally lower sampling variances and are robust against outliers.
The rth L moment is given by

$$L_r = \int_0^1 \sum_{k=0}^{r-1} (-1)^{r-1-k} \binom{r-1}{k} \binom{r-1+k}{k} u^k Q(u) du \quad (20)$$

For the model (7), the first L moment L_1 is the mean of the distribution.

$$L_1 = \int_0^1 Q(u)du = \frac{\alpha\beta}{1+\beta} + \sigma\Gamma(\frac{1}{\lambda} + 1) \quad (21)$$

The second L-moment for the family is obtained as

$$L_2 = \int_0^1 (2u-1)Q(u)du = \frac{\alpha\beta}{1+3\beta+2\beta^2} + \sigma\Gamma(\frac{1}{\lambda} + 1)(1 - 2^{-\frac{1}{\lambda}}) \quad (22)$$

which is twice the mean differences of the population.
The third and fourth L-moments are obtained as

$$L_3 = \int_0^1 (6u^2 - 6u + 1)Q(u)du$$
$$= \frac{\alpha\beta}{\beta+1} - \frac{6\alpha\beta^2}{1+5\beta+6\beta^2} + \sigma\Gamma(\frac{1}{\lambda} + 1)(1 - 32^{-\frac{1}{\lambda}} + 23^{-\frac{1}{\lambda}}) \quad (23)$$

and

$$L_4 = \int_0^1 (20u^3 - 30u^2 + 12u - 1)Q(u)du$$
$$= \frac{20\alpha\beta}{1+4\beta} - \frac{30\alpha\beta}{1+3\beta} + \frac{12\alpha\beta}{1+2\beta} - \frac{\alpha\beta}{1+\beta} + \sigma\Gamma(\frac{1}{\lambda} + 1)(1 - 32^{1-\frac{1}{\lambda}} + 103^{-\frac{1}{\lambda}} - 54^{-\frac{1}{\lambda}}) \quad (24)$$

The L-coefficient of variation (τ_2), analogous to the coefficient of variation based on ordinary moments is given by,

$$\tau_2 = \frac{L_2}{L_1} = \frac{\frac{\alpha\beta}{1+3\beta+2\beta^2} + \sigma\Gamma(\frac{1}{\lambda} + 1)(1 - 2^{-\frac{1}{\lambda}})}{\frac{\alpha\beta}{1+\beta} + \sigma\Gamma(\frac{1}{\lambda} + 1)}. \quad (25)$$

L-coefficient of skewness (τ_3) for the PW quantile function is obtained as

$$\tau_3 = \frac{L_3}{L_2} = \frac{\frac{\alpha\beta}{\beta+1} - \frac{6\alpha\beta^2}{1+5\beta+6\beta^2} + \sigma\Gamma(\frac{1}{\lambda}+1)(1-32^{-\frac{1}{\lambda}}+23^{-\frac{1}{\lambda}})}{\frac{\alpha\beta}{1+3\beta+2\beta^2} + \sigma\Gamma(\frac{1}{\lambda}+1)(1-2^{-\frac{1}{\lambda}})}. \tag{26}$$

L-coefficient of kurtosis (τ_4) for the PW quantile function is obtained as

$$\tau_4 = \frac{L_4}{L_3} = \frac{\frac{20\alpha\beta}{1+4\beta} - \frac{30\alpha\beta}{1+3\beta} + \frac{12\alpha\beta}{1+2\beta} - \frac{\alpha\beta}{1+\beta} + \sigma\Gamma(\frac{1}{\lambda}+1)(1-32^{1-\frac{1}{\lambda}}+103^{-\frac{1}{\lambda}}-54^{-\frac{1}{\lambda}})}{\frac{\alpha\beta}{\beta+1} - \frac{6\alpha\beta^2}{1+5\beta+6\beta^2} + \sigma\Gamma(\frac{1}{\lambda}+1)(1-32^{-\frac{1}{\lambda}}+23^{-\frac{1}{\lambda}})}. \tag{27}$$

6. Order statistics

If $X_{r:n}$ is the rth order statistic in a random sample of size n, then the density function of $X_{r:n}$ can be written as

$$f_r(x) = \frac{1}{B(r, n-r+1)} f(x)F^{r-1}(x)(1-F(x))^{n-r} \tag{28}$$

From (9), we have

$$f_r(x) = \frac{1}{B(r, n-r+1)} \frac{\beta\lambda F^{r-1}(x)(1-F(x))^{n-r+1}}{\alpha\lambda(1-F(x))F(x)^{\frac{1}{\beta}-1} + \sigma\beta(-log^{\frac{1}{\lambda}-1}(1-F(x)))}. \tag{29}$$

Hence,

$$\mu_{r:n} = E(X_{r:n}) = \int x f_r(x)dx = \frac{1}{B(r, n-r+1)} \int_0^\infty x \frac{\beta\lambda F^{r-1}(x)(1-F(x))^{n-r+1}}{\alpha\lambda(1-F(x))F(x)^{\frac{1}{\beta}-1} + \sigma\beta(-log^{\frac{1}{\lambda}-1}(1-F(x)))} dx. \tag{30}$$

In quantile terms, we have

$$E(X_{r:n}) = \frac{1}{B(r, n-r+1)} \int_0^1 Q(u) \frac{\beta\lambda u^{r-1}(1-u)^{n-r+1}}{\alpha\lambda(1-u)u^{\frac{1}{\beta}-1} + \sigma\beta(-log^{\frac{1}{\lambda}-1}(1-u))} du. \tag{31}$$

For the class of distributions (7), the first-order statistic $X_{1:n}$ has the quantile function

$$Q_1(u) = Q(1-(1-u)^{\frac{1}{n}}) = \alpha[1-(1-u)^{\frac{1}{n}}]^{\frac{1}{\beta}} + \sigma[-log(1-u)^{\frac{1}{n}}]^{\frac{1}{\lambda}}, \tag{32}$$

and the nth order statistic $X_{n:n}$ has the quantile function

$$Q_n(u) = Q(u^{\frac{1}{n}}) = \alpha u^{\frac{1}{n\beta}} + \sigma(-log(1-u^{\frac{1}{n}}))^{\frac{1}{\lambda}}. \tag{33}$$

7. Hazard quantile function

One of the basic concepts employed for modeling and analysis of lifetime data is the hazard rate. In a quantile setup, Nair and Sankaran [7] defined the hazard quantile function, which is equivalent to the hazard rate. The hazard quantile function $H(u)$ is defined as

$$H(u) = h(Q(u))) = (1-u)^{-1} fQ(u) = [(1-u)q(u)]^{-1}. \tag{34}$$

Thus $H(u)$ can be interpreted as the conditional probability of failure of a unit in the next small interval of time given the survival of the unit until $100(1-u)\%$ point of the distribution. Note that $H(u)$ uniquely determines the distribution using the identity,

$$Q(u) = \int_0^u \frac{dp}{(1-p)H(p)}. \tag{35}$$

The hazard quantile functions of Power and Weibull distribution is given by

$$H_1(u) = \beta\alpha^{-1}(1-u)^{-1}u^{1-\frac{1}{\beta}} \tag{36}$$

and

$$H_2(u) = \lambda\sigma^{-1}(-log(1-u))^{1-\frac{1}{\lambda}}. \tag{37}$$

Since the proposed class of distributions is the sum of quantile functions of power and Weibull quantile functions, (34) and (35) give

$$\frac{1}{H(u)} = \frac{1}{H_1(u)} + \frac{1}{H_2(u)} \tag{38}$$

where $H(u)$,$H_1(u)$ and $H_2(u)$ are the hazard quantile functions of the proposed class of distributions, power, and Weibull quantile functions, respectively.

For the PW quantile function (7), we have

$$H(u) = \frac{1}{\beta^{-1}\alpha(1-u)u^{\frac{1}{\beta}-1} + \lambda^{-1}\sigma(-log(1-u))^{\frac{1}{\lambda}-1}} \tag{39}$$

with $H(0) = \infty$ and $H(1) = 0$. Plots of hazard quantile function for different values of parameters are given in figure (2).

The shape of the hazard function is determined by the derivative of $H(u)$, which is obtained as

$$H^{'}(u) = \frac{\beta^{-1}\alpha u^{\frac{1}{\beta}-1}[1 + (\frac{1}{u}-1)(1-\frac{1}{\beta})] + \lambda^{-1}\sigma(1-\frac{1}{\lambda})(1-u)^{-1}(-log(1-u))^{\frac{1}{\lambda}-2}}{[\beta^{-1}\alpha(1-u)u^{\frac{1}{\beta}-1} + \lambda^{-1}\sigma(-log(1-u))^{\frac{1}{\lambda}-1}]^2}. \tag{40}$$

Since $[\beta^{-1}\alpha(1-u)u^{\frac{1}{\beta}-1} + \lambda^{-1}\sigma(-log(1-u))^{\frac{1}{\lambda}-1}]^2 > 0$. For all values of the parameters, the sign of $H^{'}(u)$ depends only on

$$g(u) = \beta^{-1}\alpha u^{\frac{1}{\beta}-1}[1 + (\frac{1}{u}-1)(1-\frac{1}{\beta})] + \lambda^{-1}\sigma(1-\frac{1}{\lambda})(1-u)^{-1}(-log(1-u))^{\frac{1}{\lambda}-2}. \tag{41}$$

The parameters α, σ being always > 0 do not affect the sign of the two terms in g(u). Now we consider the following cases.

Case 1. $o < \beta < 1$ and $o < \lambda < 1$.
$g(u) < 0$ and distribution has an decreasing hazard rate (DHR).

Case 2. $\beta = 1$ and $\lambda = 1$.
$g(u) = \alpha$ and distribution has an increasing hazard rate (IHR).

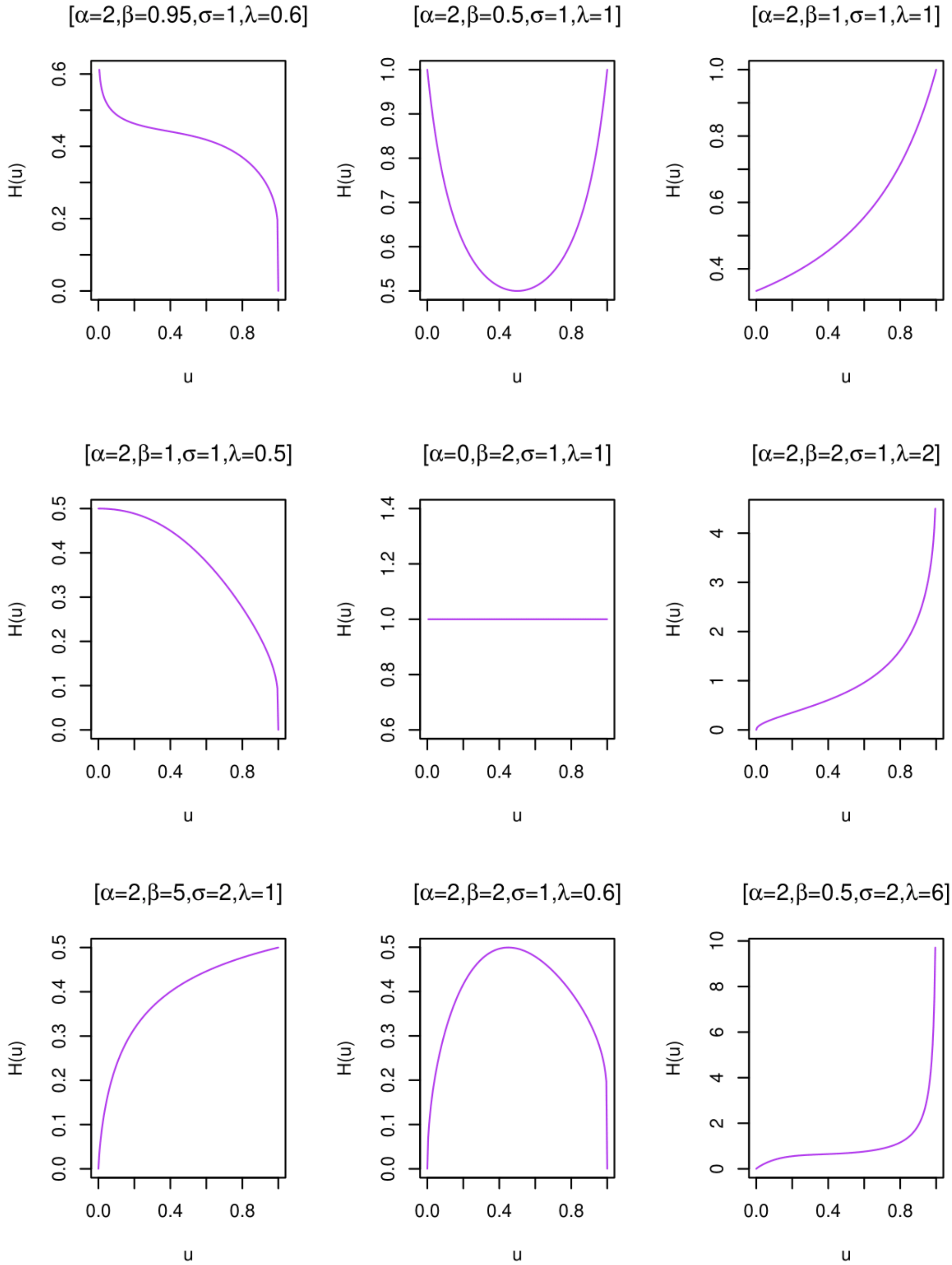

Figure 2: *Plots of hazard quantile function*

Case 3. $\beta = 1$ and $\lambda > 1$.
$g(u) > 0$ and distribution has an increasing hazard rate (IHR).

Case 4. $\beta > 1$ and $\lambda = 1$.
The first term in $g(u)$ is positive and second term is zero, so that $g(u) > 0$ and distribution has an increasing hazard rate (IHR).

Case 5. $\beta > 1$ and $\lambda > 1$.
The first and second term in $g(u)$ is positive, so that $g(u) > 0$ and distribution has an increasing hazard rate (IHR).

Case 6. $\beta = 1, 0 < \lambda < 1$.
Distribution has an decreasing hazard rate (DHR).

Case 7. $o < \beta < 1$ and $\lambda = 1$.
$H(u)$ attains a minimum at $u_0 = 1 - \beta$ and therefore $H(u)$ is bathtub shaped.

For the remaining cases, ($o < \beta < 1$, $\lambda > 1$) and ($\beta > 1$, $0 < \lambda < 1$), one term in $g(u)$ is positive and the other is negative so that $g(u)$ can be zero. From (40) and(41) we obtain the first derivative of $H(u)$,

$$H^{'}(u) = \frac{g(u)}{[\beta^{-1}\alpha(1-u)u^{\frac{1}{\beta}-1} + \lambda^{-1}\sigma(-log(1-u))^{\frac{1}{\lambda}-1}]^2}. \tag{42}$$

For further analysis, we take the second derivative of $H(u)$, the sign of $H^{''}(u)$ depends on,

$$g^{'}(u)\left(\beta^{-1}\alpha(1-u)u^{\frac{1}{\beta}-1} + \lambda^{-1}\sigma(-log(1-u))^{\frac{1}{\lambda}-1}\right)^2 + 2g(u)\left(\beta^{-1}\alpha(1-u)u^{\frac{1}{\beta}-1} + \lambda^{-1}\sigma(-log(1-u))^{\frac{1}{\lambda}-1}\right)g(u).$$

Let u_0 be the solution of the equation $g(u) = 0$. Then the sign of $H^{''}(u)$ at u_0 depends on $g^{'}(u_0)$. Since u_0 is a solution of $g(u) = 0$.

$$g^{'}(u) = \beta^{-1}\alpha(\frac{1}{\beta}-1)u^{\frac{1}{\beta}-2}\left(1 + (\frac{1}{u}-1)(1-\frac{1}{\beta}) + u^{-1}\right) + \lambda^{-1}\sigma(\frac{1}{\lambda}-1)(1-u)^{-2} (-log(1-u))^{\frac{1}{\lambda}-2}\left(1 - (\frac{1}{\lambda}-2)(-log(1-u))^{-1}\right).$$

Then,

$$g^{'}(u_0) = \beta^{-1}\alpha(\frac{1}{\beta}-1)u_0^{\frac{1}{\beta}-2}\left(1 + (\frac{1}{u_0}-1)(1-\frac{1}{\beta}) + u_0^{-1}\right) + \lambda^{-1}\sigma(\frac{1}{\lambda}-1)(1-u_0)^{-2} (-log(1-u_0))^{\frac{1}{\lambda}-2}\left(1 - (\frac{1}{\lambda}-2)(-log(1-u_0))^{-1}\right). \tag{43}$$

Case 8. $0 < \beta < 1$ and $\lambda > 1$.
$H(u)$ has an increasing hazard rate (IHR).

Case 9. $\beta > 1$ and $0 < \lambda < 1$.
$H(u)$ has an upside-down bathtub-shaped hazard quantile function.

The patterns of $H(u)$ for various parameter values are summarized in table 1.

Table 1: *Behavior of the hazard quantile function for different regions of parameter space.*

No.	Parameter region	Shape of hazard quantile function
1	$o < \beta < 1$ and $o < \lambda < 1$	DHR
2	$\beta = 1$ and $\lambda = 1$	IHR
3	$\beta = 1$ and $\lambda > 1$	IHR
4	$\beta > 1$ and $\lambda = 1$	IHR
5	$\beta > 1$ and $\lambda > 1$	IHR
6	$\beta = 1, 0 < \lambda < 1$	DHR
7	$o < \beta < 1$ and $\lambda = 1$	Bathtub
8	$o < \beta < 1$ and $\lambda > 1$	IHR
9	$\beta > 1$ and $0 < \lambda < 1$	Upside-down Bathtub

8. Mean residual quantile function

Another concept used in reliability is that of residual life $X_t = (X - t|X > t)$ with survival function

$$\bar{F}_t(x) = \bar{F}(t+x)/\bar{F}(t), \quad x \geq 0, o < t < T.$$

The mean residual life function is then

$$m(t) = E(X_t) = [\bar{F}(t)]^{-1} \int_t^{\infty} \bar{F}(x)dx.$$

Accordingly, the mean residual quantile function is defined by Nair and Sankaran [7] as

$$M(u) = mQ(u) = (1-u)^{-1} \int_u^1 (Q(t) - Q(u))dt \tag{44}$$

which is the average remaining life beyond the $100(1-u)\%$ point of the distribution. For the class of distributions (7), $M(u)$ has the form

$$M(u) = \frac{\alpha(1 - u^{\frac{1}{\beta}+1})}{(1-u)(\frac{1}{\beta}+1)} - \alpha u^{\frac{1}{\beta}} + \sigma(1-u)^{-1}\Gamma\big(\frac{1}{\lambda}+1, -log(1-u)\big) - \sigma(-log(1-u))^{\frac{1}{\lambda}}. \tag{45}$$

9. Residual variance quantile function

The quantile form of variance residual function, the residual variance quantile function is defined as

$$V(u) = (1-u)^{-1} \int_u^1 Q^2(p)dp - (M(u) + Q(u))^2. \tag{46}$$

In the above equation, the variance residual life function is obtained by letting $Q(u) = x$.
Nair and Sankaran [7] derived the relationship between $M(u)$ and $V(u)$ as

$$M^2(u) = V(u) - (1-u)V^{'}(u) \tag{47}$$

or

$$V(u) = (1-u)^{-1} \int_u^1 M^2(p)dp. \tag{48}$$

Since $M(u)$ characterizes the distribution, from above equations it follows that $V(u)$ also characterizes the distribution.

For the PW qantile function, residual variance quantile function is

$$V(u)=\frac{1}{1-u}\left\{\frac{\beta\alpha^2(1-u^{\frac{2+\beta}{\beta}})}{2+\beta}+\sigma^2\Gamma(\frac{2}{\lambda}+1,-log(1-u))+2\alpha\sigma\int_u^1 p^{1/\beta}\right.$$
$$\left.(-log(1-p))^{1/\lambda}dp\right\}-\left\{\frac{\alpha(1-u^{\frac{1}{\beta}+1})}{(1-u)(1/\beta+1)}+\sigma(1-u)^{-1}\Gamma(\frac{1}{\lambda}+1,\right.$$
$$\left.-log(1-u))\right\}^2 \tag{49}$$

10. Reversed hazard quantile function

The reversed hazard quantile function [8] is defined by

$$A(u)=\frac{1}{uq(u)} \tag{50}$$

and it determines the distribution through the formula

$$Q(u)=\int_0^u \frac{1}{pA(p)}dp. \tag{51}$$

For power-Weibull distribution,

$$A(u)=\Big[\frac{\alpha u^{\frac{1}{\beta}}}{\beta}+\frac{\sigma u(-log(1-u))^{\frac{1}{\lambda}-1}}{\lambda(1-u)}\Big]^{-1}. \tag{52}$$

11. Data Analysis

There are different methods for the estimation of parameters of the quantile function. The method of percentiles, method of L-moments, method of minimum absolute deviation, method of least squares, and method of maximum likelihood are commonly used techniques. To estimate the parameters of (7), we use the method of maximum likelihood estimation procedure.

To illustrate the application of the proposed class of distributions we consider a real data set reported in Zimmer *et al.* [15]. The data consist of times to first failure of 20 electric carts used for internal transportation and delivery in a manufacturing company. The estimates of the parameters are obtained using R software as,

$$\hat{\alpha}=0.421,\ \hat{\beta}=2.088,\ \hat{\sigma}=0.027\quad\text{and}\quad\hat{\lambda}=0.312.$$

To examine the adequacy of the model, we use chi-squared goodness of fit. The test gives the p-value 0.082. This indicates the adequacy of proposed model for the given data set.

12. Summary and Conclusion

In this paper, we introduced a class of distributions (7), which is the sum of the quantile function of the power and Weibull distributions known as Power-Weibull (PW) quantile function and its graphical representation of density function is included. We have identified several well-known distributions which are either the members of the proposed class of distributions and also through suitable transformations such as Weibull distribution, power distribution, generalized exponential distribution, etc. Various distributional characteristics and L-moments are discussed. The hazard quantile function and its shape in various parameter region are analysed. Increasing, decreasing, bathtub and upside-down bathtub hazard quantile function are obtained. Mean residual quantile function of PW quantile function was studied. Finally PW model is applied to a real life data set, and parameters are estimated by using maximum likelihood estimation procedure and model adequacy is checked by chi-squared goodness of fit test using the R software.

There are several properties and extensions for the PW quantile function not considered in this article, such as parameter estimation using L-moments, stochastic orderings and generalization of PW quantile function.

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