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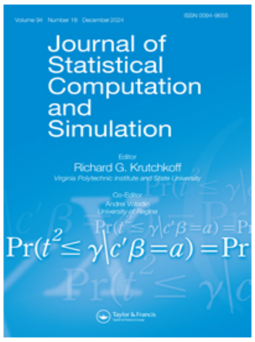

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Goodness-of-fit tests for inverse Gaussian distribution in the presence and absence of censoring

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ABSTRACT

In this article, we use the fixed point characterization for inverse Gaussian distribution to develop goodness of fit tests for the same. First, we propose a test for inverse Gaussian distribution when the data is complete. We then discuss, how the test procedure can be modified to incorporate right-censored observations. We use *U*-statistics theory to develop the test statistic. The large sample behaviour of the proposed test statistics for both uncensored and censored data are studied. We conduct extensive Monte Carlo simulation studies to validate the finite sample behaviour of the proposed tests. The practical usefulness of the tests is illustrated using real data sets. We also propose a new jackknife empirical likelihood ratio test for the inverse Gaussian distribution with unit parameters.

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Goodness-of-fit test; inverse Gaussian distribution; Stein's identity; right censoring ; *U*-statistics

1. Introduction

The lifetime data can be analysed using either parametric or non-parametric approaches. If the data can be assured to follow a known lifetime distribution, the parametric approach yields better results than the non-parametric approach. The goodness of fit tests are employed to test whether the data follows a specific lifetime model. The inverse Gaussian (IG) distribution is an important parametric model used for the analysis of lifetime data. A positive random variable X is said to follow an IG distribution, with parameters μ and θ, if the density function is given by

$$f(x) = \sqrt{\frac{\theta}{2\pi x^3}} \exp\left(-\frac{\theta(x-\mu)^2}{2\mu^2 x}\right); \quad x > 0, \ \ \mu, \theta > 0. \tag{1}$$

Here $\mu > 0$ represents the mean and $\theta > 0$ is the shape parameter of the distribution. Inverse Gaussian originates, as the first passage time of Brownian motion with positive drift, which later found applications in various fields. For example, IG distribution has been used in modelling data from stock market prices, submicroscopic particles, biology, hydrology, meteorology, labour relation, and reliability analysis among others [1,2]. The lifetime data modelling with IG distribution is extensively studied by Chhikara and Folks

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[3] and further explored by Bhattacharyya and Fries [4] and Whitmore [5]. An important property of IG distribution is that it is closed under scale transformation as $X \sim \mathrm{IG}(\mu, \theta)$, implies $cX \sim \mathrm{IG}(c\mu, c\theta), c > 0$.

The extensive use of IG distribution in the modelling of lifetime data motivates researchers to develop goodness of fit tests for IG distribution. Some earlier tests for the same can be found in O'Reilly and Rueda [6] and Pavur et al. [7]. Later Ducharme [8] provided a smooth, consistent test statistic for testing the IG distribution. They also discussed the test statistic in right-censored case without illustration. A test statistic based on the empirical Laplace transform for testing IG distribution is developed by Henze and Klar [9]. Vexler et al. [10] proposed an empirical likelihood ratio test for IG distribution. A comprehensive study of goodness of fit tests using R packages can be found in Rayner et al. [11], which includes IG distribution as well. It is well known that lifetime data suffers from censoring, quite often. IG distribution is also used in literature, as a model to explore lifetime data subjected to right censoring [1,5,12,13]. However, most of the tests, except Ducharme [8] considered the complete data. Even though Ducharme [8] addressed the problem of censoring, their test procedures are very complicated. The goodness of fit tests developed for complete data can not be directly applied to the right-censored samples. One way to tackle this problem is to use the pseudo-complete data generation method proposed by Balakrishnan et al. [14] or other similar techniques. Since these methods do not allow us to use the original data, information loss may occur. To fill this gap, we develop an easy-to-implement goodness of fit test for IG distribution, incorporating right-censored observations. Our test is based on a characterization that uses Stein's type identity.

In 1972, Stein introduced a natural identity for a random variable whose distribution belongs to an exponential family [15]. Specifically, let X be a continuous random variable with finite mean μ and variance σ^2. Let $c(x)$ be a continuous function having first derivative, then X has a normal distribution with mean μ and variance σ^2 if and only if

$$E(c(X)(X-\mu)) = \sigma^2 E(c'(X)),$$

provided the above expectation exists. This identity is known in literature as Stein's identity or Stein's lemma. Stein's identity and its applications in inferential procedures have been extensively studied in the literature. The approximations of normal, Poisson, exponential, and geometric distributions using Stein's method are discussed in Ross [16]. Stein's type identity for a general class of probability distributions and related characterizations see Kattumannil [17], Kattumannil and Tibiletti [18] and Kattumannil and Dewan [19]. Using Stein's type identity Betsch and Ebner [20] developed fixed point characterizations for continuous distributions.

Recently, using fixed point characterization, Sreedevi and Kattumannil [21], Vaisakh et al. [22] and Vaisakh et al. [23] proposed goodness of fit tests based on U-statistics for uniform, Rayleigh and gamma distributions, respectively, for complete data as well as right-censored data. We use Stein's type identity for IG distribution, to develop a goodness of fit test for IG distribution. Then we discuss, how the testing procedure can be extended to incorporate right-censored observations.

The rest of the article is organized as follows. In Section 2, Using the Stein's type identity for IG distribution, we develop a U-statistic-based goodness of fit test for IG distribution with shape parameter θ, when we have uncensored data. We also propose a jackknife empirical likelihood ratio test for the inverse Gaussian distribution when $\theta = 1$.

In Section 3, we discuss how to modify the test statistic to accommodate right-censored observations. The asymptotic properties of the test statistics are studied in detail. Extensive Monte Carlo simulation studies are carried out in Section 4 to assess the finite sample performance of the proposed tests in the presence and absence of censoring. The practical applicability of the proposed tests are illustrated using real-life datasets in Section 5. Finally, we conclude the study with a discussion on future works in Section 6.

2. Test statistic: complete data

In this Section, we propose a new goodness of fit test to assess whether the data follows an IG distribution with shape parameter θ. We use the following characterization theorem stated in Betsch and Ebner [20] for developing the test statistic.

Theorem 1: *If $f(.)$ is a probability density function with the support $[0,\infty)$, and X is a real-valued random variable which is continuously differentiable on $[0,\infty)$ with $\int_0^\infty x|f'(x)|\,\mathrm{d}x < \infty$, then X is said to follow the distribution function $F(x)$ with density function $f(x)$ if and only if, $F(t) = E[-\frac{f'(X)}{f(X)}\min\{X,t\}]$.*

In view of Theorem 1, a random variable $X \sim \mathrm{IG}(1,\theta)$ if and only if

$$F(t) = E\left[\frac{1}{2}\left(\theta + \frac{3}{X} - \frac{\theta}{X^2}\right)\min\{X,t\}\right], \quad t > 0. \tag{2}$$

2.1. U-statistics-based test

Based on a random sample $X_1,\ldots,X_n$ from F, we are interested in testing the null hypothesis

$$H_0 : F \text{ follows inverse Gaussian distribution}$$

against

$$H_1 : F \text{ does not follow inverse Gaussian distribution.}$$

For testing the above hypothesis first, we define a departure measure that discriminates between the null and the alternative hypothesis. In view of (2), we consider the departure measure

$$\Delta(F) = \int_0^\infty \left(E\left[\frac{1}{2}\left(\theta + \frac{3}{X} - \frac{\theta}{X^2}\right)\min\{X,t\}\right] - F(t)\right)\mathrm{d}F(t). \tag{3}$$

Our aim is to simplify $\Delta(F)$, in terms of expectations of the function of random variables. Consider

$$\begin{aligned}
\Delta(F) &= \int_0^\infty \left(E\left[\frac{1}{2}\left(\theta + \frac{3}{X} - \frac{\theta}{X^2}\right)\min\{X,t\}\right] - F(t)\right) \mathrm{d}F(t) \\
&= \frac{\theta}{2}\int_0^\infty\int_0^\infty \min\{x,t\}\,\mathrm{d}F(x)\,\mathrm{d}F(t) + \frac{3}{2}\int_0^\infty\int_0^\infty \frac{1}{x}\min\{x,t\}\,\mathrm{d}F(x)\,\mathrm{d}F(t) \\
&\quad - \frac{\theta}{2}\int_0^\infty\int_0^\infty \frac{1}{x^2}\min\{x,t\}\,\mathrm{d}F(x)\,\mathrm{d}F(t) - \frac{1}{2} \\
&= \Delta_1 + \Delta_2 - \Delta_3 - \frac{1}{2}, \text{(say)}.
\end{aligned} \tag{4}$$

Now,

$$\Delta_1 = \frac{\theta}{2}\int_0^\infty\int_0^\infty \min\{x,t\}\,\mathrm{d}F(x)\,\mathrm{d}F(t) = \frac{\theta}{2}E[\min\{X_1,X_2\}], \tag{5}$$

and

$$\begin{aligned}
\Delta_2 &= \frac{3}{2}\int_0^\infty\int_0^\infty \frac{1}{x}\min\{x,t\}\,\mathrm{d}F(x)\,\mathrm{d}F(t) \\
&= \frac{3}{2}\left[\int_0^\infty\int_x^\infty \mathrm{d}F(t)\,\mathrm{d}F(x) + \int_0^\infty\int_0^\infty \frac{t}{x}I(t<x)\,\mathrm{d}F(x)\,\mathrm{d}F(t)\right] \\
&= \frac{3}{4} + \frac{3}{2}E\left[\frac{X_2}{X_1}I(X_2<X_1)\right],
\end{aligned} \tag{6}$$

where $I(A)$ denotes the indicator function of a set A. Also

$$\begin{aligned}
\Delta_3 &= \frac{\theta}{2}\int_0^\infty\int_0^\infty \frac{1}{x^2}\min\{x,t\}\,\mathrm{d}F(x)\,\mathrm{d}F(t) \\
&= \frac{\theta}{2}\left[\int_0^\infty \frac{1}{x}\int_x^\infty \mathrm{d}F(t)\,\mathrm{d}F(x) + \int_0^\infty\int_0^\infty \frac{t}{x^2}I(t<x)\,\mathrm{d}F(x)\,\mathrm{d}F(t)\right] \\
&= \frac{\theta}{4}E\left[\frac{1}{\min\{X_1,X_2\}}\right] + \frac{\theta}{2}E\left[\frac{X_2}{X_1^2}I(X_2<X_1)\right].
\end{aligned} \tag{7}$$

Substituting (5)–(7) in (4), we obtain

$$\begin{aligned}
\Delta(F) &= \frac{\theta}{2}E\left[\min\{X_1,X_2\} - \frac{1}{2\min\{X_1,X_2\}}\right] + \frac{3}{2}E\left[\frac{X_2}{X_1}I(X_2<X_1)\right] \\
&\quad - \frac{\theta}{2}E\left[\frac{X_2}{X_1^2}I(X_2<X_1)\right] + \frac{1}{4}.
\end{aligned} \tag{8}$$

Now we use U-statistics theory to develop the test statistic. Define a symmetric kernel $h_1(X_1,X_2) = \min\{X_1,X_2\} - \frac{1}{2\min\{X_1,X_2\}}$. Then a U-statistic defined as

$$U_1 = \frac{1}{\binom{n}{2}}\sum_{i=1}^{n}\sum_{j=1,j<i}^{n} h_1(X_i,X_j)$$

is an unbiased estimate for $E[\min\{X_1, X_2\} - \frac{1}{2\min\{X_1,X_2\}}]$. Consider another symmetric kernel $h_2(X_1, X_2) = \frac{1}{2}[\frac{X_1}{X_2} I(X_1 < X_2) + \frac{X_2}{X_1} I(X_2 < X_1)]$, then a U-statistic defined by

$$U_2 = \frac{1}{\binom{n}{2}} \sum_{i=1}^{n} \sum_{j=1, j<i}^{n} h_2(X_i, X_j) \tag{9}$$

is an unbiased estimate for $E[\frac{X_2}{X_1} I(X_2 < X_1)]$.

Let $h_3(X_1, X_2) = \frac{1}{2}[\frac{X_1}{X_2^2} I(X_1 < X_2) + \frac{X_2}{X_1^2} I(X_2 < X_1)]$, then a U-statistic defined by

$$U_3 = \frac{1}{\binom{n}{2}} \sum_{i=1}^{n} \sum_{j=1, j<i}^{n} h_3(X_i, X_j) \tag{10}$$

is an unbiased estimate for $E[\frac{X_2}{X_1^2} I(X_2 < X_1)]$.

Let $\hat{\theta} = (\frac{1}{n} \sum_{i=1}^{n} (\frac{1}{X_i} - 1))^{-1}$ is an unbiased and consistent estimator of θ. Then, the test statistic $\hat{\Delta}$ is given by

$$\hat{\Delta} = \frac{\hat{\theta}}{2} U_1 + \frac{3}{2} U_2 - \frac{\hat{\theta}}{2} U_3 + \frac{1}{4}. \tag{11}$$

We reject the null hypothesis H_0, against the alternative hypothesis H_1 for large values of $\hat{\Delta}$. We now study the asymptotic properties of the statistic $\hat{\Delta}$. Since U_1 and U_2 are consistent estimators, we have $\hat{\Delta}$ converges in probability to Δ as $n \to \infty$.

Next, we find the asymptotic distribution of the test statistics. Define

$$\hat{\Delta}' = \frac{\theta}{2} U_1 + \frac{3}{2} U_2 - \frac{\theta}{2} U_3 + \frac{1}{4}.$$

Since $\hat{\theta}$ is a consistent estimator of θ, then by Slutsky's theorem the asymptotic distributions of $\sqrt{n}(\hat{\Delta} - \Delta)$ and $\sqrt{n}(\hat{\Delta}' - E(\Delta'))$ are the same. Also as $n \to \infty$, $\sqrt{n}(\hat{\Delta} - \Delta)$ converges in distribution to normal random variable with mean 0 and variance σ^2, where $\sigma^2 = \mathrm{Var}(E[h(X_1, X_2)|X_1])$ and $h(X_1, X_2)$ is a symmetric kernel of degree 2 given by

$$\begin{aligned} h(X_1, X_2) = \frac{1}{2} \Bigg[& 2\min\{X_1, X_2\} - \frac{1}{\min\{X_1, X_2\}} + \frac{X_1}{X_2} I(X_1 < X_2) + \frac{X_2}{X_1} I(X_2 < X_1) \\ & - \frac{X_1}{X_2^2} I(X_1 < X_2) - \frac{X_2}{X_1^2} I(X_2 < X_1) \Bigg]. \end{aligned} \tag{12}$$

To calculate σ^2, consider

$$\begin{aligned} E[\min\{X_1, X_2\}|X_1 = x] &= E[xI(x < X_2) + X_2 I(X_2 \le x)] \\ &= x \int_x^{\infty} \mathrm{d}F(y) + \int_0^x y \, \mathrm{d}F(y) \\ &= x\bar{F}(x) + \int_0^x y \, \mathrm{d}F(y). \end{aligned} \tag{13}$$

Also,

$$E\left[\frac{1}{\min\{X_1,X_2\}}|X_1=x\right]=\frac{1}{x}\bar{F}(x)+\int_0^x y\,\mathrm{d}F(y)$$

$$E\left[\frac{X_1}{X_2}I(X_1<X_2)|X_1=x\right]=x\int_x^{\infty}\frac{1}{y}\,\mathrm{d}F(y)$$

$$E\left[\frac{X_2}{X_1}I(X_2<X_1)|X_1=x\right]=\frac{1}{x}\int_0^x y\,\mathrm{d}F(y)$$

$$E\left[\frac{X_1}{X_2^2}I(X_1<X_2)|X_1=x\right]=x\int_x^{\infty}\frac{1}{y^2}\,\mathrm{d}F(y)$$

$$E\left[\frac{X_2}{X_1^2}I(X_2<X_1)|X_1=x\right]=\frac{1}{x^2}\int_0^x y\,\mathrm{d}F(y).$$

Then, we have

$$\begin{aligned}\sigma^2 = Var\Bigg[&\frac{\theta}{2}\left(X\bar{F}(X)+\int_0^X y\,\mathrm{d}F(y)\right)-\frac{\theta}{4}\left(\frac{1}{X}\bar{F}(X)+\int_0^X y\,\mathrm{d}F(y)\right)\\ &+\frac{3}{4}\left(X\int_X^{\infty}\frac{1}{y}\,\mathrm{d}F(y)+\frac{1}{X}\int_0^X y\,\mathrm{d}F(y)\right)\\ &-\frac{\theta}{4}\left(X\int_X^{\infty}\frac{1}{y^2}\,\mathrm{d}F(y)-\frac{1}{X^2}\int_0^X y\,\mathrm{d}F(y)\right)\Bigg]. \end{aligned} \tag{14}$$

Note that $\Delta=0$ under null hypothesis H_0. Hence, under H_0, as $n\to\infty$, $\sqrt{n}\hat{\Delta}$ converges in distribution to a normal random variable with mean zero and variance σ_0^2, where σ_0^2 is the value of σ^2 evaluated under the null hypothesis.

Let $\hat{\sigma}_0^2$ be a consistent estimator of σ_0^2, then at a chosen significance level, α, we reject H_0 against H_1 if

$$\frac{\sqrt{n}|\hat{\Delta}|}{\hat{\sigma}_0^2}>Z_{\frac{\alpha}{2}},$$

where Z_α is the upper α-percentile point for the standard normal distribution. Finding a consistent estimator of the null variance σ_0^2 is difficult. Hence we find the critical region of the proposed test using Monte Carlo simulation. We determine lower (c_1) and upper (c_2) quantiles in such a way that $P(\hat{\Delta}<c_1)=P(\hat{\Delta}>c_2)=\alpha/2$. The finite sample behaviour of the test is evaluated through an extensive Monte Carlo simulation study. The results of the simulation study are reported in Section 4.

Remark 2.1: We can note that, the assumption $\mu=1$, does not make any impact on the testing procedure, since the family of inverse Gaussian distribution is closed under scale transformations as $X\sim \mathrm{IG}(\mu,\theta)$, implies $\frac{X}{\mu}\sim \mathrm{IG}(1,\frac{\theta}{\mu})$.

2.2. Jackknife Empirical Likelihood (JEL) based test

We also develop a JEL based goodness-of-fit test for IG distribution with unit parameters. For developing JEL based test, consider the jackknife pseudo values corresponding to the test statistic given in (11). The jackknife pseudo values, $\beta_i,\ i = 1, 2, \ldots, n$ are defined as

$$\beta_i = n\hat{\Delta} - (n-1)\hat{\Delta}_i, \quad i = 1, 2, \ldots, n$$

where $\hat{\Delta}_i$ is the value of the test statistic by deleting the i^{th} observation from the sample $X_1, \ldots, X_n$. Let $q = (q_1, \ldots, q_n)$ be a probability vector, then $\prod_{i=1}^{n} q_i$ subject to $\sum_{i=1}^{n} q_i = 1$ attains its maximum value at n^{-n} at $q_i = \frac{1}{n}$. Then the JEL ratio for testing the IG distribution with parameters $(1, 1)$, based on the departure measure $\Delta(F)$ is defined as

$$l(\Delta) = \max\left\{\prod_{i=1}^{n} nq_i; \sum_{i=1}^{n} q_i = 1, \sum_{i=1}^{n} q_i\beta_i = 0\right\}.$$

Then using the Lagrange multipliers method, we obtain q_i as

$$q_i = \frac{1}{n(1+\nu\beta_i)},$$

and ν satisfies

$$\nu = \frac{1}{n}\sum_{i=1}^{n} \frac{\beta_i}{1+\nu\beta_i} = 0,$$

provided

$$\min_{1\le k\le n} \beta_k < \hat{\Delta} < \min_{1\le k\le n} \beta_k.$$

For more details on this, see Jing et al. [24]. Hence the jackknife empirical log-likelihood ratio is given by

$$\ln\ l(\Delta) = -\ln\sum_{i=1}^{n} \ln(1+\nu\beta_i).$$

Now we reject the null hypothesis against the alternate hypothesis for large values of $\ln\ l(\Delta)$. To construct the critical region of the JEL-based test, we find the asymptotic null distribution of the jackknife empirical log-likelihood ratio. Then using Wilk's theorem, we reject the null hypothesis against the alternate at a significance level, α, if

$$-2\ \ln\ l(\Delta) > \chi^2_{1,\alpha},$$

where $\chi^2_{1,\alpha}$ is the upper α-percentile point of the χ^2 distribution with one degree of freedom.

3. Test statistic: censored data

Next we discuss how the right-censored observations can be incorporated in the proposed testing method. Consider the right-censored data (Y, δ), with $Y = \min(X, C)$ and $\delta = I(X \le C)$, where C is the censoring time. We assume censoring times and lifetimes are independent. Now we are interested to test the hypothesis discussed in Section 2 based on n independent and identical observation $\{(Y_i, \delta_i), 1 \le i \le n\}$. As we developed the test based on U-statistics for right-censored data, we used the same departure measure $\Delta(F)$ given in (1). For that purpose, we consider (8) given in Section 2

$$\Delta(F) = \frac{\theta}{2}E\left[\min(Y_1, Y_2) - \frac{1}{2\min(Y_1, Y_2)}\right] + \frac{3}{2}E\left[\frac{Y_2}{Y_1}I(Y_2 < Y_1)\right]$$
$$- \frac{\theta}{2}E\left[\frac{Y_2}{Y_1}I(Y_2 < Y_1)\right] + \frac{1}{4}.$$

To develop the test statistic for right-censored case, we estimate each quantity in $\Delta(F)$ using U-statistics for right-censored data [25].

An estimator of $E(\min(Y_1, Y_2) - \frac{1}{\min(Y_1,Y_2)}$ is given by

$$\widehat{U}_{1c} = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j<i;j=1}^{n}\frac{\left(\min(Y_i, Y_j) - 1/\min(Y_i, Y_j)\right)\delta_i\delta_j}{\widehat{K}_c(Y_i)\widehat{K}_c(Y_j)}, \tag{15}$$

provided $\widehat{K}_c(Y_i) > 0$ and $\widehat{K}_c(Y_j) > 0$, with probability 1 and $\widehat{K}_c$ is the Kaplan-Meier estimator of K_c, the survival function of C. Again, an estimator of $E[\frac{Y_2}{Y_1}I(Y_2 < Y_1)]$ is given by

$$\widehat{U}_{2c} = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j<i;j=1}^{n}\frac{(\frac{Y_i}{Y_j}I(Y_i < Y_j) + \frac{Y_j}{Y_i}I(Y_j < Y_i))\delta_i\delta_j}{\widehat{K}_c(Y_i)\widehat{K}_c(Y_j)}. \tag{16}$$

Similarly, an estimator of θ in the right-censored case is given by

$$\theta = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{\delta_i}{Y_i\widehat{K}_c(Y_i)} - 1\right)^{-1}. \tag{17}$$

Using the estimators given in Eqs. (15)–(17), we obtain the test statistic as

$$\widehat{\Delta}_c = \frac{\widehat{\theta}}{2}\widehat{U}_{1c} + \frac{2}{3}\widehat{U}_{2c} - \frac{\widehat{\theta}}{2}\widehat{U}_{1c} + \frac{1}{4}. \tag{18}$$

Hence in the right-censored case, we reject H_0 in favour of H_1 for large values of $\widehat{\Delta}_c$.

To obtain the limiting distribution of $\widehat{\Delta}_c$, let $N_i^c(t) = I(Y_i \le t, \delta_i = 0)$ be the counting process corresponds to the censoring variable C_i. Denote $R_i(t) = I(Y_i \ge t)$. Also let σ_c^2 be the hazard rate of C. The martingale associated with this counting process $N_i^c(t)$ is given by

$$M_i^c(t) = N_i^c(t) - \int_0^t R_i(u)\sigma_c^2(u)\,\mathrm{d}u.$$

Let $G(x, y) = P(X_1 \le x, Y_1 \le y, \delta = 1)$, $x \in \mathcal{X}$, $\bar{H}(t) = P(Y_1 > t)$ and

$$w(t) = \frac{1}{\bar{H}(t)}\int_{\mathcal{X}\times[0,\infty)}\frac{h_1(x)}{K_c(y-)}I(y > t)\,\mathrm{d}G(x, y),$$

where $h_1(x) = E(h(X_1, X_2)|X_1 = x)$. The proof of next result follows from Theorem 1 of Datta et al. [25] for a particular choice of the kernel

$$h(X_1, X_2) = \frac{\theta}{2}\left[\min(X_1, X_2) - \frac{1}{2\min(X_1, X_2)}\right] + \frac{3}{2}\left[\frac{X_2}{X_1} I(X_2 < X_1)\right] - \frac{\theta}{2}\left[\frac{X_2}{X_1} I(X_2 < X_1)\right] + \frac{1}{4}.$$

Theorem 2: *Let*

$$h_1(x) = \frac{\theta}{2} E\left[\min(x, Y_2) - \frac{1}{2\min(x, Y_2)}\right] + \frac{3}{2} E\left[\frac{Y_2}{x} I(Y_2 < x)\right] - \frac{\theta}{2} E\left[\frac{Y_2}{x} I(Y_2 < x)\right] + \frac{1}{4}.$$

Suppose the conditions

$$E\left[\frac{\theta}{2} E\left[\min(Y_1, Y_2) - \frac{1}{2\min(Y_1, Y_2)}\right] + \frac{3}{2} E\left[\frac{Y_2}{Y_1} I(Y_2 < Y_1)\right] - \frac{\theta}{2} E\left[\frac{Y_2}{Y_1} I(Y_2 < Y_1)\right] + \frac{1}{4}\right]^2 < \infty,$$

$\int_{\mathcal{X}\times[0,\infty)} \frac{h_1^2(x)}{K_c^2(y)} dG(x, y) < \infty$ *and* $\int_0^\infty w^2(t)\sigma_c^2(t)dt < \infty$ *holds.*

As $n \to \infty$, $\sqrt{n}(\widehat{\Delta}_c - \Delta(F))$ *converges in distribution to Gaussian random variable with mean zero and variance* $4\sigma_c^2$, *where* σ_c^2 *is given by*

$$\sigma_c^2 = Var\left(\frac{h_1(X)\delta_1}{K_c(Y_1-)} + \int w(t) dM_1^c(t)\right).$$

Next we find an estimator of σ_c^2 using the reweighed techniques. An estimator of σ_c^2 is given by

$$\widehat{\sigma}_c^2 = \frac{4}{(n-1)} \sum_{i=1}^{n} (V_i - \bar{V})^2,$$

where

$$V_i = \frac{\widehat{h}_1(X_i)\delta_i}{\widehat{K}_c(Y_i)} + \widehat{w}(X_i)(1-\delta_i) - \sum_{j=1}^{n} \frac{\widehat{w}(X_i) I(X_i > X_j)(1-\delta_i)}{\sum_{i=1}^{n} I(X_i > X_j)},$$

$$\bar{V} = \frac{1}{n}\sum_{i=1}^{n} V_i, \quad \widehat{h}_1(X) = \frac{1}{n}\sum_{i=1}^{n} \frac{h(X, Y_i)\delta_i}{\widehat{K}_c(Y_i-)}, \quad R(t) = \frac{1}{n}\sum_{i=1}^{n} I(Y_i > t)$$

and

$$\widehat{w}(t) = \frac{1}{R(t)} \sum_{i=1}^{n} \frac{\widehat{h}_1(X_i)\delta_i}{\widehat{K}_c(Y_i)} I(X_i > t).$$

Let $\widehat{\sigma}_{0c}^2$ be the value of $\widehat{\sigma}_c^2$ evaluated under H_0. Under right censored situation, we reject the null hypothesis H_0 against the alternative H_1 at a significance level α, if

$$\frac{\sqrt{n}|\widehat{\Delta}_c|}{\widehat{\sigma}_{0c}} > Z_{\alpha/2}.$$

The results of the Monte Carlo simulation which assess the finite sample performance of the test is also reported in Section 4.

4. Simulation study

The finite sample performance of the proposed test procedures is evaluated through extensive Monte Carlo simulation studies using R software (version 4.1.3). We compare the empirical type I error and power of our test with the existing tests for complete data, to show the competitiveness of the proposed tests, in both uncensored and censored cases. The power of the tests is evaluated against various alternatives, which are the commonly employed parametric lifetime models. In the censored case, we evaluate the empirical power at various censoring percentages, to depict the effect of censoring on the proposed test.

4.1. Uncensored case

We find the empirical type I error and empirical power of the proposed test and other competitive tests to conduct a comprehensive study. We simulate random samples of different sizes ($n = 10, 20, 30, 40, 50$) from IG distribution with parameter value $\theta = 1$ or 2. We then calculate the test statistic value for the simulated random samples and based on the value of it, we accept or reject the null hypothesis. This procedure is repeated ten thousand times and the proportion of times, the null hypothesis rejected is observed. This value gives the empirical type I error of the proposed test.

We follow a similar procedure to calculate the empirical power of the proposed test. To find the empirical power, lifetime random variables are generated from different choices of alternatives including Weibull, gamma, log-normal, Pareto, and half-normal distributions. The choices of alternative distributions along with the corresponding cumulative distribution functions (CDF), employed in this study are listed in Table 1. To calculate the empirical power of the test, first, we generate lifetime data from the desired alternative and calculate the test statistic. We then estimate the maximum likelihood estimator $\hat{\theta}$ of the parameter θ, of the generated data. Next, to compute the simulated critical points, we generate a sample of size n from inverse Gaussian distribution with parameter $\hat{\theta}$, 10,000 times. Based on the test statistic value and the simulated critical points, we determine whether the null hypothesis is to be rejected or not. The entire procedure is repeated 10000 times and the empirical power is computed as the proportion of rejections of the test.

We test the competitiveness of our test with the classical tests for goodness of fit include Kolmogorov–Smirnov (KS), Cramer–von Mises (CvM) and Anderson–Darling (AD) tests. Moreover, we compare the competitiveness of our test with the another three tests proposed specifically for testing IG distribution, given by Henze and Klar [9] and González-Estrada and Villaseñor [26]. In Henze and Klar [9], they developed a test statistic for goodness of fit for IG distribution as the weighted integrals over the squared modulus of a measure of

Table 1. Choices of alternative distributions with the corresponding CDFs.

Distribution	CDF
Weibull	$F_1(x) = 1 - e^{(-\frac{x}{\lambda})^k}, x > 0, k, \lambda > 0$
Rayleigh	$F_2(x) = 1 - \exp\{\frac{-x^2}{2\lambda^2}\}, x > 0, \lambda > 0$
Chi-square	$F_3(x) = \frac{1}{\Gamma(\frac{k}{2})}\gamma(\frac{k}{2}, \frac{x}{2}), x > 0, k > 0$ and $\gamma(k, \frac{x}{\lambda})$ is the lower incomplete gamma function
Gamma	$F_4(x) = \frac{1}{\Gamma(k)}\gamma(k, \frac{x}{\lambda}), x > 0, k, \lambda > 0$ and $\gamma(k, \frac{x}{\lambda})$ is the lower incomplete gamma function
Log-normal	$F_5(x) = \Phi(\frac{\ln x - \mu}{\lambda}), x > 0, -\infty < \mu < \infty, \lambda > 0$ where $\Phi(x)$ is the cumulative distribution function of the standard normal random variable
Half-normal	$F_6(x) = erf(\frac{x}{\lambda\sqrt{2}}), x > 0, \lambda > 0$, where erf is the error function
Pareto	$F_7(x) = (\lambda/x)^k x > 0, k, \lambda > 0$

deviation of the empirical distribution of given data from the family of inverse Gaussian laws, expressed by means of the empirical Laplace transform. A variance ratio test for the goodness of fit test for IG distribution is proposed in Henze and Klar [9] by exploiting the relation between gamma and IG distributions, while Villaseñor et al. [27] developed by transforming IG variables to normal variables and then by employing Shapiro-Wilk test for normality. Illustration of both these tests are explained in Gonzáalez-Estrada and Villaseñor[26]. We use the above three tests to compare the performance of the proposed test statistic in simulation studies. We denote the test by Henze and Klar [9] as 'HK' and the tests by Henze and Klar [9] and Villaseñor et al. [27] as 'EV1' and 'EV2' in the tables presenting simulation results. The expressions for the classical test statistics KS, CvM and AD are also summarized below.

The Kolmogorov–Smirnov test statistic is given by KS = $\max\{D^+, D^-\}$ where

$$D^+ = \max_{i=1,2,\ldots,n}\left(\frac{i}{n} - F_0(X_{(i)})\right) \quad \text{and} \quad D^- = \max_{i=1,2,\ldots,n}\left(F_0(X_{(i)}) - \frac{i-1}{n}\right);$$

the Cramer-von Mises test statistic is given by

$$CvM = \frac{1}{12n} + \sum_{i=1}^{n}\left(F_0(X_{(i)}) - \frac{2i-1}{2n}\right)^2;$$

and the Anderson-Darling test statistic is given by $AD = -n - S$, where

$$S = \sum_{i=1}^{n}\frac{2i-1}{n}[\ln F_0(X_{(i)}) + \ln(1 - F_0(X_{n+1-i}))],$$

where $F_0(.)$ is the specified distribution function.

The calculated empirical type I error of the proposed test and other competing tests are reported in Table 2. The same is visually represented in Figure 1.

From Table 2 and Figure 1, we can observe that as n increases, the test $\widehat{\Delta}$ stabilizes near the chosen level of significance for both $\alpha = 0.05$ and $\alpha = 0.01$. This observation is true for both choices of parameters 1 and 2. The tests HK, EV1, and EV2 exhibit similar behaviour. The classical tests KS, CvM, and AD provide type I error values slightly higher than the desired significance level for both choices of α.

The results of the simulation study for power comparison are tabulated in Tables 3 and 4. Table 3 gives the power of $\widehat{\Delta}$ for different alternatives at significance level $\alpha = 0.01$ and

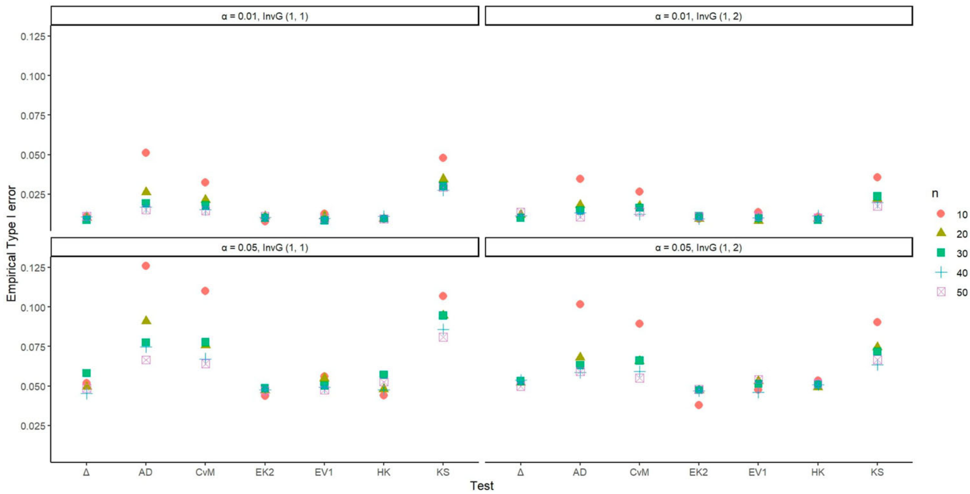

Figure 1. Plot of empirical type I error at $\alpha = 0.01$ and $\alpha = 0.05$.

Table 2. Comparison of empirical type I error at $\alpha = 0.01$ and $\alpha = 0.05$.

	n	α	$\hat{\Delta}$	HK	EV1	EV2	KS	CvM	AD
InvG (1, 1)	10	0.01	0.0110	0.0094	0.0128	0.0077	0.0478	0.0323	0.0510
	20		0.0101	0.0093	0.0112	0.0110	0.0345	0.0214	0.0262
	30		0.0088	0.0096	0.0085	0.0100	0.0301	0.0178	0.0191
	40		0.0106	0.0109	0.0099	0.0104	0.0276	0.0153	0.0168
	50		0.0110	0.0094	0.0097	0.0112	0.0293	0.0146	0.0153
InvG (1, 1)	10	0.05	0.0519	0.0441	0.0559	0.0437	0.1067	0.1102	0.1259
	20		0.0495	0.0481	0.0549	0.0474	0.0945	0.0759	0.0909
	30		0.0582	0.0570	0.0502	0.0485	0.0945	0.0778	0.0775
	40		0.0455	0.0475	0.0492	0.0476	0.0857	0.0672	0.0747
	50		0.0486	0.0526	0.0474	0.0472	0.0810	0.0641	0.0666
InvG (1, 2)	10	0.01	0.0109	0.0108	0.0137	0.0093	0.0355	0.0265	0.0347
	20		0.0121	0.0093	0.0082	0.0095	0.0220	0.0176	0.0182
	30		0.0101	0.0088	0.0099	0.0110	0.0237	0.0164	0.0146
	40		0.0113	0.0113	0.0100	0.0093	0.0198	0.0122	0.0134
	50		0.0138	0.0102	0.0111	0.0109	0.0176	0.0139	0.0105
InvG (1, 2)	10	0.05	0.0526	0.0535	0.0476	0.0380	0.0903	0.0894	0.1017
	20		0.0522	0.0493	0.0532	0.0474	0.0745	0.0657	0.0681
	30		0.0531	0.0509	0.0513	0.0474	0.0716	0.0660	0.0631
	40		0.0538	0.0508	0.0460	0.0470	0.0635	0.0593	0.0586
	50		0.0499	0.0515	0.0543	0.0481	0.0668	0.0550	0.0591

Table 4 represents the same when $\alpha = 0.05$. The results in Tables 3 and 4 are portrayed visually in Figures 2 and 3. Tables 3 and 4 and Figures 2 and 3 clearly show that, the newly developed test achieves a higher empirical power compared to all the other goodness of fit tests. Additionally, the power of all tests increases with sample size. Notably, the proposed test demonstrates high power even for small sample sizes, highlighting its efficiency. Furthermore, the proposed test attains the maximum power against the alternatives, Rayleigh, lognormal and Pareto distributions for the chosen parameter settings. Among the tests, the HK test exhibits the lowest power. In contrast, EV1 and EV2 deliver comparable results. The classical goodness-of-fit tests generally have low power for small sample sizes ($n = 10$),

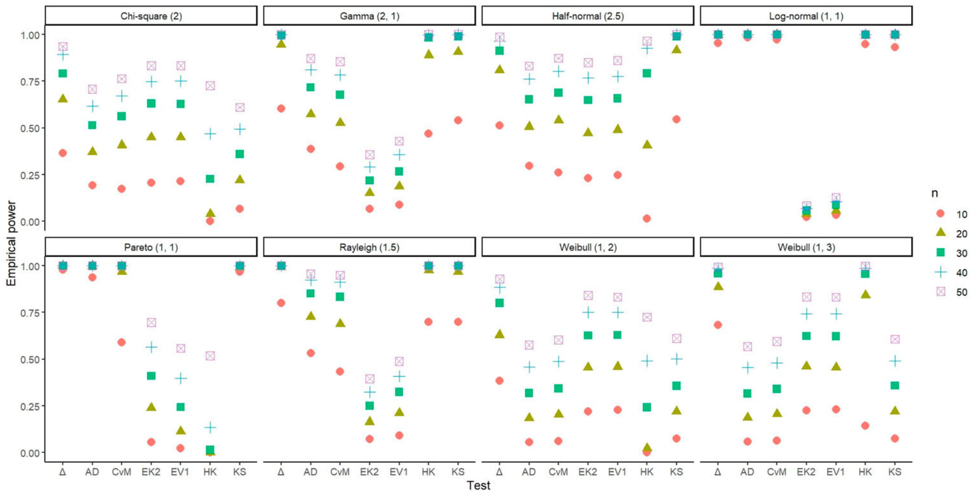

Figure 2. Plot of empirical power when $\alpha = 0.01$.

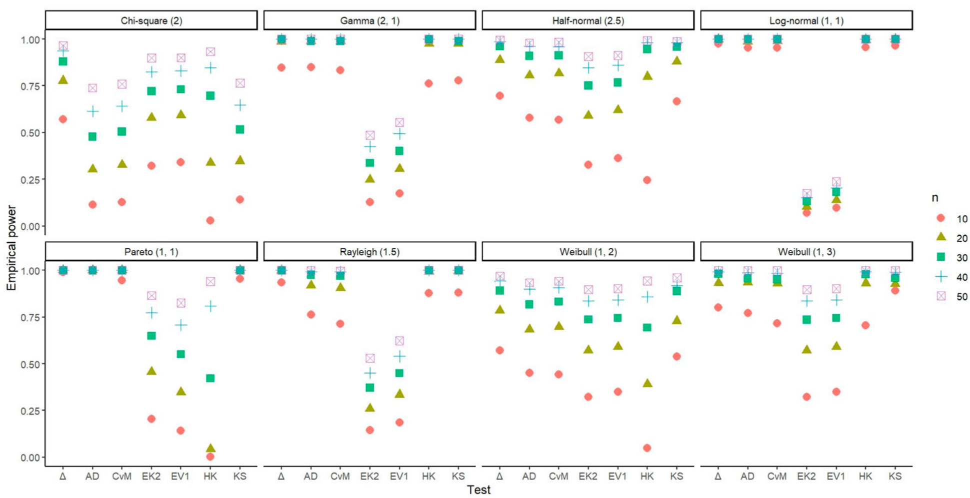

Figure 3. Plot of empirical power when $\alpha = 0.05$.

though their performance improves significantly with larger sample sizes. This could be attributed to the higher rejection rates under the null hypothesis associated with these classical tests. The proposed test consistently demonstrates robust power, even with small sample sizes, confirming its effectiveness.

4.2. Censored case

We carry out Monte Carlo simulation studies to calculate the empirical type I error and the empirical power of the test statistic proposed for right-censored data. We generate random samples of sizes $n = 50$, 75, 100, 200 and 400 to evaluate the empirical type I error

Table 3. Comparison of empirical power for different alternatives ($\alpha = 0.01$).

	n	$\hat{\Delta}$	HK	EV1	EVf2	KS	CvM	AD
Weibull (1,2)	10	0.3839	0.0008	0.2274	0.2208	0.0742	0.0621	0.0567
	20	0.6293	0.0235	0.4595	0.4557	0.2204	0.2045	0.1843
	30	0.7999	0.2424	0.6297	0.6269	0.3571	0.3438	0.3180
	40	0.8840	0.4924	0.7519	0.7507	0.5019	0.4876	0.4592
	50	0.9297	0.7253	0.8324	0.8409	0.6109	0.6032	0.5765
Weibull (1,3)	10	0.6825	0.1433	0.2305	0.2253	0.0747	0.0652	0.0579
	20	0.8860	0.8416	0.4546	0.4612	0.2202	0.2060	0.1879
	30	0.9584	0.9564	0.6230	0.6237	0.3586	0.3401	0.3158
	40	0.9878	0.9878	0.7440	0.7421	0.4914	0.4813	0.4560
	50	0.9939	0.9990	0.8314	0.8339	0.6078	0.5952	0.5684
Rayleigh (1.5)	10	0.8000	0.6986	0.0923	0.0736	0.6993	0.4328	0.5333
	20	0.9924	0.9749	0.2110	0.1639	0.9683	0.6887	0.7271
	30	0.9996	0.9972	0.3244	0.2504	0.9982	0.8341	0.8516
	40	1.0000	0.9998	0.4093	0.3248	0.9999	0.9133	0.9232
	50	1.0000	1.0000	0.4885	0.3958	1.0000	0.9493	0.9559
Chi-Square (2)	10	0.3654	0.0012	0.2141	0.2072	0.0665	0.1731	0.1915
	20	0.6525	0.0381	0.4507	0.4493	0.2203	0.4072	0.3699
	30	0.7911	0.2267	0.6271	0.6310	0.3594	0.5621	0.5137
	40	0.8928	0.4702	0.7516	0.7482	0.4946	0.6706	0.6156
	50	0.9353	0.7269	0.8323	0.8330	0.6099	0.7631	0.7071
Gamma (2, 1)	10	0.6020	0.4683	0.0895	0.0676	0.5404	0.2951	0.3868
	20	0.9444	0.8891	0.1869	0.1518	0.9076	0.5236	0.5732
	30	0.9942	0.9838	0.2672	0.2189	0.9883	0.6764	0.7168
	40	0.9996	0.9982	0.3564	0.2920	0.9990	0.7846	0.8103
	50	0.9998	0.9997	0.4291	0.3566	0.9998	0.8552	0.8706
Log-normal (1,1)	10	0.9531	0.9494	0.0344	0.0234	0.9303	0.9720	0.9823
	20	0.9979	0.9958	0.0571	0.0396	0.9995	0.9999	0.9999
	30	0.9999	0.9999	0.0881	0.0587	1.0000	1.0000	1.0000
	40	1.0000	1.0000	0.1051	0.0690	1.0000	1.0000	1.0000
	50	1.0000	1.0000	0.1286	0.0833	1.0000	1.0000	1.0000
Half-normal (2.5)	10	0.5140	0.0156	0.2468	0.2309	0.5447	0.2606	0.2957
	20	0.8094	0.4056	0.4894	0.4719	0.9152	0.5398	0.5058
	30	0.9122	0.7915	0.6580	0.6486	0.9898	0.6873	0.6525
	40	0.9653	0.9270	0.7764	0.7687	0.9989	0.8028	0.7615
	50	0.9857	0.9652	0.8599	0.8500	0.9999	0.8727	0.8321
Pareto (1, 1)	10	0.9791	0.0001	0.0240	0.0566	0.9675	0.5886	0.9372
	20	0.9994	0.0004	0.1123	0.2401	0.9892	0.9671	0.9988
	30	0.9999	0.0152	0.2431	0.4107	0.9999	0.9988	0.9999
	40	1.0000	0.1348	0.3972	0.5661	1.0000	1.0000	1.0000
	50	1.0000	0.5183	0.5582	0.6958	1.0000	1.0000	1.0000

and power of the proposed test in the presence of censoring. Lifetimes are generated from inverse Gaussian distribution with parameters $(1, 1)$ to calculate the empirical type I error. The computational procedures given in the absence of censoring to calculate empirical type I error/ power is used to calculate the same in the presence of censoring also. We consider different choices of alternatives, as in uncensored case for finding the empirical power. The percentage of censoring is chosen to be 20% or 40%, to examine the effect of censoring on the proposed test statistic. In all cases, the censoring random variable C is generated from an exponential distribution with parameter b, where b is chosen such that $P(T > C) = 0.2$ or 0.4. Re-weighting technique explained in Section 3 is used to estimate the variance of $\widehat{\Delta}_c$.

Table 4. Comparison of empirical power for different alternatives ($\alpha = 0.05$).

	n	$\hat{\Delta}$	HK	EV1	EV2	KS	CvM	AD
Weibull (1,2)	10	0.5705	0.0501	0.3502	0.3234	0.5388	0.4438	0.4498
	20	0.7848	0.3908	0.5901	0.5709	0.7283	0.6976	0.6837
	30	0.8915	0.6932	0.7452	0.7360	0.8886	0.8320	0.8166
	40	0.9427	0.8578	0.8409	0.8375	0.9187	0.9080	0.8989
	50	0.9687	0.9426	0.9021	0.8969	0.9595	0.9420	0.9338
Weibull (1,3)	10	0.7995	0.7056	0.3496	0.3225	0.8905	0.7161	0.7696
	20	0.9321	0.9306	0.5900	0.5714	0.9262	0.9304	0.9357
	30	0.9816	0.9785	0.7450	0.7358	0.9598	0.9515	0.9568
	40	0.9919	0.9915	0.8407	0.8370	0.9881	0.9840	0.9873
	50	0.9988	0.9985	0.9020	0.8968	0.9971	0.9979	0.9980
Rayleigh (1.5)	10	0.9343	0.8786	0.1857	0.1435	0.8810	0.7122	0.7612
	20	0.9982	0.9935	0.3345	0.2599	0.9940	0.9049	0.9191
	30	1.0000	0.9995	0.4497	0.3713	0.9999	0.9703	0.9757
	40	1.0000	1.0000	0.5397	0.4517	1.0000	0.9909	0.9929
	50	1.0000	1.0000	0.6228	0.5300	1.0000	0.9965	0.9969
Chi-Square (2)	10	0.5721	0.0303	0.3422	0.3216	0.1431	0.1281	0.1140
	20	0.7773	0.3391	0.5935	0.5797	0.3482	0.3284	0.3034
	30	0.8804	0.6972	0.7304	0.7218	0.5167	0.5049	0.4781
	40	0.9385	0.8463	0.8313	0.8262	0.6473	0.6420	0.6152
	50	0.9656	0.9326	0.8999	0.8983	0.7643	0.7602	0.7388
Gamma (2, 1)	10	0.8481	0.7622	0.1732	0.1277	0.7783	0.8339	0.8493
	20	0.9859	0.9749	0.3063	0.2486	0.9766	0.9886	0.9886
	30	0.9985	0.9984	0.4016	0.3375	0.9889	0.9897	0.9896
	40	1.0000	0.9997	0.4937	0.4257	1.0000	1.0000	1.0000
	50	1.0000	1.0000	0.5545	0.4867	1.0000	1.0000	1.0000
Log-normal (1,1)	10	0.9749	0.9558	0.0976	0.0694	0.9638	0.9531	0.9549
	20	0.9987	0.9958	0.1384	0.1025	0.9986	0.9898	0.9857
	30	0.9999	1.0000	0.1812	0.1320	1.0000	1.0000	1.0000
	40	1.0000	1.0000	0.2055	0.1527	1.0000	1.0000	1.0000
	50	1.0000	1.0000	0.2366	0.1760	1.0000	1.0000	1.0000
Half-normal (2.5)	10	0.6978	0.2459	0.3618	0.3283	0.6665	0.5686	0.5780
	20	0.8884	0.7993	0.6206	0.5897	0.8799	0.8158	0.8066
	30	0.9606	0.9453	0.7670	0.7509	0.9579	0.9129	0.9100
	40	0.9864	0.9802	0.8612	0.8482	0.9798	0.9600	0.9606
	50	0.9960	0.9921	0.9127	0.9064	0.9872	0.9834	0.9797
Pareto (1, 1)	10	0.9886	0.0015	0.1406	0.2061	0.9528	0.9468	0.9980
	20	0.9998	0.0435	0.3462	0.4566	0.9999	0.9999	1.0000
	30	1.0000	0.4225	0.5504	0.6489	1.0000	1.0000	1.0000
	40	1.0000	0.8100	0.7077	0.7733	1.0000	1.0000	1.0000
	50	1.0000	0.9407	0.8247	0.8646	1.0000	1.0000	1.0000

As mentioned earlier, the existing goodness of fit tests in literature, consider complete data only. Hence, to compare the efficiency of the proposed test, we generate pseudo-complete random sample using the method proposed by Balakrishnan et al. [14]. Following Balakrishnan et al. [14], to generate the pseudo-complete sample, for each Y_i with $\delta_i = 0$, a value $\hat{Y}_i$ is generated as $\hat{Y}_i = F_0^{-1}(\zeta_i)$, where $\zeta_i \sim U[F_0(C_i), 1)$, where $F_0(.)$ is the corresponding distribution function and C_i is the censoring time for $i = 1, 2, \ldots, n$. The corresponding values of censored observations in the original sample are then replaced by these generated values to obtain the pseudo-complete sample. Goodness of fit tests developed for complete sample can be then applied to the pseudo-complete sample.

To compare the proficiency of the proposed test for censored samples, we consider the same tests described in Section 4.2. We perform the simulation study for all the choices of alternatives, since the results are similar, we present the results for gamma, log-normal, Weibull and Rayleigh distributions only. First, we generate a censored sample from the chosen alternative with 20% or 40% of censored observations, while the censored observations are generated from an exponential distribution as mentioned above. The censored sample is then converted to pseudo-complete sample using the method explained above. Results of the simulation study when the censoring percentage is chosen as 20% are presented in Table 5 and the same for 40% censoring is shown in Table 6. In Tables 5 and 6, values corresponding to inverse Gaussian distribution give the empirical type I error and all the values corresponding to all other choices of alternatives give the empirical power of the test.

From Tables 5 and 6, it is evident that, as n increases, the empirical type I error of the test converges to the specified significance levels $\alpha = 0.01$ or $\alpha = 0.05$. Additionally, for all alternative distributions considered, the proposed test demonstrates superior power compared to tests based on the pseudo-complete sample. Specifically, the tests EV1 and EV2 exhibit lower power relative to others, while the HK test and the classical tests KS, CvM, and AD provide higher power than EV1 and EV2 but fall short of the performance of the newly proposed test. This pattern holds true across both significance levels and under various censoring scenarios. This can be explained by the fact that the proposed test incorporates the censored lifetimes as such in computing the test statistic, while the other tests based on pseudo-complete sample need to convert the censored lifetimes into observed lifetimes. The empirical power of the test increases with increase in n, while it shows a decreasing tendency with the increase in censoring percentage.

To depict the results of the simulation study, we plot the values of empirical type I error and power in Figures 4–6. It is evident from Figure 4 that, the empirical type I error is approaching the chosen significance level as n increases. Figures 5 and 6 clearly depict the supremacy of the proposed in test in terms of power, compared to the competitors.

5. Data analysis

We illustrate the applicability of the proposed test procedures using several real datasets in this Section.

Uncensored Case, Illustration 1

We consider the data on active repair times in hours for an airborne communication transceiver reported in Chhikara and Folks [3]. The data consists of 46 observations. Jayalath and Chhikara [13] used this data set to illustrate the Gibbs sampling approach for IG distribution in the presence of right censoring. We now standardize the data to apply the testing procedures discussed in Section 3, to test whether this dataset follows an IG distribution. We obtain the test statistic as $\hat{\Delta} = 0.1870$. Accordingly, we accept the null hypothesis that the data follows an inverse Gaussian distribution at both 1% (critical values are −0.0.5583 and 0.5583) and 5% (critical values are −0.4248 and 0.4248) level of significance.

Table 5. Comparison of empirical power for different alternatives : 20% censoring.

	n	α	$\hat{\Delta}$	HK	EV1	EV2	KS	CvM	AD
Inv Gauss (1,1)	50	0.01	0.0147	0.0088	0.0192	0.0199	0.0165	0.0178	0.0148
	75		0.0136	0.0092	0.0163	0.0181	0.0145	0.0151	0.0129
	100		0.0119	0.0121	0.0152	0.0168	0.0132	0.0142	0.0121
	200		0.0113	0.0118	0.0139	0.0150	0.0121	0.0129	0.0115
	400		0.0106	0.0110	0.0117	0.0141	0.0113	0.0115	0.0109
Inv Gauss (1,1)	50	0.05	0.0527	0.0537	0.0467	0.0460	0.0671	0.0652	0.0617
	75		0.0518	0.0530	0.0472	0.0592	0.0591	0.0598	0.0581
	100		0.0488	0.0525	0.0479	0.0577	0.0582	0.0562	0.0562
	200		0.0493	0.0519	0.0481	0.0565	0.0558	0.0551	0.0545
	400		0.0495	0.0511	0.0488	0.0534	0.0535	0.0540	0.0532
Gamma (2,1)	50	0.01	0.7823	0.7286	0.2248	0.2072	0.6993	0.5428	0.6533
	75		0.8482	0.7649	0.4586	0.4483	0.7683	0.6887	0.7271
	100		0.8921	0.7892	0.6221	0.6327	0.7982	0.8041	0.7816
	200		0.9282	0.8298	0.7518	0.7485	0.8999	0.8133	0.8532
	400		0.9521	0.8808	0.8345	0.8349	0.9213	0.8993	0.8656
Gamma (2,1)	50	0.05	0.7965	0.7512	0.5123	0.5336	0.7665	0.7311	0.7715
	75		0.8625	0.7891	0.6610	0.5639	0.8209	0.7872	0.7996
	100		0.8911	0.8267	0.7221	0.6527	0.8494	0.8221	0.8137
	200		0.9828	0.8701	0.8293	0.7985	0.8794	0.8307	0.9159
	400		0.9953	0.9169	0.8545	0.8349	0.9599	0.9201	0.8872
Log-normal (1,1)	50	0.01	0.8612	0.8183	0.1095	0.0676	0.7404	0.7915	0.8167
	75		0.9448	0.8891	0.1869	0.1896	0.8076	0.8461	0.8721
	100		0.9742	0.9439	0.2877	0.3889	0.8585	0.8962	0.9168
	200		0.9996	0.9772	0.3964	0.4520	0.9390	0.9546	0.9821
	400		1.0000	0.9998	0.5191	0.4961	0.9998	1.0000	1.0000
Log-normal (1,1)	50	0.05	0.9531	0.9494	0.1344	0.1024	0.9303	0.9422	0.9524
	75		0.9979	0.9958	0.2571	0.2398	0.9995	0.9999	0.9999
	100		0.9999	0.9999	0.4083	0.4087	1.0000	1.0000	1.0000
	200		1.0000	1.0000	0.4854	0.4990	1.0000	1.0000	1.0000
	400		1.0000	1.0000	0.5912	0.5632	1.0000	1.0000	1.0000
Weibull (3,3)	50	0.01	0.8145	0.5057	0.4465	0.4309	0.3447	0.3606	0.3857
	75		0.8498	0.6256	0.5394	0.4719	0.4652	0.5398	0.5058
	100		0.9125	0.7915	0.6580	0.6486	0.6198	0.6873	0.6525
	200		0.9659	0.9270	0.7764	0.7687	0.7289	0.7428	0.7615
	400		0.9971	0.9652	0.8599	0.8500	0.8599	0.8828	0.8722
Weibull (3,3)	50	0.05	0.9113	0.5801	0.5243	0.5266	0.4617	0.5286	0.5891
	75		0.9494	0.6281	0.6229	0.6012	0.6399	0.6872	0.6888
	100		0.9869	0.8152	0.7331	0.7410	0.6791	0.7911	0.8091
	200		0.9999	0.9448	0.8179	0.8068	0.7942	0.8522	0.8763
	400		1.0000	0.9883	0.8989	0.8901	0.9021	0.9265	0.9348
Rayleigh (1.5)	50	0.01	0.9492	0.9029	0.3440	0.3766	0.7672	0.5886	0.6384
	75		0.9795	0.9252	0.4608	0.4913	0.8382	0.6676	0.7590
	100		0.9997	0.9367	0.5731	0.5701	0.8890	0.7987	0.8677
	200		1.0000	0.9981	0.7172	0.7111	0.9387	0.8251	0.8801
	400		1.0000	1.0000	0.7952	0.8058	0.9843	0.9082	0.9356
Rayleigh (1.5)	50	0.05	0.9791	0.9283	0.4243	0.4566	0.9675	0.5886	0.6377
	75		0.9994	0.9288	0.5123	0.5405	0.9892	0.9671	0.8281
	100		0.9999	0.9557	0.6431	0.6507	0.9999	0.9988	0.8991
	200		1.0000	0.9992	0.7672	0.7861	1.0000	1.0000	0.9212
	400		1.0000	1.0000	0.8482	0.8358	1.0000	1.0000	0.9671

Uncensored Case, Illustration 2:

Here the data of age at death of 141 Roman era Egyptian mummies is considered. The data can be found in *egypt* data in library *univariateML* of statistical package R. After standardizing the data, we obtain the test statistic as $\hat{\Delta} = 0.3245$. Accordingly, we reject

Table 6. Comparison of empirical power for different alternatives : 40% censoring.

	n	α	$\hat{\Delta}$	HK	EV1	EV2	KS	CvM	AD
Inv Gauss (1,1)	50	0.01	0.0188	0.0201	0.0222	0.0009	0.0212	0.0209	0.0198
	75		0.0167	0.0191	0.0195	0.0039	0.0201	0.0192	0.0184
	100		0.0142	0.0165	0.0176	0.0042	0.0187	0.0179	0.0153
	200		0.0139	0.0150	0.0154	0.0065	0.0166	0.0166	0.0142
	400		0.0125	0.0142	0.0143	0.0076	0.0146	0.0140	0.0134
Inv Gauss (1,1)	50	0.05	0.0617	0.0688	0.0410	0.0674	0.0647	0.0652	0.0617
	75		0.0591	0.0612	0.0422	0.4612	0.0622	0.0631	0.0602
	100		0.0573	0.0583	0.0562	0.0597	0.0593	0.0601	0.0581
	200		0.0555	0.0564	0.0554	0.0565	0.0569	0.0577	0.0564
	400		0.0532	0.0541	0.0545	0.0551	0.0554	0.0569	0.0548
Gamma (2,1)	50	0.01	0.7203	0.6586	0.1928	0.1837	0.5826	0.5028	0.5228
	75		0.7965	0.7049	0.3512	0.3398	0.6683	0.5871	0.6841
	100		0.8341	0.7477	0.5644	0.4561	0.7188	0.6864	0.7054
	200		0.8867	0.7961	0.6987	0.6890	0.7699	0.7865	0.7751
	400		0.9054	0.8114	0.7854	0.7981	0.8298	0.8235	0.8310
Gamma (2,1)	50	0.05	0.6563	0.6989	0.4314	0.5089	0.6465	0.5721	0.6321
	75		0.7825	0.7437	0.4507	0.6193	0.7263	0.7174	0.7399
	100		0.8390	0.7983	0.6271	0.7110	0.7591	0.7622	0.7637
	200		0.8998	0.8341	0.7516	0.7182	0.8441	0.8206	0.8454
	400		0.9355	0.8862	0.8323	0.7901	0.8992	0.8733	0.8471
Log-normal (1,1)	50	0.01	0.6028	0.5672	0.0795	0.0467	0.6501	0.6551	0.6868
	75		0.7843	0.7802	0.1386	0.1518	0.7274	0.7235	0.7734
	100		0.8814	0.8486	0.2577	0.2881	0.7901	0.7847	0.8104
	200		0.9190	0.8910	0.3564	0.3922	0.8790	0.8671	0.9028
	400		0.9518	0.9522	0.4291	0.4264	0.9209	0.9152	0.9306
Log-normal (1,1)	50	0.05	0.7513	0.6954	0.1147	0.0935	0.7801	0.7620	0.7823
	75		0.8461	0.8231	0.1957	0.2096	0.8322	0.8451	0.8652
	100		0.9312	0.8992	0.2982	0.3105	0.9182	0.8851	0.9021
	200		0.9629	0.9337	0.3877	0.3869	0.9462	0.9342	0.9487
	400		0.9987	0.9765	0.5183	0.5333	0.9712	0.9763	0.9877
Weibull (3,3)	50	0.01	0.7245	0.4391	0.3826	0.3708	0.3289	0.3060	0.3289
	75		0.7991	0.5473	0.4994	0.4114	0.4015	0.4622	0.4461
	100		0.8622	0.6311	0.6078	0.5889	0.4980	0.5672	0.5921
	200		0.9320	0.8192	0.7216	0.6467	0.6500	0.6892	0.6981
	400		0.9561	0.8892	0.8014	0.7901	0.8109	0.8321	0.7905
Weibull (3,3)	50	0.05	0.8213	0.4981	0.4788	0.4562	0.3981	0.4590	0.4781
	75		0.8618	0.5790	0.5671	0.5712	0.5899	0.6172	0.6380
	100		0.9347	0.7234	0.6891	0.6902	0.6399	0.7241	0.7691
	200		0.9785	0.8214	0.7966	0.7709	0.7645	0.8129	0.8435
	400		0.9941	0.9023	0.8691	0.8501	0.8792	0.8763	0.9021
Rayleigh (1.5)	50	0.01	0.8346	0.7928	0.2549	0.2671	0.6355	0.4238	0.5567
	75		0.9049	0.8562	0.3890	0.3875	0.8382	0.6092	0.6792
	100		0.9562	0.9021	0.4904	0.4860	0.8692	0.7333	0.7826
	200		0.9766	0.9673	0.5891	0.6233	0.9077	0.7905	0.8409
	400		0.9912	0.9873	0.7561	0.7348	0.9561	0.8762	0.9045
Rayleigh (1.5)	50	0.05	0.8993	0.8567	0.3879	0.3907	0.7874	0.7423	0.7278
	75		0.9456	0.8934	0.4677	0.4992	0.9092	0.9230	0.7972
	100		0.9879	0.9310	0.5875	0.5881	0.9578	0.9651	0.8588
	200		0.9993	0.9762	0.6981	0.6861	0.9802	0.9908	0.8921
	400		1.0000	0.9995	0.7652	0.7881	0.9982	1.0000	0.9499

the null hypothesis that the data follows an inverse Gaussian distribution at both 1% (critical values are −0.2869 and 0.2869) and 5% (critical values are −0.2183 and 0.2183) level of significance.

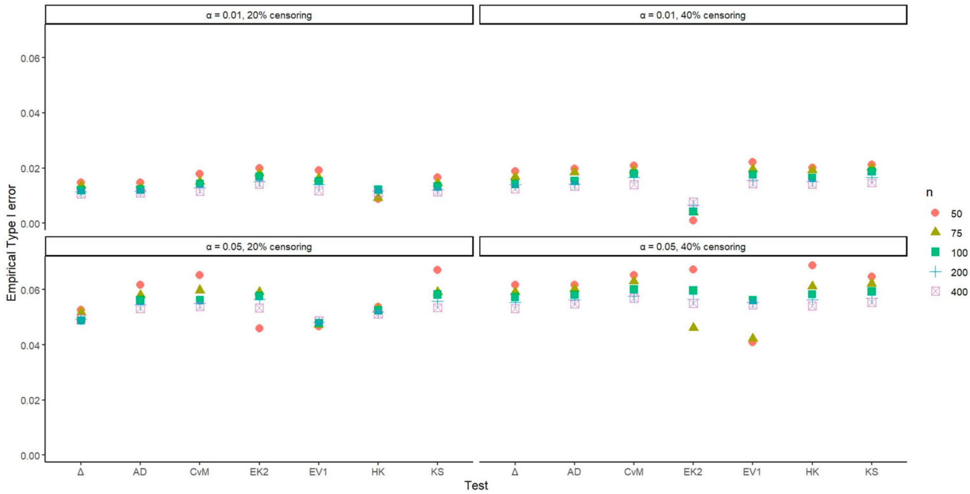

Figure 4. Plot of empirical type I error at $\alpha = 0.01$ and $\alpha = 0.05$ for IG (1, 1).

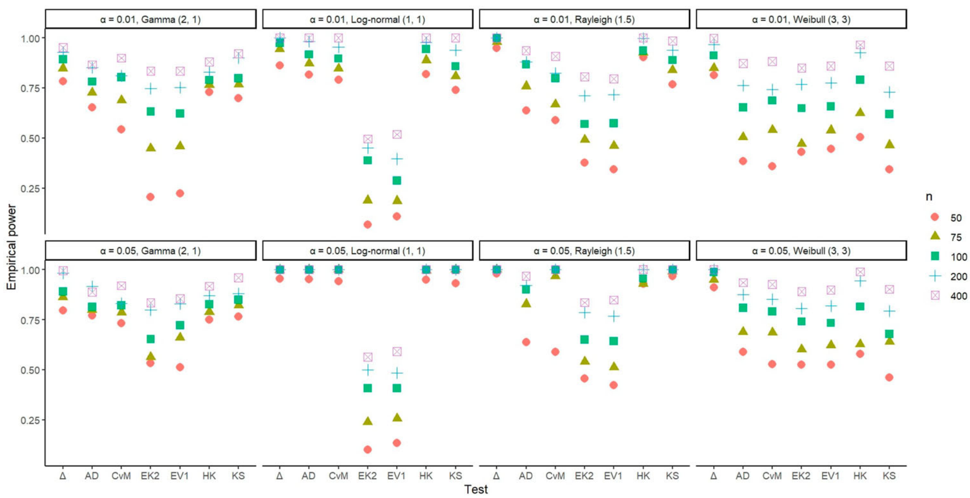

Figure 5. Plot of empirical power for 20% censoring.

Censored Case, Illustration 1:

We consider the same data used for the uncensored case and following Jayalath and Chhikara [13], we randomly select 6 observations out 46, to be randomly right censored and the right censoring times are generated using a random number generator. This generates, randomly right-censored data where 13% of observations are censored. We now standardize the data to apply the testing procedures discussed in Section 3, to test whether this dataset follows an IG distribution. When we use the same censoring points as in Jayalath and Chhikara [13] We obtain the test statistic as $\hat{\Delta}_c = -1.7341$. Accordingly, we accept the null hypothesis that the data follows an inverse Gaussian distribution at both

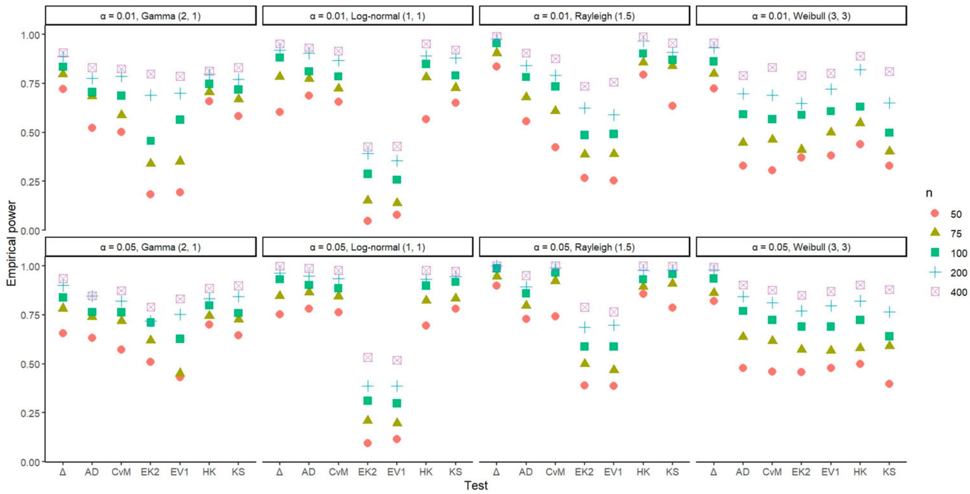

Figure 6. Plot of empirical power for 40% censoring.

1% and 5% levels of significance. We can also note that, when we use random censoring points also, we accept H_0.

Censored Case, Illustration 2:

We examine the data on survival times for patients with bile duct cancer who took part in a study to determine whether a combination of radiation treatment (RoRx) and the drug 5-fluorouracil (5-FU) prolonged survival [28]. Survival times in days for 47 patients are given, which include 3 censoring times. The complete data is given Lawless [29] (Page 372, Example 7.4). We analyse the data using the proposed procedures and the test statistic is obtained as -3.6553. The obtained test statistic value suggests that we can reject the null hypothesis that the data follows inverse Gaussian distribution at both 5% and 1% levels of significance.

6. Conclusion

In this article, we developed new goodness of fit tests for the inverse Gaussian distribution in the presence and absence of censoring. We used the fixed point characterization for inverse Gaussian distribution by Betsch and Ebner [20] to develop the tests. First, we proposed a test static based on U-Statistic theory for IG distribution, when the data is complete, and studied the asymptotic properties of the test statistic in detail. Several tests are available for IG distribution when the data is complete. For right-censored data, Ducharme [8] proposed a goodness of fit test for IG distribution, for which the computational procedures are complex. Inspired by this, we developed a new goodness of fit test for IG distribution for right-censored data which is easy to implement. We showed that the asymptotic distribution of the proposed test statistic is normal. A consistent estimator of the asymptotic variance is also obtained. Further, we also developed a JEL ratio test to test IG distribution with unit parameters.

An extensive Monte Carlo simulation study is conducted to evaluate the finite sample performance of the proposed tests. In the complete data scenario, we compared the performance of our test with the other tests available in the literature for IG distribution. For right-censored data, we generated pseudo-complete sample from the original right-censored data, using the method proposed by Balakrishnan et al. [14] and then applied the classical goodness of fit tests and other tests proposed for inverse Gaussian distribution. The proposed test procedures have a well-controlled type I error rate and good power against different alternatives. The practical applicability of the proposed procedures is well exemplified by applying it to four real datasets.

The test considered in this article deals with the right censoring scenario. In lifetime data analysis, the data may also suffer from several other forms of censoring and truncation. The proposed procedures can be modified to accommodate left truncated data. Further, we can develop the goodness of fit tests using similar fixed point characterization for other important lifetime distributions, in both continuous and discrete cases.

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Conflicts of Interest

The authors declare, no conflict of financial or non-financial interests that are directly or indirectly related to this research work.

Ethical statement

The submitted work is original and has not been published elsewhere.

Disclosure statement

No potential conflict of interest was reported by the authors.

Data availability statement

The source of the dataset used for illustration purposes is mentioned in the manuscript.

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