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ORIGINAL RESEARCH

Unit elements in the path algebra of an acyclic quiver

S. Karthika · M. Viji

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Abstract We investigate the algebraic properties of a particular non- commutative algebra, the path algebra, associated with a quiver. Quiver was initially introduced by Peter Gabriel. In this paper, we obtain a characterization for the invertibility of an element in the path algebra of an acyclic quiver. The study is an extension of the invertibility condition in a unique path quiver to acyclic quivers.

Keywords Acyclic quiver · Path algebra · Unit element

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1 Introduction

Quivers and its representation theory plays an important role in the study of non-commutative algebra. Peter Gabriel used the term Quiver [5] in representation theory for what is refered to as directed graph in most other areas. The reason for using the term quiver rather than graph is that the latter word already has too many concepts connected to it and is used in different senses as it can be oriented or not,with or without multiple arrows or loops,which may lead to certain ambiguity.

Path algebras were introduced and studied by Joachim Cuntz and Daniel Quillen [3]. It is an excellent tool for the study of non commutative algebra. Viji and Chakravarti [4] characterized the invertibility of an element in the path algebra of a unique path quiver. This motivated us to examine the form of a unit element in case of an acyclic quiver. Our main aim is to investigate the algebraic properties of path algebra associated to a quiver.

A *quiver* is a directed graph in which we allow multiple edges and loops. A *quiver* [2] $Q = (Q_0, Q_1, s, t)$ is a quadruple consisting of two sets: Q_0(whose elements are called *points*, or *vertices*) and Q_1 (whose elements are called *arrows*) and two maps $s, t : Q_1 \to Q_0$ which associates to each arrow $\alpha \in Q_1$ its source $s(\alpha) \in Q_0$ and its target $t(\alpha) \in Q_0$, respectively. Hereafter, we use the notation $Q = (Q_0, Q_1)$ or simply Q to denote a quiver.

Definition 1 A *path* of length l in Q is an expression of the form $(x|\alpha_1, \alpha_2, \ldots, \alpha_l|y)$ where $x, y \in Q_0$ and $\alpha_i \in Q_1$ such that $s(\alpha_1) = x, t(\alpha_i) = s(\alpha_i + 1)$ for $1 \leq i < l$ and $t(\alpha_l) = y$. If $l = 0$ we impose the condition that x = y. (Note that $x = y$ need not imply $l = 0$)

S. Karthika (✉) · M. Viji
Department of Mathematics, St.Thomas' College, Thrissur 680001, Kerala, India
E-mail: karthikasantharam@gmail.com

M. Viji
E-mail: vijigeethanjaly@gmail.com

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A stationary path is a path of length 0. It is of the form $(x|x)$ for $x \in Q_0$ and is denoted by ϵ_x. A path is a *cycle* if its source and target coincides. A quiver Q is said to be *Acyclic* if it contains no cycles.

Definition 2 Let Q be a quiver. The *Path Algebra KQ*, of Q is the $K-$algebra, whose underlying $K-$vector space has as a basis, the set of all paths $(a|\alpha_1, \alpha_2, ..., \alpha_l|b)$ of length $l \geq 0$. The product of two basis elements $(a|\alpha_1, \alpha_2, ..., \alpha_l|b)$ and $(c|\beta_1, \beta_2, ..., \beta_m|d)$ of KQ is defined as,

$$(a|\alpha_1, \alpha_2, ..., \alpha_l|b).(c|\beta_1, \beta_2, ..., \beta_m|d) = \delta_{bc}(a|\alpha_1, ..., \alpha_l, \beta_1, ...\beta_m|d).$$

Let KQ_l be the subspace of KQ generated by the set Q_l of all paths of length l, where $l \geq 0$. It is clear that $(KQ_n).(KQ_m) \subseteq (KQ_{n+m})$ and we have the direct sum decomposition

$$KQ = KQ_0 \oplus KQ_1 \oplus ... \oplus KQ_l \oplus ....$$

KQ is an associative algebra. It has an identity if and only if Q_0 is finite and acyclic.

Path Algebra: A Generalized Definition
In the paper, ' *On Quivers and Incidence Algebras*', Viji and Chakravarti gave a genaralised definition of path algebra. This definition is in agreement with the existing results and also led to new findings in infinite dimensional algebra.

Definition 3 Let Q be a quiver, and let P be the set of all paths in Q. A *Path Algebra* $\overline{KQ}$ ([4]) of Q is defined as $\left\{\sum_{\alpha\in P} c_\alpha \alpha \mid c_\alpha \in K, \alpha \in P\right\}$. Addition and scalar multiplication is defined componentwise. If $(a \mid \alpha_1, \alpha_2, ..., \alpha_l \mid b)$ and $(c \mid \beta_1, \beta_2, ..., \beta_m \mid d)$ are any paths in Q, their product is defined as,
$(a \mid \alpha_1, \alpha_2, ..., \alpha_l \mid b).(c \mid \beta_1, \beta_2, ..., \beta_m \mid d) = \delta_{bc}(a \mid \alpha_1, ..., \alpha_l, \beta_1, ...\beta_m \mid d)$.
The product of two arbitrary elements of $\overline{KQ}$ can be defined by assuming distributivity of multiplication of paths over arbitrary summation.

$$\therefore \left(\sum_{\alpha\in P} c_\alpha \alpha\right)\left(\sum_{\beta\in P} d_\beta \beta\right) = \sum_{\alpha,\beta\in P} c_\alpha d_\beta \alpha\beta$$

For a finite acyclic quiver Q, the set of all its paths P, will serve as a basis for $\overline{KQ}$.

By this generalized definition of path algebra, the following results were extended to infinite dimensional algebras.

Proposition 1 Let Q be a quiver and $\overline{KQ}$ be the corresponding path algebra.Then,

(a) $\overline{KQ}$ is an associative algebra
(b) The element $\sum_{i\in Q_0} \varepsilon_a$ is the identity in $\overline{KQ}$.
(c) $\overline{KQ}$ is finite dimensional if and only if Q is finite and acyclic.

Definition 4 **Unique Path Quiver**: If a quiver $Q = (Q_0, Q_1)$ is such that there exists atmost one path from x to y for each pair $x, y \in Q_0$, then we call Q a *unique path quiver*([4]).

Proposition 2 Let $Q = (Q_0, Q_1)$ be a unique path quiver then an element $a \in \overline{KQ}$ is a unit if and only if the coefficient of the stationary path ε_x is nonzero for all $x \in Q_0$.

2 Main Result

In previous section, we have seen the invertibility condition of an element in the path algebra of a unique path quiver. The proof of proposition 2 had the advantage that for any chosen pair of vertices x,y there was one and only one path from x to y. We needed some level of uniqueness to extend this result to an acyclic quiver. So we decided to treat each path $(\alpha_1, \alpha_2, ..., \alpha_n)$ as different unless they had the same constituent edges. In this section, we show that Proposition 2 holds for every acyclic quiver.

Theorem 1 *Let Q be any acyclic quiver. Then an element $a \in \overline{KQ}$ is a unit if and only if the co-efficient of ϵ_x is non-zero for all $x \in Q_0$.*

Proof Let $a = \sum_{\alpha \in P} a_\alpha \alpha \in \overline{KQ}$ be a unit. Then there exists an element $b = \sum_{\beta \in P} b_\beta \beta \in \overline{KQ}$ such that

$$ab = \sum_{x \in Q_0} \epsilon_x$$

$$\left(\sum_{\alpha \in P} a_\alpha \alpha\right)\left(\sum_{\beta \in P} b_\beta \beta\right) = \sum_{x \in Q_0} \epsilon_x$$

$$\sum_{\alpha, \beta \in P} (a_\alpha b_\beta)\alpha\beta = \sum_{x \in Q_0} \epsilon_x \tag{1}$$

For each $x \in Q_0$, $\alpha\beta = \epsilon_x$ iff $\alpha = \epsilon_x$ and $\beta = \epsilon_x$.

Coefficient of ϵ_x in LHS of (1) is $a_{\epsilon_x} b_{\epsilon_x}$.

Equating coefficients of ϵ_x on both sides of (1),

$$a_{\epsilon_x} b_{\epsilon_x} = 1 \Rightarrow a_{\epsilon_x} \neq 0$$

Thus the coefficient of stationary paths in the element a is non zero.

Conversely, let $a = \sum_{\alpha \in P} a_\alpha \alpha \in \overline{KQ}$ such that for all $x \in Q_0, a_{\epsilon_x} \neq 0$. Define $b = \sum_{\alpha \in P} b_\alpha \alpha \in \overline{KQ}$ by defining b_α inductively on the length of α as follows :

Let $b_{\epsilon_x} = a_{\epsilon_x}^{-1}$ for all $x \in Q_0$. This defines b_α on all paths α of length 0. Assume by induction that b_α has been defined whenever the length of α is less than l. Now take an α of length $l \geq 1$, say $\alpha = (x|\alpha_1, \alpha_2, ..., \alpha_l|y)$. Define

$$b_\alpha = \frac{-1}{a_{\epsilon_x}} \sum_{i=1}^{l} a_{(x|\alpha_1,...,\alpha_i|t(\alpha_i))} b_{(t(\alpha_i)|\alpha_{i+1},...,\alpha_l|y)}$$

Coefficient of ϵ_x in ab is

$$a_{\epsilon_x} b_{\epsilon_x} = a_{\epsilon_x} . \frac{1}{a_{\epsilon_x}} = 1 \, \forall x \in Q_0$$

For $l \geq 1$, coefficient of $\alpha = (x|\alpha_1, ..., \alpha_l|y)$in ab is

$$a_{\epsilon_x} b_\alpha + \sum_{i=1}^{l} a_{(x|\alpha_1,...,\alpha_i|t(\alpha_i))} b_{(t(\alpha_i)|\alpha_{i+1},...,\alpha_l|y)} = 0$$

Thus $ab = \sum_{x \in Q_0} \epsilon_x$ which implies a is a unit. □

The theorem gives a complete characterisation of a unit element in the path algebra of an acyclic quiver. In the case of path algebra of any quiver, an element can be a unit only if the coefficients of stationary paths are non zero.

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