TYPE-1 BETA DISTRIBUTION AND ITS CONNECTIONS TO LIKELIHOOD RATIO TEST

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2 authors:

Nicy Sebastian
St. Thomas College, Thrissur

T. Princy
Cochin University of Science and Technology

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Nicy Sebastian [1]
Department of Statistics, St Thomas College Thrissur, Calicut University, Kerala, India
T. Princy
Department of Statistics, Cochin University of Science and Technology, Kerala, India

SUMMARY

In many cases involving hypothesis testing for parameters in multivariate Gaussian populations and certain other populations, likelihood ratio criteria, or their one-to-one functions, can be expressed in terms of the determinant of a real type-1 beta matrix. In geometrical probability problems, when the random points are type-1 beta distributed, the volume content of the parallellotope generated by these points is also associated with the determinant of a real type-1 beta matrix. These problems in the corresponding complex domain do not seem to have been discussed in the literature. It is well-known that the determinant of a real type-1 beta matrix can be written as a product of statistically independently distributed real scalar type-1 beta random variables. This paper addresses the general h-th moments of a scalar random variable, in either the real or complex domain, for any arbitrary h. The structure of these moments is quite general, and the paper provides exact distribution results, asymptotic gamma function results, and asymptotic normal results for both the real and complex domains.

Keywords: Likelihood ratio criteria; Type-1 beta matrix; General structures; Real and complex cases; Asymptotic chi-square; Asymptotic normal; Exact distribution.

1. INTRODUCTION

The likelihood ratio test (LRT) is an important statistical tool used to test hypotheses about the parameters of a statistical model. Its importance stems from the fact that it is a powerful and versatile test that can be used in a wide range of applications. The likelihood ratio test is an important tool in statistical analysis, offering increased power, robustness, and model selection capabilities. It is widely used in many fields, including biology, economics, engineering, and the social sciences. The foundational principles

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[1] Corresponding Author. E-mail: nicycms@gmail.com

and interpretation of specific test criteria for statistical inference are outlined in Neyman and Pearson (1928). Further insights into the most efficient tests of statistical hypotheses are provided in Neyman and Pearson (1933) and Lehmann (2012). Anderson (1958) presents a comprehensive exploration of multivariate statistical analysis, delving into the theory and applications of the likelihood ratio test in various multivariate analysis contexts. The comparison of sample covariance matrices through likelihood ratio tests is detailed by Manly and Rayner (1987). Pioneering the incorporation of Random Matrix Theory, Bai *et al.* (2009) scrutinized two Likelihood ratio Tests and elucidated the limiting distributions of the associated test statistics.

Lim *et al.* (2010) introduced likelihood ratio tests for correlated multivariate samples. Central limit theorems for classical likelihood ratio tests applied to high-dimensional normal distributions is developed by Jiang and Yang (2013). Subsequently, other researchers expanded on these findings in various ways. For example, Jiang and Qi (2015) relaxed assumptions on the parameters and Jiang and Wang (2017) established a moderate deviation principle for these likelihood ratio tests. Furthermore, investigations into likelihood ratio tests under model uncertainty have been conducted by various authors, including Luo and Tsai (2012) and Lemonte (2013) and Lemonte (2016). More recently, Dette and Dörnemann (2020) and Dörnemann (2023) derived insightful distributional results relevant to high-dimensional settings. These results specifically involve likelihood ratio statistics that can be expressed in terms of products and ratios of determinants, incorporating sample sum of squares and cross-product matrices.

Likelihood ratio criteria associated with testing hypotheses on the parameters of one or more real multivariate normal populations, as well as some non-normal populations, often involve type-1 beta matrices and their determinants. The presence of real matrix-variate type-1 beta distribution is prevalent in various problem domains. In the area of geometrical probabilities dealing with type-1 beta distributed random points, the volume content of a p-parallellotope generated by these random points is associated with the determinant of a real type-1 beta matrix (see Mathai, 1999). Mathai and Provost (2022) obtained the density functions of both types of real and complex singular matrix-variate beta random variables using a technique that relies on a series of successive transformations. Notably, there seems to be a gap in the literature regarding corresponding hypothesis tests for distribution parameters in the complex domain, with limited discussion in the recent book by Mathai *et al.* (2022). In these complex cases, likelihood ratio criteria are observed to be functions of complex type-1 beta matrices and their determinants.

In many physical situations, variables appear in pairs, such as time and phase. Hence, in such situations, it is found that a more appropriate representation of such random phenomena is through random variables in the complex domain. Scalar, vector, and matrix-variate distributions in the complex domain find extensive applications in communication theory, engineering problems, quantum physics, and other domains. Deng (2016) specifically addresses the analysis of data related to multi-look return signals from polarimetric synthetic aperture radar. A cross-section of the return signal has two components: one is the pepper dust-like contaminants called freckle, and the other is the cross-section variable itself, called texture. Typically, freckle is modeled using a matrix-variate random variable in the complex domain, while texture is represented by either a scalar positive variable or a positive definite matrix variable. These models are respectively termed the scalar texture model and matrix texture model. Deng (2016) provides a comprehensive list of scalar variable distributions and matrix-variate distributions in the complex domain commonly employed in radar data analysis. In a recent paper by Benavoli *et al.* (2016), physicists illustrate and demonstrate with examples that Quantum Mechanics can be viewed as a Bayesian analysis of Hermitian positive definite matrices in a Hilbert space. Consequently, matrix-variate distributions in the complex domain play a crucial role in quantum physics. Hence, the results obtained in the present paper will have relevance and use in communication theory, engineering problems, quantum physics, and related areas, apart from their significance in the statistical literature within the complex domain.

This paper is organized as follows: Section 2 deals with the likelihood ratio criteria, their connection to the real type-1 beta matrix, the structural representation of the determinant of a type-1 beta matrix, and a general moment structure. In Section 3, approximations and asymptotic gamma form are presented. Section 4 is about asymptotic normality, dealing with a novel method of deriving such results. Concluding remarks are given in Section 5.

The following subsection introduces the notations and definitions used throughout this article for the likelihood ratio criteria associated with the type-1 beta matrix in real and complex scenarios.

1.1. Notations

In this paper, the following notations will be used for convenience. Real scalar variables, whether mathematical variables or random variables, will be denoted by lower-case letters such as x, y, x_1, x_2, …. Real vector/matrix variables, mathematical or random, will be denoted by capital letters such as X, Y, X_1, X_2, …, whether the matrices are square or rectangular. Complex variables will be denoted with a tilde in the form $\tilde{x}$, $\tilde{y}$, $\tilde{X}$, $\tilde{Y}$, …. Constant scalars will be denoted by a, b, c, … and constant vector/matrix by A, B, …. No tilde will be used on constants. The determinant of a $p \times p$ matrix Z will be written as $|Z|$ or $\det(Z)$. In the complex domain, $\det(\tilde{Z})$ will be of the form $\det(\tilde{Z}) = a + ib, i = \sqrt{(-1)}, a, b$ real scalar. Then, the absolute value or modulus of the determinant of $\tilde{Z}$ will be

$$|\det(\tilde{Z})| = +\sqrt{a^2 + b^2} = +\sqrt{\det(\tilde{Z})\det(\tilde{Z}^*)} = +\sqrt{\det(\tilde{Z}\tilde{Z}^*)},$$

where $\tilde{Z}^*$ means the complex conjugate transpose of $\tilde{Z}$. A prime will be used to indicate the transpose such as A'. When a $m \times n$ real matrix $X = (x_{ij})$, where the x_{ij}'s are distinct (functionally independent) real scalar variables, then the wedge product of their differentials $\mathrm{d}x_{ij}$'s will be written as $\mathrm{d}X = \wedge_{i=1}^{m} \wedge_{j=1}^{n} \mathrm{d}x_{ij}$. When $Y = (y_{ij})$ is a $p \times p$

real symmetric matrix, $Y = Y'$, then $\mathrm{d}Y = \wedge_{i\le j}\mathrm{d}y_{ij} = \wedge_{i\ge j}\mathrm{d}y_{ij}$ since there are only $p(p+1)/2$ distinct elements due to symmetry. For a $p \times q$ matrix $\tilde{Y}$ in the complex domain, $\tilde{Y}$ can be written as $\tilde{Y} = Y_1 + iY_2, i = \sqrt{(-1)}, Y_1, Y_2$ real, then $\mathrm{d}\tilde{Y}$ will be defined as $\mathrm{d}\tilde{Y} = \mathrm{d}Y_1 \wedge \mathrm{d}Y_2$. In this paper, a vector will mean a $n \times 1$ or $1 \times n$ matrix for $n > 1$ and when $n = 1$ these will be called scalars. A statistical density will be defined as a real-valued scalar function $f(X)$, where the argument X can be scalar, vector, square or rectangular matrix, in the real or complex domain, such that $f(X) \ge 0$ for all X and $\int_X f(X)\mathrm{d}X = 1$. In the following discussion, all matrices appearing are $p \times p$ real positive definite, or Hermitian positive definite in the complex domain, unless stated otherwise. Also, it is understood that the density functions are zero outside the support stated with each function.

1.2. *The Matrix-Variate Gamma Distribution*

A real matrix-variate gamma density with shape parameter α and scale parameter matrix $B > O$, denoted by $f_1(X)$, is defined as the following:

$$f_1(X) = \frac{|B|^{\alpha}}{\Gamma_p(\alpha)}|X|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(BX)}, X = X' > O, B = B' > O, \Re(\alpha) > \frac{p-1}{2} \tag{1}$$

and zero elsewhere, where $X > O, B > O$ mean that the $p \times p$ matrices X and B are real positive definite, B is a constant matrix, $\mathrm{tr}(\cdot)$ means the trace of $(\cdot)$, $\Re(\cdot)$ means the real part of $(\cdot)$ and $\Gamma_p(\alpha)$ is the real matrix-variate gamma given by

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}}\Gamma(\alpha)\Gamma(\alpha - \frac{1}{2})\ldots\Gamma(\alpha - \frac{p-1}{2}), \Re(\alpha) > \frac{p-1}{2}.$$

It is called the real matrix-variate gamma because it is associated with a real matrix-variate gamma integral, namely

$$\Gamma_p(\alpha) = \int_{X>O}|X|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(X)}\mathrm{d}X, \Re(\alpha) > \frac{p-1}{2}.$$

The corresponding gamma density in the complex domain denoted by $\tilde{f}_1(\tilde{X})$, complex matrix-variate gamma and complex matrix-variate gamma integral are the following (see also Mathai *et al.*, 2022):

$$\tilde{f}_1(\tilde{X}) = \frac{|\det(B)|^{\alpha}}{\tilde{\Gamma}_p(\alpha)}|\det(\tilde{X})|^{\alpha-p}\mathrm{e}^{-\mathrm{tr}(B\tilde{X})}, \quad \tilde{X} = \tilde{X}^* > O, \quad B = B^* > O, \quad \Re(\alpha) > p-1, \tag{2}$$

$$\tilde{\Gamma}_p(\alpha) = \pi^{\frac{p(p-1)}{2}}\Gamma(\alpha)\Gamma(\alpha-1)\ldots\Gamma(\alpha-p+1), \quad \Re(\alpha) > p-1,$$
$$\tilde{\Gamma}_p(\alpha) = \int_{\tilde{X}>O} |\det(\tilde{X})|^{\alpha-p}\mathrm{e}^{-\mathrm{tr}(\tilde{X})}\mathrm{d}\tilde{X}, \quad \Re(\alpha) > p-1,$$

where $\tilde{X}$ and B are Hermitian positive definite with B being a constant matrix, $|\det(\cdot)|$ means the absolute value of the determinant of $(\cdot)$, a tilde is used on the functions in the complex domain, a letter c is attached to the section number to indicate equation numbers in the complex domain in order to avoid too many equation numbers.

In a statistical density, usually, the parameters are real. But the mathematical properties hold for parameters in the complex domain also. Hence, the relevant conditions are stated for the parameters in the complex domain.

2. Likelihood Ratio Criteria associated with Type-1 Beta Matrix and Structural Representations

Consider the case of a $p \times 1$ real vector variable X having a real p-variate nonsingular Gaussian distribution, that is, $X \sim N_p(\mu, \Sigma), \Sigma > O$. Let X be partitioned as $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ where X_1 is $p_1 \times 1$ and X_2 is $p_2 \times 1$ with $p_1 + p_2 = p$. Consider the hypothesis that X_1 and X_2 are independently distributed. Consider testing this hypothesis by using a simple random sample of size n from this normal population. Let λ be the likelihood ratio criterion for testing this hypothesis of independence. Let $u = \lambda^{\frac{2}{n}}$. Then, under the null hypothesis one can see that the h-th moment of u, for arbitrary h, is of the form

$$E[u^h] = c\,\frac{\prod_{j=p_1+1}^{p}\Gamma(\frac{m}{2} - \frac{j-1}{2} + h)}{\prod_{j=1}^{p_2}\Gamma(\frac{m}{2} - \frac{j-1}{2} + h)}, \quad \Re(h) > -\frac{m}{2}, \tag{3}$$

where c is the normalizing constant such that $E[u^h] = 1$ when $h = 0$. In the complex case, consider the p-variate complex nonsingular Gaussian population $\tilde{X} \sim \tilde{N}_p(\tilde{\mu}, \Sigma)$, $\Sigma = \Sigma^* > O$. Consider the partitioning of $\tilde{X}$ as in the real case and consider the hypothesis that the subvectors are independently distributed. Then, it is shown that the h-th moment of $\tilde{u} = \tilde{\lambda}^{\frac{2}{n}}$, for arbitrary h and under the null hypothesis, is of the form

$$E[\tilde{u}^h] = \tilde{c}\,\frac{\prod_{j=p_1+1}^{p}\Gamma(m+h-j+1)}{\prod_{j=1}^{p_2}\Gamma(m+h-j+1)}, \quad \Re(h) > -m. \tag{4}$$

It can be seen that $\tilde{u}$ in the complex case is also real and for consistency of the notation we have used a tilde. In the real and complex cases in Eq. (3) and Eq. (4), it can be seen that the h-th moments of u and $\tilde{u}$ are coming from the determinants of a real type-1 beta matrix and a complex type-1 beta matrix respectively. Those beta matrices are coming from the following properties. From Mathai (1999) it follows that when X_1 and X_2 are two $p \times p$ real matrix-variate random variables, independently distributed as the gamma distribution in Eq. (2) with the parameters (α, B) and (β, B), respectively, with the same scale parameter matrix $B > O$, then $U = (X_1 + X_2)^{-\frac{1}{2}} X_1 (X_1 + X_2)^{-\frac{1}{2}}$ is type-1 real matrix-variate beta distributed with the density

$$f_2(U) = \{\frac{\Gamma_p(\alpha+\beta)}{\Gamma_p(\alpha)\Gamma_p(\beta)}\}|U|^{\alpha-\frac{p+1}{2}}|I-U|^{\beta-\frac{p+1}{2}}, \quad O < U < I, \quad \Re(\alpha), \quad \Re(\beta) > \frac{p-1}{2}. \tag{5}$$

Here $(\cdot)^{\frac{1}{2}}$ means the positive definite square root of the positive definite matrix $(\cdot)$ and the notation $O < U < I$ means $U > O$ and $I - U > O$. It is well-known that since the density in Eq. (5) is a function of the determinant, $|U|$, the h-th moment of the determinant of this real type-1 beta matrix in Eq. (5) is available from the normalizing constant in Eq. (5) and it has the form

$$\begin{aligned} E[|U|^h] &= \frac{\Gamma_p(\alpha+h)}{\Gamma_p(\alpha)} \frac{\Gamma_p(\alpha+\beta)}{\Gamma_p(\alpha+\beta+h)}, \quad \Re(\alpha+h) > \frac{p-1}{2} \\ &= c_p \prod_{j=1}^{p} \frac{\Gamma(\alpha+h-\frac{j-1}{2})}{\Gamma(\alpha+\beta+h-\frac{j-1}{2})}, \quad c_p = \prod_{j=1}^{p} \frac{\Gamma(\alpha+\beta-\frac{j-1}{2})}{\Gamma(\alpha-\frac{j-1}{2})} \\ &= E[u_1^h]E[u_2^h]\ldots E[u_p^h], \; E[u_j^h] \\ &= \frac{\Gamma(\alpha+h-\frac{j-1}{2})}{\Gamma(\alpha-\frac{j-1}{2})} \frac{\Gamma(\alpha+\beta-\frac{j-1}{2})}{\Gamma(\alpha+\beta+h-\frac{j-1}{2})}, \end{aligned} \tag{6}$$

where $u_1, \ldots, u_p$ are independently distributed real scalar type-1 beta variables with the parameters $(\alpha - \frac{j-1}{2}, \beta)$, for $j = 1, \ldots, p$. Hence, $|U|$ has the structural representation

$$|U| = u_1 \ldots u_p.$$

It can be seen that in the complex case also the transformation $\tilde{U} = (\tilde{X}_1 + \tilde{X}_2)^{-\frac{1}{2}} \tilde{X}_1 (\tilde{X}_1 + \tilde{X}_2)^{-\frac{1}{2}}$ holds and $\tilde{U}$ has a complex matrix-variate type-1 beta distribution when $\tilde{X}_1$ and $\tilde{X}_2$ are independently distributed complex matrix-variate gamma variables with the densities as in Eq. (2) with the parameters (α, B) and (β, B), respectively, with the same $B = B^* > O$, where the square root means the Hermitian positive definite square root of the Hermitian positive definite matrix. Then, following the steps parallel to those in the real case, it can be seen that the density of $\tilde{U}$, denoted by $\tilde{f}_2(\tilde{U})$, is the following:

$$\tilde{f}_2(\tilde{U}) = \{\frac{\tilde{\Gamma}_p(\alpha+\beta)}{\tilde{\Gamma}_p(\alpha)\tilde{\Gamma}_p(\beta)}\}|\det(\tilde{U})|^{\alpha-p}|\det(I-\tilde{U})|^{\beta-p},\ O<\tilde{U}<I,\ \Re(\alpha),\ \Re(\beta)>p-1,$$

where $O < \tilde{U} < I$ means $\tilde{U} = \tilde{U}^* > O$ and $I - \tilde{U} > O$. In the complex case the h-th moment of the absolute value of the determinant of $\tilde{U}$, namely $E[|\det(\tilde{U})|^h]$, has the following form:

$$\begin{aligned} E[|\det(\tilde{U})|^h] &= \frac{\tilde{\Gamma}_p(\alpha+h)}{\tilde{\Gamma}_p(\alpha)}\frac{\tilde{\Gamma}_p(\alpha+\beta)}{\tilde{\Gamma}_p(\alpha+\beta+h)}, \quad \Re(\alpha+h)>p-1 \\ &= \tilde{c}_p\prod_{j=1}^{p}\frac{\Gamma(\alpha+h-j+1)}{\Gamma(\alpha+\beta+h-j+1)}, \quad \tilde{c}_p=\prod_{j=1}^{p}\frac{\Gamma(\alpha+\beta-j+1)}{\Gamma(\alpha-j+1)} \\ &= E[\tilde{u}_1^h]E[\tilde{u}_2^h]\ldots E[\tilde{u}_p^h], \quad E[\tilde{u}_j^h] \\ &= c_j^*\frac{\Gamma(\alpha+h-j+1)}{\Gamma(\alpha+\beta+h-j+1)}, \end{aligned} \tag{7}$$

for $\Re(\alpha+h)>0$, where c_j^* is the corresponding normalizing constant. Here, it can be seen that $\tilde{u}_1,\ldots,\tilde{u}_p$ are independently distributed real scalar type-1 beta variables with the parameters $(\alpha-j+1,\beta)$ for $j=1,\ldots,p$. The only difference between the real and complex cases here is that $\frac{j-1}{2}$ of the real case is replaced by $j-1$ in the complex case. For consistency of the notation, we have written $\tilde{u}_j, j=1,\ldots,p$. Then, the distributions of the determinant in U in the real case and the absolute value of the determinant of $\tilde{U}$ in the complex case, can be studied by using the structural representations in Equations (6) and (7).

3. Approximations and Asymptotic Gamma and Chisquare

Let us consider a more general form of the gamma structure so that Equations (6) and (7) will be available as special cases therein. Let v be a real scalar random variable with the h-th moment of the form

$$E[v^h] = c\prod_{j=1}^{p}\frac{\Gamma(\alpha+\delta_j+\rho h)}{\Gamma(\alpha+\delta_j+\beta_j+\rho h)}, \tag{8}$$

for $\Re(\alpha) > \frac{p-1}{2}, \rho > 0, \Re(\alpha+\delta_j+\rho h) > 0, j = 1,\ldots,p$ where c is the normalizing constant so that $E[v^h]=1$ when $h=0$. Note that for $\delta_j = -\frac{j-1}{2}, \rho=1, \beta_j=\beta$ in Eq. (8) yields the h-th moment of the real case in Eq. (6). For $\delta_j=-(j-1), \beta_j=\beta, \rho=1$ give the h-th moment in Eq. (7). Hence, we will deal with the moment in Eq. (8) and study some properties of the real scalar random variable v. Consider $E[v^{\alpha h}]$ coming from Eq. (8) and consider the approximation of all gammas in Eq. (8) by using the first term approximation of gamma functions or Stirling's formula, given by

$$\Gamma(z+\delta)\approx\sqrt{2\pi}z^{z+\delta-\frac{1}{2}}e^{-z},\quad \text{for } |z|\to\infty \text{ and } \delta \text{ bounded.}$$

Then, under Stirling's formula we have

$$\prod_{j=1}^{p}\frac{\Gamma(\alpha+\delta_j+\beta_j)}{\Gamma(\alpha)}\approx\frac{\sqrt{2\pi}\alpha^{\alpha+\delta_j+\beta_j-\frac{1}{2}}e^{-\alpha}}{\sqrt{2\pi}\alpha^{\alpha+\delta_j-\frac{1}{2}}e^{-\alpha}}=\alpha^{\sum_j\beta_j}$$

$$\begin{aligned}\prod_{j=1}^{p}\frac{\Gamma(\alpha+\delta_j+\alpha\rho h)}{\Gamma(\alpha+\delta_j+\beta_j+\alpha\rho h)}&=\prod_{j=1}^{p}\frac{\Gamma(\alpha(1+\rho h)+\delta_j)}{\Gamma(\alpha(1+\rho h)+\delta_j+\beta_j)}\\&\approx\prod_{j=1}^{p}\frac{\sqrt{2\pi}[\alpha(1+\rho h)]^{\alpha(1+\rho h)+\delta_j-\frac{1}{2}}e^{-\alpha(1+\rho h)}}{\sqrt{2\pi}[\alpha(1+\rho h)]^{\alpha(1+\rho h)+\delta_j+\beta_j-\frac{1}{2}}e^{-\alpha(1+\rho h)}}\\&=\alpha^{-\sum_j\beta_j}(1+\rho h)^{-\sum_j\beta_j}.\end{aligned}$$

Therefore,

$$E[v^{\alpha h}]=E[e^{-h(-\alpha\ln v)}]\to(1+\rho h)^{-\sum_j\beta_j},\quad \text{for } 1+\rho h>0,\quad |\alpha|\to\infty$$

and all other parameters are bounded. Observe that $(1+\rho h)^{-\sum_j\beta_j}$ is the moment generating function of a real scalar gamma variable with the scale parameter $\rho>0$ and shape parameter $\sum_j\beta_j$ with h replaced by $-h$ and it is the Laplace transform of the gamma density. Hence, we have the following Theorem.

THEOREM 1. *Consider the real scalar variable v having the moment structure in Eq. (8) for an arbitrary h. Then, when $|\alpha|\to\infty$, δ_j's, β_j's and $\rho>0$ (real positive) are bounded, then $-\alpha\ln v$ goes to a real scalar gamma variable with scale parameter ρ and shape parameter $\sum_j\beta_j$. Convergence in distribution is considered here. When $\sum_j\beta_j=\frac{\gamma}{2}$, $\gamma=1,2,\ldots$ and $\rho=2$, then $-\alpha\ln v\to\chi^2_\gamma$, when $|\alpha|\to\infty$ and where χ^2_γ is a real scalar central chisquare with γ degrees of freedom.*

In statistical distributions, usually the parameters are real but the results in this paper hold good for complex parameters as well. Hence, the conditions are stated for complex parameters. When the parameters are real, then in all the conditions delete "$\Re$" notation. As, mentioned in Section 2, Theorem 1 holds for real and complex cases of gamma products coming from likelihood ratio criteria and other similar situations or from the determinants of type-1 beta matrix in the real case and the absolute value of the determinant of type-1 beta matrix in the complex domain.

4. Approximations and Asymptotic Normality

For obtaining an asymptotic normal form, we will appeal to the general asymptotic expansion of gamma functions, namely

$$\Gamma(z+\delta)=\sqrt{2\pi}z^{z+\delta-\frac{1}{2}}\mathrm{e}^{-z-\sum_{k=1}^{\infty}\frac{(-1)^k B_{k+1}(\delta)}{k(k+1)z^k}}, \quad |z|\to\infty, \quad \delta \text{ bounded},$$

where $B_k(\cdot)$ is Bernoulli polynomial of order k and the first two Bernoulli polynomials are $B_2(\delta)=\delta^2-\delta+\frac{1}{6}$ and $B_3(\delta)=\delta^3-\frac{3}{2}\delta^2+\frac{1}{2}\delta$, see for example Mathai (1993). Note that the Stirling's formula is the case $k=0$ or the first term approximation of the gamma function. Now, let us look at the expansion of all gammas in $E[v^h]$ of Eq. (8) for $|\alpha|\to\infty$ and all other parameters bounded. The following techniques are adopted from Mathai (1999). From Section 3 we have, for $k=0$ or under Stirling's formula $E[v^h]\approx 1$ for $|\alpha|$ large. Now, we look at the additional terms for $k=1,2,\ldots$. For $k=1$ the additional term coming from $E[v^h]$ is the following:

$$\prod_{j=1}^{p}\frac{\Gamma(\alpha+\delta_j+\beta_j)}{\Gamma(\alpha+\delta_j)}\to\frac{\mathrm{e}^{\frac{1}{2\alpha}[(\delta_j+\beta_j)^2-(\delta_j+\beta_j)+\frac{1}{6}]}}{\mathrm{e}^{\frac{1}{2\alpha}[\delta_j^2-\delta_j+\frac{1}{6}]}}$$
$$=\mathrm{e}^{\frac{1}{2\alpha}[\sum_j(2\delta_j\beta_j+\beta_j^2-\beta_j)]}.$$

and

$$\prod_{j=1}^{p}\frac{\Gamma(\alpha+\delta_j+\rho h)}{\Gamma(\alpha+\delta_j+\beta_j+\rho h)}\to\frac{\mathrm{e}^{\frac{1}{2\alpha}[(\delta_j+\rho h)^2-(\delta_j+\rho h)+\frac{1}{6}]}}{\mathrm{e}^{\frac{1}{2\alpha}[(\beta_j+(\delta_j+\rho h))^2-(\beta_j+(\delta_j+\rho h))+\frac{1}{6}]}}$$
$$=\mathrm{e}^{-\frac{1}{2\alpha}\sum_j[\beta_j^2+2\beta_j\delta_j-\beta_j+2\beta_j\rho h]}.$$

Therefore the additional factor for $k=1$ gives

$$E[v^h]\to\mathrm{e}^{-\frac{1}{\alpha}\sum_j(\beta_j)\rho h}.$$

For $k=2$ the additional factor leads to the following:

$$\prod_{j=1}^{p}\frac{\Gamma(\alpha+\delta_j+\beta_j)}{\Gamma(\alpha+\delta_j)}\to\frac{\mathrm{e}^{-\frac{1}{6\alpha^2}[(\delta_j+\beta_j)^3-\frac{3}{2}(\delta_j+\beta_j)^2+\frac{1}{2}(\delta_j+\beta_j)]}}{\mathrm{e}^{-\frac{1}{6\alpha^2}[\delta_j^3-\frac{3}{2}\delta_j^2+\frac{1}{2}\delta_j]}}$$
$$=\mathrm{e}^{-\frac{1}{6\alpha^2}\sum_j[3\delta_j^2\beta_j+3\delta_j\beta_j^2+\beta_j^3-\frac{3}{2}(\beta_j^2+2\beta_j\delta_j)+\frac{1}{2}\beta_j]}.$$

and

$$\prod_{j=1}^{p}\frac{\Gamma(\alpha+\delta_j+\rho h)}{\Gamma(\alpha+\beta_j+\delta_j+\rho h)}\to\frac{\mathrm{e}^{-\frac{1}{6\alpha^2}[(\delta_j+\rho h)^3-\frac{3}{2}(\delta_j+\rho h)^2+\frac{1}{2}(\delta_j+\rho h)]}}{\mathrm{e}^{-\frac{1}{6\alpha^2}[(\beta_j+(\delta_j+\rho h))^3-\frac{3}{2}(\beta_j+(\delta_j+\rho h))^2+\frac{1}{2}(\beta_j+(\delta_j+\rho h))]}}$$
$$=\mathrm{e}^{\frac{1}{6\alpha^2}\sum_j[\beta_j^3+3\beta_j^2(\delta_j+\rho h)+3\beta_j(\delta_j+\rho h)-\frac{3}{2}\beta_j^2-3\beta_j(\delta_j+\rho h)+\frac{1}{2}\beta_j]}.$$

Hence the additional factor corresponding to $k=2$ goes to the following:

$$E[v^h]\to e^{\frac{1}{2\alpha^2}\sum_j(\beta_j\rho^2h^2+\epsilon)},$$

where ϵ contains only a linear term of h. Now, combining the cases $k=0,1,2$ the h-th moment of v goes to the following:

$$E[v^h]\to e^{-\frac{1}{\alpha}\sum_j(\beta_j)\rho h+\frac{1}{2\alpha^2}(\sum_j\beta_j\rho^2h^2)+\frac{1}{2\alpha^2}(\epsilon)}.$$

If $h=it, i=\sqrt{(-1)}$ and if h is replaced by αh then we have the following:

$$\begin{aligned}E[v^{\alpha it}]&=E[e^{it\alpha\ln v}]=E[e^{-it(-\alpha\ln v)}]\\&\to e^{(\sum_j\beta_j)\rho(-it)-\frac{1}{2}(\sum_j\beta_j\rho^2)t^2}.\end{aligned}$$

Observe that for $k=3,\dots$ the additional factors in the asymptotic formula for gamma functions produce e^0 when $|\alpha|\to\infty$. Hence, for $|\alpha|\to\infty$ and all other parameters bounded, $\rho>0$

$$-\alpha\ln v-\rho(\sum_j\beta_j)\approx N_1(0,\rho^2\sum_j\beta_j)\Rightarrow$$

$$\frac{1}{\sqrt{\rho^2\sum_j\beta_j}}[-\alpha\ln v-\rho\sum_j\beta_j]\to N_1(0,1).$$

Hence, we have the following asymptotic normality Theorem

THEOREM 2. *Consider the real scalar variable defined in Eq. (8). For $|\alpha|\to\infty$ and all other parameters bounded with $\rho>0$ (real positive),*

$$\frac{1}{\sqrt{\rho^2\sum_j\beta_j}}[-\alpha\ln v-\rho\sum_j\beta_j]\to N_1(0,1),$$

a real scalar standard normal variable. Convergence in distribution is considered here.

5. CONCLUDING REMARKS

Observe that in the illustrative example of testing independence in a p-variate Gaussian considered in Section 2, the h-th moment in Eq. (3) is in fact coming from the h-th moment of the determinant of a real type-1 beta matrix. For example, take $\delta_j=-(\frac{p_1}{2}+\frac{j-1}{2})$ and $\beta_j=\frac{p_1}{2}$ for $j=1,\dots,p_2$, then the general moment structure in (8) agrees with that in Eq. (3). Therefore, in Eq. (3) the shape parameter in the asymptotic gamma is $\rho\sum_j\beta_j=\rho\sum_j\frac{p_1}{2}=\rho[\frac{1}{2}(p_1p_2)]$. Then, when $\rho=2$ the asymptotic chisqure has the degrees of freedom $2[\frac{1}{2}(p_1p_2)]=p_1p_2=$ the number of parameters restricted by the hypothesis there. Note that sections 3 and 4 results also hold good for the shape parameter being α_j, depending on j, and $|\alpha_j| \to \infty$ for each α_j, provided α_j can be taken as $\alpha_j = \alpha + \gamma_j$ where $|\alpha| \to \infty$ whereas γ_j is bounded. In a practical situation, a parameter can be coming from two sources where the part coming from one source can go to infinity whereas the other part is always bounded and the part going to infinity may be common to all α_j's. The h-th moment in (8) corresponds to the representation of the real scalar variable v in the form $v = v_1^\delta \dots v_p^\delta$ where $v_j, j = 1, \dots, p$ are mutually independently distributed real scalar type-1 beta variables. As seen before, in the complex case also the individual variables are real type-1 beta, the only difference being the $\frac{j-1}{2}$ in the real case is replaced by $j-1$ in the complex case.

If we had started with real and complex Wishart distributions, instead of matrix-variate gamma distributions, then the changes required are the following: In the real case, replace α by $\frac{n-1}{2}$ and B by $\frac{1}{2}\Sigma^{-1}, \Sigma > O$ where n is the sample size, $n > p$ where p is the order of the Wishart matrix. In the complex case, replace α by $n-1$ and B by $\Sigma^{-1}, \Sigma = \Sigma^* > O$. These results on the complex domain will become handy when tests of hypotheses, geometrical probability problems and other similar situations are developed in the complex domain in future. The methods used in deriving asymptotic results are novel and applicable to other similar problems.

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